Chapter 13

Structure-Genetic Didactical Analyses—Empirical Research “of the First Kind”

Abstract In mathematics education, theories of teaching and learning based on disciplines different from mathematics (“imported” theories) are widely dominating the field. This imbalance greatly reduces the impact of mathematics education both on teacher education and on the teaching practice. In order to return to a balanced situation it is necessary to pay more attention to theories which are based on mathematics. As an example of such a “homegrown” theory, the paper presents the structure-genetic didactical analysis, the research method of mathematics education conceived of as a “design science”.

Keywords Design science · Substantial learning environments · Didactical analysis · Empirical research

AMS (2000) Subject Classification: Primary 97C · Secondary 97D

A comparison of the papers published in journals and proceedings in the 1970s and early 1980s (see, for example, the pace-setting paper Krygowska 1972) with the papers in the new millennium shows that over the past two decades the coordinate system of mathematics education has shifted massively *away from*

- the subject matter mathematics,
- the teaching practice and
- the critical examination of educational foundations concerning the subject,

towards

- qualitative and quantitative empirical studies of learning and teaching processes,
- the development and application of tests and
- theories of learning mathematics based on ideas imported from other disciplines.

This shift consciously or unconsciously involved a break from the tradition of mathematics education. Nevertheless, this tradition is still alive. In recent decades a branch of mathematics education has developed that explicitly builds on the tradition of “subject matter didactics” as it has been common in the past in many countries. This “mathematics education *emerging from the subject*”, as it has been called, continues to carry the teaching of mathematics and teacher education and has created

a scientific basis of its own. The internationally known project Mathe 2000 may serve as an example (Wittmann 2012). “Mathematics education *emerging from the subject*” constitutes by no means “didactics from the armchair” for which its predecessor had been criticized. On the contrary, it is supported empirically in its own way. Its specific feature is that it rests on theories of teaching and learning that are *implicit* in the subject mathematics itself. This will be shown in this paper, which is structured as follows: in the first three sections three central themes of the curriculum will be considered both from the position of the present mainstream in mathematics education and from the position of mathematics education *emerging from the subject*. In the fourth section the research method of the latter, the *structure-genetic didactical analysis*, will be characterized and it will be indicated what can be achieved by this method.

1 Introduction of the Multiplication Table in Grade 2

In the curricula of many countries multiplication is introduced as “repeated addition” and the multiplication table is accordingly learned row by row. The last decade has seen a vivid discussion in the Anglo-Saxon countries about what multiplication is about. The empirical analysis of (Park and Núñez 2001) fits into this context. The authors compared two hypotheses of concept formation for multiplication: multiplication as “repeated addition” and multiplication as a “schema of correspondences”. What the latter means, however, remains unclear in that paper. It is likely that the authors allude to the interpretation of multiplication as a linear function: for a fixed multiplier c we have a mapping that assigns the product $x \cdot c (= c \cdot x)$ to any number x . As a result of their research the authors arrive at the conclusion that “repeated addition” should not be used for defining multiplication, but only for calculating the results.

From the perspective of mathematics education *emerging from the subject* multiplication in grade 2 can be approached in the following way: multiplication is defined as “abridged” addition, as it is common in mathematics. For calculating the results it is natural to refer to the laws of multiplication: among the multiples

$1 \cdot m$, $2 \cdot m$, $3 \cdot m$, $4 \cdot m$, $5 \cdot m$, $6 \cdot m$, $7 \cdot m$, $8 \cdot m$, $9 \cdot m$ and $10 \cdot m$

there are four multiples that are trivial or easy to calculate:

$1 \cdot m$, $2 \cdot m$ (double of $1 \cdot m$), $10 \cdot m$ and $5 \cdot m$ (half of $10 \cdot m$).

Other multiples from $3 \cdot 7$ to $9 \cdot 7$ can be derived from easy ones by means of the distributive law:

$$\begin{aligned}
 3 \cdot m &= 2 \cdot m + 1 \cdot m, \\
 4 \cdot m &= 2 \cdot m + 2 \cdot m \text{ (or } 5 \cdot m - 1 \cdot m), \\
 6 \cdot m &= 5 \cdot m + 1 \cdot m, \\
 7 \cdot m &= 5 \cdot m + 2 \cdot m, \\
 8 \cdot m &= 10 \cdot m - 2 \cdot m, \\
 9 \cdot m &= 10 \cdot m - 1 \cdot m.
 \end{aligned}$$

This approach has been elaborated by Arnold Fricke in his “operative didactics” and is widespread in German primary schools (Fricke 1968). In the early eighties Heinrich Winter went one step further: In line with his general postulate to look at arithmetic from the point of view of algebra he suggested to use rectangular arrays of dots for representing multiplication (Winter 1984). This proposal is also found in Courant and Robbins (1996, p. 3), a classic among mathematical textbooks, and in Freudenthal (1983, pp. 109–110). In (Penrose 1994, pp. 51–53) it is even stated that rectangular arrays of dots are the most efficient means to explain what multiplication is about.

The preference of eminent mathematicians for these arrays underlines the fact that this representation of multiplication is not just a visual aid which has been invented for the purpose of teaching, but that is fundamentally interwoven in the epistemological structure of mathematics. The great advantage of this representation is that the commutative law, the associative law and the distributive law can be derived in an operative way and used in teaching (see, for example, Wittmann and Müller 2017, pp. 201–211). This is not possible with other representations of multiplication.

Later in the curriculum arrays of dots pass into the representation of a product as the area of a rectangle and this representation reaches up to the integral. It is a fundamental idea of algebra and calculus.

Comparison: What multiplication is about and how it should be introduced in the classroom, cannot be decided by means of empirical methods imported from psychology, but should be based on a sound mathematical and epistemological analysis. This, however, is not to say that empirical investigations of learning processes are superfluous (see Sect. 3).

2 Designing a Substantial Learning Environment for Practicing Long Addition

While the first example deals with the didactical foundation of some topic the second example leads to the very core of teaching. The natural way to help learners to master some piece of knowledge or some skill is to offer them substantial learning environments that stimulate mathematical activities. Here the practice of skills plays a crucial role. Heinrich Winter introduced the concept of “productive practice” which means a type of practice in which contents and general objectives of mathematics

teaching (mathematizing, exploring, reasoning and communicating) are combined (Winter 1984).

In order to design a substantial learning environment for practicing long addition in our project Mathe 2000 we had to browse elementary mathematics for patterns that involve long addition. We had to check whether children’s knowledge in grade 3 is sufficient for understanding and solving the intended tasks, for exploring, discovering and describing patterns and for explaining them by using familiar means with some support of the teacher.

Our analyses led us to the following learning environment that is based on the famous rule “casting out nines” (Wittmann and Müller 2012, pp. 85).

The guiding problem posed to students is as follows:

Form two three-digit numbers with the six digit cards 2, 3, 4, 5, 6, and 7 and add these two numbers.

- (a) Find different results.
- (b) Try to reach results as near as possible to 600, 700, 800, 900, 1000, 1100, 1200 and 1300.
- (c) Try to find results between 900 and 1000.

The subtasks (b) and (c) are intended as hints for discovering the underlying pattern.

Guy Brousseau’s theory of didactical situations provides a natural framework for the teacher in putting a learning environment into practice (Brousseau 1997).

Here this theory can be applied as follows: In the first situation the problem is introduced to students, best by means of examples.

In the second situation students work on their own, individually or in groups. The teacher serves as an advisor.

In the third situation the results are collected and compared. The teacher is free to add some more examples, and to give hints that stimulate students to discover the underlying pattern. Subtask (b) is particularly helpful as the optimal results 603, 702, 801, 900, 999, 1008, 1098, 1107, 1197, 1206, 1296, 1305 reveal a striking pattern: The total of the digits of these numbers is 9, 18 or 27.

The results in subtask (c) support these findings. Possible results are 900, 909, 918, 927, 936, 963, 972, 981, 990, 999.

A check with other examples will confirm this pattern. Of course some students will offer calculations with results that seem to violate this pattern. However, checks will reveal mistakes in the calculations.

In this way the conjecture is formed that for this problem only results are possible for which the total of the digits is a multiple of 9.

Situation 4 in Brousseau’s classification requires the explanation of this pattern. The place value chart with which students in grade 3 are familiar, serves this purpose perfectly (Wittmann and Müller 2013, 120–121): Some examples are represented by means of counters on the place value chart. It is interesting to note that in this context the total of the digits of a number has a very concrete meaning: It denotes the number of counters that are necessary for representing the number on the place value chart.

Figure 1 shows two examples:

In the second example again 27 counters are needed to represent the sum. We have a carry from the Ones to the Tens column and a second carry from the Hundreds to the Thousands column. According to the two carries the total of digits of the result 1161 is $27 - 2 \cdot 9 = 9$.

The fifth and final didactical situation is “institutionalization”. Here the teacher’s task is to summarize in a concise way what has been discovered. This might include the information that the operation of “casting out nines” is independent of the special numbers used here: For any sum of two or more numbers the sum of the totals of the digits of the numbers differs from the total of the digits of the result by a multiple of 9. The reason is that any carry involves a “loss” of 9 counters.

Comparison. In this example the “home-grown” approach is unrivaled. It is obvious that theories of mathematics education imported from elsewhere, as well as empirical methods, are blunt when it comes to designing substantial learning environments. Only a thorough knowledge of mathematical structures and processes connected with curricular expertise will lead to solutions, and this knowledge is also essential for the teacher in doing her or his job.

3 Nets of a Cube

Nets of the cube are a standard topic of mathematics teaching at the secondary level. In this section two approaches to this topic are compared.

Susanne Prediger and Claudia Scherres have conducted guided clinical interviews with pairs of students in grade 5 (Prediger and Scherres 2012). The objective of this study has been to investigate in some depth how students proceed when trying to find as many different nets as possible. The authors applied quite a number of empirical instruments in order to obtain a differentiated picture of the processes occurring during the collaboration. The results of this study are very complex and therefore cannot be summarized in short terms. For the following comparison two findings are relevant (Prediger and Scherres 2012, p. 171):

1. Pairs of students can often exhaust their potential only through the intervention of the teacher.
2. The cooperation for exploiting the potential fully is enhanced when this cooperation is guided by mathematical considerations.

From the perspective of developmental research the first objective of a didactical analysis concerning the topic “nets of the cube” is to find out at which place of the curriculum students are in a position to respond to the requirements that certain treatment of this topic involves. At the very outset it should be kept in mind that any beautiful and important topic might allow for different approaches suitable for different places in the curriculum.

In the Mathe 2000 curriculum nets of the cube are embedded in the fundamental idea of “dissecting and recombining figures”, which is systematically developed along grade levels. An easy way of determining all possible nets is revealed in connection with polyominoes, a rich topic that was introduced by Golomb (1962) and elaborated for the primary level in Besuden (1984). A polyomino is a composition of congruent squares edge by edge. Polyominoes that are congruent are considered as equal. It is easy to see that there is only one domino (with two squares), but that there are two different triominoes (with three squares). Children in grade 3 easily find all 5 tetrominoes (with 4 squares) by adding one square to triominoes, and also all 12 pentominoes (with 5 squares) by extending tetrominoes. It is a stimulating task for kids to determine the 8 pentominoes that can be folded into an open cube.

In a textbook for grade 3, the 11 nets of a cube are obtained in the following way (Wittmann and Müller 2013, p. 65): The children are informed that it is possible to derive all 35 hexominoes by extending the 12 pentominoes. As this process would take too much time, the 35 hexominoes are provided by the teacher (Fig. 2) and the students are asked to find out which of these hexominoes are nets of a cube. In Fig. 2 the nets are arranged in five groups of 7 nets. This suggests forming five groups of students each of which has to make their 7 hexominoes with paper squares and sellotape and to investigate which ones can be folded into a cube. All five groups have to explain the reasons why some of their hexominoes do not produce nets. So in cooperation all 11 nets are determined through cooperation in a rigorous way.

1. 2. 3. 4. 5. 6. 7.

8. 9. 10. 11. 12. 13. 14.

15. 16. 17. 18. 19. 20.

21. 22. 23. 24.

25. 26. 27. 28. 29.

30. 31. 32. 33. 34. 35.

1 a) Baut in Gruppen alle 35 Sechslinge aus Quadraten nach.
b) Findet die 11 Sechslinge heraus, aus denen man Würfel falten kann. Diese Sechslinge nennt man **Würfelnetze**. Begründet eure Auswahl.

2 Zu welchen Würfelnetzen sind diese Netze deckungsgleich?

a) b) c) d) e)

Fig. 2 Selecting the nets of cubes from the set of hexominoes

An alternative approach at this level would be to start from the 8 pentominoes that can be folded into an open cube and to extend them to nets of a cube. However, as most nets can be derived from different nets of an open cube, it may be rather complicated to eliminate congruent nets.

In grade 5, the theme “nets of a cube” should be revisited. Again it seems appropriate to provide the students first with paper squares and sellotape and to stimulate them to find as many different nets as possible. Based on students’ findings the teacher can guide the students to a systematic derivation of all possible nets. A natural way is to refer to the “addition principle” of combinatorics which consists of subdividing the set of combinatorial possibilities into subsets which are easier to manage. In the case of nets of the cube the maximum number of squares in a row is an appropriate criterion for a classification as is indicated briefly.

Case 1: *6 squares in a row*

No cube is possible as there are overlays and two faces remain open.

Case 2: *5 squares in a row*

Again no cube is possible as there is one overlay and one face remains open.

Case 3: *At most 4 squares in a row*

First it must be found out where a fifth square can be added so that a net becomes possible. For each of the two possible positions of the fifth square the possible positions of the sixth square have to be determined. Some care is needed to eliminate nets that are congruent to nets that have been found before. Figure 3 shows how to proceed stepwise starting from four squares in a row. The six nets determined in this way are drawn in bold lines.

Fig. 3 Derivation of the nets of a cube where at most four squares are in a row

Case 4: *At most 3 squares in a row*

In Fig. 4 no arrows are drawn away from the four pentominoes on the right. The reason is that the extensions of these pentominoes would result in nets that were already found.

Case 5: *At most 2 squares in a row*

In this case there is essentially only one way to get a net (Fig. 5).

It is obvious that this systematic derivation of all 11 nets of the cube is not easy. However, only means are used that are accessible to students in grade 5. With the assistance of the teacher, this learning environment is good to handle.

Of course it cannot be predicted how the investigation of this learning environment might develop in a certain class. Every interaction takes place under the particular circumstances of the class. However, a teacher who knows the mathematical background

Fig. 4 Derivation of the nets of a cube where at most three squares are in a row

Fig. 5 Derivation of the only net of a cube where at most two squares are in a row

thoroughly is in a position to deal flexibly and productively with the contributions and ideas from the students. Based on their findings the teacher can introduce the classification. Different groups of students can investigate the three cases. In this way the complexity of the task is reduced to a reasonable level. The teacher can provide support where necessary.

Comparison: In this example the empirical investigation and the didactical analysis complement each other. Both are useful and instructive. There is no question that a teacher who has more insight into the processes linked to finding the various nets is more likely to interact with the students than a teacher who closely adheres to the mathematical structure and hardly leaves any room to the students. On the other hand, it is hard to imagine that a teacher who does not have a clear picture of the mathematical structure can organize a lesson solely with the spontaneous ideas of the students and with general pedagogical knowledge.

With respect to teaching and to teacher education, there are nevertheless significant differences between the two approaches. It is questionable if the “high resolution” instruments that have been employed in the empirical study by Prediger and Scherres (2012) can be communicated to teachers and students teachers in the time that is usually available in teacher education. It is also a question whether the results of this study can be integrated into teaching materials that work without the intervention of a teacher. The main findings prove the opposite.

In contrast, the didactical analysis requires only a relatively small amount of time and can be well integrated into teacher education. The language that is used is simple and easy to understand. If the nets of a cube are included in both mathematical and didactical courses in an inquiry-based way there is a good chance that the metacognitive and cooperative skills that have been found as important in the empirical study can be acquired implicitly in these courses. This, however, is not to devalue empirical studies. The aim of this paper is to plead for didactical analyses as one tool of mathematics education without excluding other tools.

4 Structure-Genetic Didactical Analyses

The approach of mathematics education *emerging from the subject* is based on the following assumptions:

1. Mathematical skills and techniques are acquired best in an active way under the guidance of mathematically experienced teachers. This refers to both teaching and teacher education. The practice of skills in its various forms plays a crucial role for successful learning.
2. The level of achievement that can be reached depends on the organization of teaching along fundamental mathematical ideas that are being revisited continuously. Only in this way is it possible to secure solid foundations for further learning and to brush up on prior knowledge. Also, only in this way it is also possible to provide mathematical structures as building blocks for modeling real situations. The development of curricula that are consistently and systematically designed accordingly and combine the orientation towards structures with the orientation towards applications is the central task of mathematics education.
3. Authentic mathematical activities in which heuristic plays a crucial role, are by their very nature social and communicative and include theories of teaching and learning *quite naturally* (implicit didactics). To make student teachers and teachers aware of these implicit theories by referring to their own mathematical experiences is the most direct and most efficient form of providing them with (explicit) didactical knowledge.

Against this background, didactical analyses as employed in the examples above are playing a fundamentally important role. This research method, which is the gold standard in mathematics education conceived of as a “design science”, is an extension of the traditional “subject matter didactics”. While the latter has been focused on the

logical analysis of subject matter and too much linked to the “broadcast” method of transmitting knowledge from the teacher to the student, the extended method emphasizes both the genesis of knowledge over the grades and individual learning processes. In order to emphasize this wider perspective, the term *structure-genetic didactical analysis* is proposed for this extended method.

The above examples show that structure-genetic didactical analyses are linked to hard facts: to the mathematical practice in exploring, describing and explaining patterns on various levels, to the prerequisite knowledge of learners, to the objectives of teaching and to the curriculum. This is all *empirical material*. Therefore, the structure-genetic didactical analysis is an empirical method. Because of its nativeness it may well be considered as empirical research of “the first kind”. The usual empirical studies are then empirical research of the “second kind”. The assertion that only empirical studies of the second kind would provide “evidence-based models” for teaching and learning is untenable.

Structure-genetic didactic analyses are of primary importance in mathematics education for the following reasons:

1. They emerge from the *mathematical practice*, that is from *doing mathematics*, at various levels.
2. They foster an *active relationship* with mathematics as a living subject.
3. They are *constructive* and therefore absolutely essential for designing substantial learning environments and curricula.
4. They are natural guidelines for teachers, as they unfold the *implicit theories of teaching and learning* of mathematics, that is, as they “unfreeze” the “didactical moments frozen in the subject” (Heintel 1978, 46).
5. They are *meaningful for teachers*, as the feedback from the field clearly demonstrates.

The examples in the first three sections show that structure-genetic didactical analyses take the following points into account:

- mathematical substance and richness in activities at different levels,
- evaluation of the cognitive load on students,
- curricular matching (with respect to contents and general objectives)
- coherence and consistency along the curriculum,
- curricular reach,
- potential for practicing skills (most important!)
- estimation of the expenditure of time.

Paradigms of structure-genetic didactical analyses are Wheeler (1967), Freudenthal (1983) and the developmental research initiated by Hans Freudenthal at the IOWO in the 1970s, the developmental research initiated by Nicolas Rouche at the CREM in Belgium, see for example (Rouche et al. 1996), as well as the work of Heinrich Winter, the German Freudenthal, in particular (Winter 2015). These paradigms demonstrate that the development of mathematics education as a research discipline also depends on the design of conceptually founded substantial learning

environments. Achievements in this direction have to be acknowledged as *results of research*.

In the context of this paper point 4 above is of particular importance and therefore deserves some elaboration. The idea that theories of teaching and learning are implicitly contained in the subject matter, and that therefore mathematics education is not completely dependent on imports of theories from other disciplines is by far not new. More than 100 years ago John Dewey has formulated this idea with a clarity that leaves nothing to be desired. In his paper there is a long enlightening section on the importance of the subject matter for teacher education (Dewey 1977, pp. 263–264):

Scholastic knowledge is sometimes regarded as if it were something quite irrelevant to method. When this attitude is even unconsciously assumed, method becomes an external attachment to knowledge of subject-matter. It has to be elaborated and acquired in relative independence from subject-matter, and *then* applied.

Now the body of knowledge which constitutes the subject-matter of the student teacher must, by the very nature of the case, be organized subject-matter. It is not a separate miscellaneous heap of scraps. Even if (as in the case of history and literature), it be not technically termed “science,” it is none the less material which has been subjected to method—has been selected and arranged with reference to controlling intellectual principles. There is, therefore, method in subject-matter itself—method indeed of the highest order which the human mind has yet evolved, scientific method.

It cannot be too strongly emphasized that this scientific method is the method of the mind itself. The classifications, interpretations, explanations, and generalizations which make subject-matter a branch of study do not lie externally in facts apart from mind. They reflect the attitudes and workings of mind in its endeavor to bring raw material of experience to a point where it at once satisfies stimulates the needs of active thought. Such being the case, there is something wrong with the “academic” side of professional training, if by means of it the student does not constantly get object-lessons of the finest type in the kind of mental activity which characterizes mental growth and, hence, the educative process. (...)

Only a teacher thoroughly trained in the higher levels of intellectual method and who thus has constantly in his own mind a sense of what adequate and genuine intellectual activity means, will be likely, in deed, not in mere word, to respect the mental integrity and force of children.

For the teaching practice this view is of fundamental importance: The ancient Greeks understood ‘*theory*’ as *view*. The Greek word for theory, *θεωρία* is derived from *θεωρεῖν*, which means *viewing, regarding, observing*. In this original sense a *theory* provides a comprehensive view of some area that allows for acting purposefully in this area while taking some circumstances and contingencies in this area into account. The natural theories of teaching and learning embedded in subject matter serve exactly this purpose: they represent practicable theories *for the teacher*, and they supply him or her with profound information or knowledge on which to base her or his actions. Whether it is to introduce children to multiplication, or to practice long addition, or to determine the nets of the cube; or to estimate students’ prerequisite knowledge, to activate their thinking, to interact and communicate with them; or to interpret students’ oral and written utterings, to assess their learning progress or to start remedial work—all this is essentially determined by the teacher’s “comprehensive view” of the topic to be learned. That teaching does not proceed smoothly, that there are breaks and obstacles in the learning processes, that students make mistakes,

have difficulties in understanding some points, forget what they have learned before, and so on: This knowledge is an essential part of the implicit theories of teaching and learning arising from *an active mastery of subject matter*.

What therefore counts most in teacher preparation is not an explicit didactical component (i.e., method courses), but the *mathematical* component, *given that* in this component mathematical activities are offered that stimulate and provide student teachers with relevant experiences in regard to learning processes, including learning difficulties, phases of confusion, confidence in overcoming difficulties and so on.

Mathematical courses organized in this way also provide the most effective *theoretical* basis for teaching. This is not to say that theories imported from other disciplines are of no use. They may be. This is also not to say that method courses are superfluous. Rather, both imported theories and method courses can significantly enhance structure-genetic didactical analyses. However, they should not replace them.

5 Conclusion

This paper is a plea for structure-genetic didactical analyses, the empirical research of the first kind. It must not be misunderstood as a plea against empirical studies of the second kind. On the contrary, such studies are indispensable, when new topics are to be introduced, for which no information on students' prerequisite knowledge is available, and when new approaches or new means of representations are used. Examples are the introduction of stochastics at the primary level or the use of digital media. Empirical research of the second kind is also very useful for investigating the processes more closely that occur when a learning environment is "staged" in the classroom. Of course these studies are all the more revealing and more meaningful, the closer they are attached to structure-genetic analyses.

It has also to be acknowledged that a wider perspective in mathematics education including imports from related disciplines significantly contributes to a better understanding of mathematics and therefore supports structure-genetic didactical analyses. In this sense the present author has greatly profited from Jean Piaget's genetic epistemology. It is no accident that the term "genetic" is a constituent of the term "structure-genetic didactical analysis".

In a position paper on the nature of mathematics education Heinz Griesel contended that in his sense "didactical analyses" would not differ from the "logical analyses" of mathematics (Griesel 1974). Heinz Steinbring rightly rejected this narrow view (Steinbring 2011). With structure-genetic didactical analyses the situation is completely different. These analyses include logical analyses, it is true, however, they involve also knowledge about mathematical processes, about the curriculum, about students' prerequisite knowledge at different levels, and about the boundary conditions of teaching. A mere knowledge of (elementary) mathematics is by far not sufficient. To put oneself in the place of a child who takes his or her first steps

in early mathematics, to look at the multiplication table with the eyes of a second grader, to find the nets of a cube with the means that are available to students at the secondary level, or to make the concept of a limit accessible to high school students, all this requires a special didactical approach and a special sensitivity for the genesis of knowledge and for the mathematical practice at the level in question.

Mathematics education has certainly been enriched enormously by contributions from other disciplines. Structure-genetic didactical analyses are nevertheless the key for developing mathematics teaching and teacher education. Without them mathematics education is in danger to degenerate into a self-referential system. Jeremy Kilpatrick’s warning of the “reasonable ineffectiveness of research in mathematics education” should, thus, be taken seriously (Kilpatrick 1981).

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