Chapter 11

Operative Proofs in School Mathematics and Elementary Mathematics

Abstract This paper gives an account of the conceptual and practical approach to “operative proofs” that has been developed in the Mathe 2000 project. By means of some typical learning environments, this notion and its theoretical background are explained.

Keywords Operative proof · Learning environments · Practicing skills · Design science

In the past few decades, the notion of “proof” has been a prominent topic of research in mathematics education, both within the German-speaking world and at the international level. Within this topic, we can distinguish between three lines of research: the philosophical and epistemological aspects of proving, the elaboration of the various functions of proving, and the empirical investigation of students’ routes to proving. Hanna and de Villiers (2012) provide an excellent overview of this research.

The present paper builds upon an independent line of research that has been evolving in German mathematics education in the context of elaborating on the operative principle and the genetic principle. This line of research is closely connected with curriculum development, and for this reason it is of particular interest for Mathe 2000, a project that is based on the following two basic assumptions:

1. A seamless learning process throughout a child’s education is only possible if the teaching of mathematics from kindergarten through the end of high school is treated as a whole and if it reflects an authentic view of mathematics as the science of patterns (Wittmann 2006).
2. Mathematics education can best serve its purpose for developing mathematics teaching if it is conceived of as a “design science” (Simon 1970), that is, if the design, the empirical investigation and the implementation of the artificial objects of the design science mathematics education, namely substantial learning environments, are put at the very core of developmental research (Wittmann 1995, 2002).

In accordance with the first assumption, the project aims at introducing fundamental ideas of mathematics early and at developing them in a *genetic* way. Proving is one of these fundamental ideas. The investigation of this idea within the framework

of ordinary teaching, namely by employing the usual means of representation and by connecting proof to the practice of skills, is a challenge on which we have focused.

It is on the second of these aspects, the practice of skills, that we have placed particular emphasis, as we see it as absolutely crucial to a successful and sustainable learning process. During the developmental research in Mathe 2000, the concept of “operative proof” has taken shape more and more. Papers by Werner Walsch and Heinrich Winter on proof, both related to curriculum development, have been important landmarks for us (see e.g. Walsch 1972; Winter 1984).

The structure of the present paper reflects the second basic assumption. The first section describes some learning environments which include “operative” proofs. These examples serve as illustrations for the second section, in which the notion of “operative proof” is explained, as well as for the last section, in which the theoretical underpinning of this notion will be described.

1 Some Learning Environments with Embedded Operative Proofs

The following four learning environments cover the spectrum from grades 1 to 6. At this level, the special features of operative proofs become particularly clear.

1.1 *Even and Odd Numbers*

Counters are a fundamental means of representing numbers in primary-school mathematics. Usually they are understood as “teaching aids” which have been specially invented for this purpose. However, their status is not primarily a didactic, but rather an epistemological one: in the time of Pythagoras there was a period in Greek arithmetic known as “*ψηφοι* arithmetic” which can be considered the cradle of arithmetic (Becker 1954, 34–41; Damerow and Lefèvre 1981).

In the Mathe 2000 curriculum, odd and even numbers are introduced in grade 1 in the ancient Greek fashion by means of special patterns of counters (Fig. 1).

Fig. 1 Representation of even and odd numbers by dot arrays

These patterns are painted on cardboard and cut out so that children can perform operations with the pieces and form sums of numbers. The initial exercises help children become familiar with the material. The next exercise asks the children to find sums with an *even* result. This is a first invitation to look at the structure more carefully. The subsequent task is more direct, as children are asked to reflect on the results of the four packages of sums in Fig. 2: “What do you notice? Can you explain it?”

$4 + 6 =$	$5 + 1 =$	$2 + 1 =$	$1 + 8 =$
$6 + 8 =$	$7 + 3 =$	$4 + 3 =$	$3 + 6 =$
$8 + 4 =$	$9 + 5 =$	$6 + 5 =$	$5 + 4 =$
$10 + 2 =$	$5 + 7 =$	$8 + 7 =$	$7 + 2 =$
$12 + 8 =$	$9 + 9 =$	$10 + 9 =$	$9 + 0 =$

Fig. 2 Pretty packages with even and odd summands

At this early level, teachers are expected to refrain from pushing the children. All they should do is listen to children’s spontaneous attempts to grasp the underlying patterns.

In grades 2 and 3, even and odd numbers are revisited using a wider range of numbers. This becomes necessary because there will inevitably be some children who will have to realize that 30, for example, is an even number although 3 is an odd number. Children are again given small packages of problems similar to those in Fig. 2 with larger numbers and asked the same questions.

At this level, the even/odd patterns are recognized more clearly and expressed in the children’s own words more precisely. In the manual, teachers are advised to be content with children’s spontaneous explanations and “warned” against demanding a “proof.”

In grade 4, however, children are expected to have enough experience with even and odd numbers and to be ready to tackle the following task, which explicitly demands a proof:

Even numbers can be represented by double rows, odd numbers by double rows and a singleton. Use this representation to prove that:

- The sum of two even numbers is always even.*
- The sum of two odd numbers is always even.*
- The sum of an even and an odd number is always odd.*

Children realize that no singletons occur when even patterns are combined and that in the case of two odd patterns, the two singletons form a pair, yielding another even result. Children also see that the singleton is preserved if an even and an odd pattern are combined and that in this case the result must be odd. The teacher’s task is to take up the children’s attempts and to assist the children in formulating coherent lines of argument.

The formal proof is addressed in higher grades, and in fact it expresses exactly the same relationships, albeit using a different language: the language of algebra. In general, operations with patterns of counters are an excellent preparation for algebraic calculations.

1.2 Multiplicative Arrow Strings

In grade 2 the multiplication of natural numbers is based on rectangular arrays of counters. The Hundred array (ten lines of ten dots, subdivided into four quadrants by the vertical and horizontal midlines) is a very convenient teaching aid. Children can easily represent and determine all products of the multiplication table. The subdivision of the field suggests the implicit use of the distributive law in calculating the results.

From the multitude of exercises, the following one is selected, in which children are offered strings of operators as in Fig. 3:

Fig. 3 Arrow strings

When reflecting on the results, students recognize that the target numbers differ from the starting numbers in a systematic way: in the first chain, the target number is always 1 more than the starting number; in the second chain, it is 2 more, etc.

An explanation of these number patterns can be given by referring to arrays of counters.

Fig. 4 An operative proof of the pattern underlying Fig. 3, a)

Figure 4 must be read as follows: We place 3 counters, double them, add two more counters, and finally divide by 2. We get one counter more than we had at the beginning. We start with 5 counters, double them, add two more counters and divide by 2. Again we get one counter more than we had before. We start with 6 counters, etc.

The repetition of the argument for several starting numbers is essential.

In grade 3, these operator chains are resumed with larger numbers; the operators $\cdot 2$, $+2$ and $:2$ are replaced by the operators $\cdot 20$, $+20$, $:20$, etc. The earlier explanations are repeated by referring again to arrays. This time, however, they are only given in shorthand notation (Fig. 5).

Fig. 5 An operative proof of the pattern with bigger numbers

The verbal description is as follows: “5 times 20 plus 20 is 6 times 20; 6 times 20 divided by 20 is 6, one more than 5” etc.

This argument is based on general relationships between numbers, not on special numbers. The approach provides efficient preparation for the transition to algebra long before variables are used.

1.3 Egyptian Fractions

This learning environment deals with a classic topic that can be found in some secondary-school textbooks, albeit without including a proof.

It is well known that the ancient Egyptians represented fractions smaller than 1 as sums of different unit fractions (with the numerator 1). To achieve this, they used a table for fractions of the type $2/2n + 1$. The mathematical question is whether any fraction smaller than 1 can be represented in this way. The answer is in the positive and the standard proof runs as follows: Let n/m be a reduced fraction, $n < m$. We choose the largest unit fraction $1/k$ smaller than n/m and subtract it from the given fraction:

$$n/m - 1/k = (n \cdot k - m)/m \cdot k.$$

The numerator $(n \cdot k - m)$ of the fraction on the right side must be smaller than n . Otherwise $1/k$ would not be the largest unit fraction smaller than n/m . Therefore, $(n \cdot k - m)/m$ is a fraction with a numerator smaller than n , and it is smaller than $1/k$. This procedure can be repeated. Step by step the numerators of the remaining fractions get smaller and smaller and, in a finite number of steps, one arrives at the numerator 1 and at a representation of n/m as a sum of different unit fractions.

There is another proof which rests on a repeated use of the formula

$$2/(2n + 1) = 1/(n + 1) + 1/(2n + 1) \cdot (n + 1).$$

The hard part, however, is showing that this algorithm terminates (see Fung 2005).

Both proofs go far beyond the secondary-school level, and again the question arises if it is possible to explain the existence of such a representation with elementary means. The following learning environment shows that it is possible.

First, students are provided with some historical background information. Then they are asked to find representations of reduced fractions of the type $2/3$, $2/5$, $2/7$, ... as sums of different unit fractions. This investigation, which involves ample practice in adding and subtracting fractions, takes some time and leads to the following pattern:

$$2/3 = 1/2 + 1/6, \quad 2/5 = 1/3 + 1/15, \quad 2/7 = 1/4 + 1/28, \quad 2/9 = 1/5 + 1/45, \quad \dots,$$

It is not important whether the students discover the underlying pattern themselves or whether the teacher provides some hints that incorporate the students' findings.

The explanation of this pattern is quite easy, as it rests on a very simple operative relationship: if the denominator of a fraction is increased, the fraction is decreased. If an arbitrary fraction of the type $2/2n + 1$ is given, say $2/31$, we increase the denominator by 1 and get a smaller fraction with an even denominator, $2/32$, and this fraction can be reduced to a unit fraction, $1/16$. Calculating the difference leads to

$$2/31 - 1/16 = (2 \cdot 16 - 31)/31 \cdot 16 = 1/31 \cdot 16 = 1/496,$$

which is a unit fraction. This procedure can be applied to *any* fraction of the type $2/2n+1$. The numerator of the difference must always be 1, as it marks the difference between an odd number and the subsequent even number. Students should verify this

fact by calculating quite a number of examples. In noting down the calculations on the blackboard, the table of the ancient Egyptians is re-established.

The next step is to look at reduced fractions of the type $3/n$, where n is not a multiple of 3. These are the fractions $3/4, 3/5, 3/7, 3/8, 3/10, 3/11, \dots$

Again, the students' calculations can be ordered with the teacher's assistance. Perhaps some students will find out by themselves that the idea they have already applied to fractions with the numerator 2 can be adapted: Take any reduced fraction with the numerator 3, say $3/31$. Increase the denominator until you get a multiple of 3. The fraction $3/33$ is smaller than the given fraction and can be reduced to a unit fraction, $1/11$.

Calculating the difference leads to

$$3/31 - 1/11 = (3 \cdot 11 - 31)/31 \cdot 11 = (33 - 31)/31 \cdot 11 = 2/31 \cdot 11 = 2/341,$$

a fraction with the numerator 2. This difference can be treated as before:

$$2/341 = 1/171 + 1/341 \cdot 171 = 1/171 + 1/58311.$$

In this context, it becomes obvious that the numerator of the difference must be smaller than the numerator of the given fraction because it measures the distance of the denominator from the next largest multiple of the numerator. If the numerator of the difference is 1, the difference is a unit fraction and we are done. If it is 2, we can apply our earlier results for fractions with the numerator 2.

In the same way, fractions of the type $4/n$ can be reduced to fractions with the numerators 3, 2 or 1, and by mathematical induction we conclude that any reduced fraction smaller than 1 can be represented as a sum of different unit fractions.

Again, students should verify the procedure for quite a number of fractions. Example:

$$\begin{aligned} 5/11 - 5/15 &= 5/11 - 1/3 = (15 - 11)/3 \cdot 11 = 4/33 \\ 4/33 - 4/36 &= 4/33 - 1/9 = (36 - 33)/297 = 3/297 = 1/99 \end{aligned}$$

Therefore: $5/11 = 1/3 + 1/9 + 1/99$.

In order to double-check the calculations, the unit fractions should be added up, a useful exercise for adding fractions.

As in the previous learning environments, the notion of proof is not present at the beginning. It is only after quite a number of calculations that patterns are recognized and verified by checking the examples, and it is not until much later that these patterns are explained by looking at the effects of the operations. Practicing skills and proof are inseparably intertwined within a truly mathematical investigation.

1.4 Fitting Polygons

The following series of learning environments is based upon “fitting,” a fundamental idea of elementary geometry (Freudenthal 1971, 422–423).

To develop the idea of “fitting” across the primary grades, the Mathe 2000 curriculum starts in grade 1 with the following activity (Fig. 6): paper squares of equal size are cut into two or four isosceles right triangles, and the parts are recombined to make other shapes.

Fig. 6 Decomposing a square into isosceles triangles and rearranging the parts

At this level, the approach is essentially experimental. Students move the parts around and see if they fit. However, it is not merely experimentation that is at work here. For example, two right angles form a straight angle by the “definition” of a right angle. One of the shapes that can be obtained in this way is a special case of the Pythagorean theorem.

In grade 2 this activity is extended: paper squares are folded and cut so that four congruent right triangles are obtained (Fig. 7). One of the shapes that can be made by re-arranging these basic forms is well known as a foundation of the Pythagorean theorem.

Fig. 7 Decomposing a square into four rectangles and rearranging the parts

Fitting regular polygons together is continued in grade 3 by means of a template for drawing squares, regular triangles, pentagons, hexagons and octagons with the same side length (Fig. 8). Children can explore, still experimentally, which shapes fit which way, a very creative exercise. They realize that there are only three regular

tessellations, discover some semi-regular tessellations and quite a number of other tessellations.

Fig. 8 Template for drawing regular polygons

Fig. 9 “clock template”

In grade 4 children make regular polygons from cardboard by means of the “clock template” (Fig. 9) and build the five Platonic solids (Winter 1986). The name “clock template” is derived from the fact that a circle is divided into 60 equal parts. As 60 is divisible by 3, 4, 5 and 6, the clock template allows for a convenient construction of squares, regular triangles, pentagons and hexagons. For example, to draw a regular polygon one must divide the circumference into five equal parts of 12 “minutes” and connect the points. When clock templates of different sizes are used, polygons of different sizes are obtained. The shapes are copied onto cardboard. The segments of the circle containing the sides of the polygons can be folded down and used for pasting the polygons together. In this way, stable models of all five Platonic solids can be made. Interestingly, the proof of the existence of at most five Platonic solids at the end of book 13 of Euclid’s *Elements of Mathematics* is fully in line with the children’s experimental findings.

In grade 5 the concept and the measure of an angle is based on cutting and fitting polygons, which once again mirrors the historical development of mathematics (Becker 1954, 27). At this level, the purely experimental approach gives way to a conceptual approach: by referring to the measure of angles and the lengths of segments,

students can explain why certain combinations of polygons must fit. Polygons cut from paper and drawings of polygons now enjoy a new status. They are no longer simple physical objects that allow for empirical experiments but rather representations of mathematical concepts that carry *theoretical* properties (see Sect. 3.2 below).

In the following grades, cutting polygons into parts and re-combining these parts is the customary approach to arriving at formulas for area and decomposition proofs of the Pythagorean theorem.

2 The Concept of Operative Proof

In the preface of Shafarevich 2005 there is an interesting statement concerning the limitations of formal definitions:

... the meaning of a mathematical notion is by no means confined to its formal definition; in fact, it may be rather better expressed by a (generally fairly small) number of basic examples, which serve the mathematicians as the motivation and the substantive definition, and at the same time as the real meaning of the notion. Perhaps the same kind of difficulty arises if we attempt to characterize in terms of general properties any phenomenon that has any degree of individuality.

So it is for good reason that the present paper starts with typical examples of operative proofs. Referring to these examples, this notion can now be described as follows:

Operative proofs

- arise from the exploration of a mathematical problem in the context of practicing skills and explaining patterns,
- are based on operations with “quasi-real” mathematical objects,
- use means of representation with which students are familiar at a given level
- are communicable in a simple problem-oriented language with little symbolism.

Strictly speaking, the term “operative proof” is not entirely correct, as it is not the proof that is “operative” but rather the whole mathematical setting. However, for the sake of brevity the term seems acceptable. Operative proofs have received growing attention since Zbigniew Semadeni’s seminal papers on “pre-mathematics” (Semadeni 1974; Semadeni 1984). His ideas were elaborated on in Germany by Kirsch (1979), Heinrich Winter (1985) and others and in Japan by Mikio Miyazaki (1997). These authors called proofs of this kind “pre-formal proofs” or “explanations by actions on manipulable things.” These descriptions indicate that the authors had some concerns about the status of such proofs while, at the same time, they also had an unquestioned respect for formal proofs. However, research in the philosophy of mathematics and a re-thinking of the role of proofs in the mathematics community has changed the situation considerably cf. the overview given in Hanna 2000.

The first example in Sect. 1.1 shows that operative proofs are the most elementary form of proof associated with the first attempts to shape the discipline known as

“mathesis.” Operative proofs refer not to symbolic descriptions of mathematical objects within a systematic-deductive theory but rather directly to these objects via representations that allow for “concrete” operations. These operations are generally applicable independently of the particular objects to which they are applied. So it is not from particular cases that the generality of a pattern is derived but from *operations* with objects (see also Kautschitsch 1989, p. 184). This fact must be kept in mind in order to avoid erroneously rejecting “operative proofs” as non-rigorous proofs.

In higher mathematics the objects and the operations are much more complicated. Nevertheless, the operative character of proofs is still present in mathematics of all levels (see, for example, operative proofs of Sperner’s Lemma in Struve and Wittmann 1984 and of the structure of the limit cycles of Bulgarian Solitaire in Wittmann 2006).

3 The Theoretical Background of Operative Proofs

The notion of operative proof is based on some theoretical positions from various disciplines. In this section, four positions will be described.

3.1 *Mathematics as the Science of Patterns*

As already mentioned at the beginning, the Mathe 2000 project has adopted the view of mathematics as the science of patterns, which has become a widely accepted view among mathematicians in the post-Bourbaki era (Sawyer 1995; Steen 1988; Devlin 1994). What matters in mathematics education, however, is not the science of ready-made and static patterns but rather the science of dynamic patterns which can be developed globally in the curriculum as well as explored, continued, re-shaped, and invented in the context of learning environments by the learners themselves. In other words, long-term and short-term mathematical processes related to patterns count much more than the finished products. The work of British, Scottish, Dutch and Japanese mathematics educators in the sixties and seventies as well as the pioneering work of Heinrich Winter, the “German Freudenthal,” have all served as models (Fletcher 1965; Wheeler 1967; IOWO 1976; Becker and Shimada 1997; Winter 1984; 2015).

In order for students to understand mathematics, it is important that they become aware of mathematical patterns as early as possible. The ability to see something general in something particular is essential for appreciating and understanding mathematics at any level, particularly as far as the role of proof is concerned.

3.2 *The Quasi-empirical Nature of Mathematics*

Operative proofs depend on appropriate representations of mathematical objects. It was Imre Lakatos who first pointed out the fact that mathematical theories are always developed in close relationship with the construction of the objects to which they refer (Lakatos 1976). Graph theory emerges with the construction of graphs, group theory emerges with the construction of groups, theories of coding emerge with the construction of new codes, etc. In each theory the mathematical objects form a kind of “quasi-reality” which permits the researcher to conduct experiments similar to scientific experiments. In recent decades, the importance of this “quasi-empirical” perspective for mathematics education has gained more and more recognition.

At the school level, *informal* representations of mathematical objects are indispensable as they provide a “quasi-reality” that is easily accessible. Patterns become “visible” and manageable when informal representations like counters, the number line, the place value chart, calculations with numbers and constructions of geometric figures are used.

Representations of mathematical objects, both informal and formal, form an interface between pure mathematics and manageable applications. They can be seen as concretizations of abstract mathematical concepts on the one hand and as representations of real objects on the other hand. Compared with the abstract objects, these representations are more concrete than the mathematical objects which they represent, and compared with the real objects they model, they are more abstract.

The “quasi-reality” of mathematical objects forms a world of its own which Yuri Manin in a letter to ICME 7 aptly called a “mathscape.” As the theoretical nature of mathematical objects is imposed on these representations, this mathscape is well suited to support the building of theories at whatever level by conveying meaning, stimulating ideas and providing data for checking mathematical arguments. Unlike Hilbert’s fictitious mathematician who has cut all ontological links, the working mathematician and the learner act inside a “visible” mathscape. The following statement by D. Gale summarizes this position very neatly (Gale 1990, 4):

The main goal of all science is first to observe and then to explain phenomena. In mathematics the explanation is the proof.

3.3 *The Operative Principle*

In Jean Piaget’s epistemology, knowledge is seen as a construction that results from the interaction of the individual with the environment: the individual acts upon the environment, notices the effects of her or his actions, and fits them into growing and changing cognitive schemata. According to Piaget, mathematical knowledge is not derived from the objects themselves, but *from operations with objects* in the process of reflective abstraction (“abstraction réfléchissante,” Beth and Piaget 1961, 217–223). Operations involve general patterns for the following reason: when it is

intuitively clear that the operations applied to a particular object are applicable to all objects of a certain class to which the particular object belongs, the relationships which can be derived from these operations are recognized as generally valid.

Quite a number of German mathematics educators have contributed to applying Piaget's epistemology to mathematics education. Over the course of time, this has led to the following formulation of what is referred to as the "operative principle" (Wittmann 1996, 154–161):

To understand mathematical objects means to explore how they are constructed and how they behave if they are subjected to operations (actions, constructions, transformations, ...). Therefore students must be stimulated in a systematic way

- (1) to explore which operations can be performed and how they are linked to one another,
- (2) to find out which properties and relationships are imprinted into the objects through construction,
- (3) to observe which effects, properties, and relationships are brought about by the operations according to the guiding question "What happens with ... if ...?"

The relationship of this principle to operative proofs is obvious: operative proofs depend on the effects of operations applied to the objects in question. Because of the general nature of the operations, operative proofs are rigorous proofs with a clear foundation. At this level, the effects of the operations take over the role that axioms play at higher levels.

3.4 *Practicing Skills in a Productive Way*

When Mathe 2000 was founded 20 years ago, it was a conscious decision to pay particular attention to basic skills in order to escape the fate of many curriculum projects in the sixties and seventies which had failed because they neglected basic skills. Traditionally, "practice" is linked to the proverbial "drill and practice," which of course is not compatible with the objectives of mathematics teaching as we see them today. So a new approach to practice had to be developed which deliberately combines the practice of skills with higher objectives like mathematizing, exploring, reasoning and communicating. This type of practice has been called "productive practice" (Wittmann and Müller 1990/1992). The basic idea is quite simple: for practicing skills, appropriate mathematical patterns are used as contexts.

Learning environments designed accordingly always start with extended calculations, constructions or experiments. In this way a "quasi-reality" is created, allowing students to observe phenomena, discover patterns, formulate conjectures, and finally to explain, i.e. prove, patterns. The operations on which these operative proofs rest are introduced in this first phase in a natural way. Reference to this quasi-reality is made continuously while the environment is explored more and more deeply. In checking and verifying arguments, skills are practiced again.

The aspect of “practice” comes in a second time at a higher level. The ability to understand an argument in a proof depends on repetition as much as it does on the mastery of a skill. So it is very important that explanations are not only repeated several times within a given learning environment by referring to a series of different examples. It is equally important that coherent sequences of learning environments within the curriculum provide continued opportunities for repeating explanations. Heinz Steinbring’s studies of the Mathe 2000 learning environments strongly confirm this fact (Steinbring 2005, Chap. 3). We cannot expect students to become familiar with operative proofs on the spot; students need continued opportunities for improving and refining their arguments. Developmental research in the Mathe 2000 project has shown that the addition table, the multiplication table, and the standard algorithms for addition, subtraction, multiplication and division are so rich in patterns that there is no need to introduce additional content for developing the higher objectives of mathematics teaching. It is crucial, however, to select representations of numbers that incorporate fundamental mathematical relationships and so to allow for operations upon which operative proofs can be built (Wittmann 1998). In arithmetic, counters provide the representation of choice. For example, rectangular arrays of counters allow the multiplication of natural numbers to be represented in a fashion that contains and supports the arithmetical laws. Section 1.2 provides some insight into the power of this representation.

4 Concluding Remarks

Operative proofs are not restricted to school mathematics but rather reach far into those parts of elementary mathematics that are accepted as the background of school mathematics and should form the subject matter of teacher education. The textbook “Arithmetic as a Process” (Müller et al. 2004), which was inspired by Mathe 2000, makes systematic use of informal representations and operative proofs within a process-led approach to the science of patterns. For example, in the chapter on number theory, all theorems up to Euler’s generalization of Fermat’s “little theorem” are explained by referring to “quasi-realities” represented by arrays of counters, the number line and arrays of numbers (Müller et al. 2004, 255–290).

In teacher education, the operative approach offers a dual advantage: this approach not only helps student teachers learn and understand mathematics better, but it also provides them with first-hand professional knowledge in dealing with means of representation and communication that are appropriate for the classroom. Mathematical courses designed accordingly provide an excellent basis for courses in mathematics education in which the underlying didactic principles can be made explicit by referring to student teachers’ own mathematical experiences.

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