Modified Bacterial Foraging Optimization for Fuzzy Mean-Semivariance-Skewness Portfolio Selection

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Abstract. In this paper, a novel bacterial foraging optimization with decreasing chemotaxis step combined with sine function is employed to solve a fuzzy portfolio optimization with a modified mean-semivariance-skewness model which includes the transaction fee and no short sales. First of all, a decreasing chemotaxis step combined with sine function (BFO-SDC) takes the place of constant chemotaxis step size. It is a nonlinear decreasing strategy at every iteration of the algorithm. And then, the variance is replaced by semivariance and skewness is taken into account in order to generate asymmetry of return distributions to overcome the inadequacy of the standard mean-variance model. Finally, fuzzy variables are used to express the uncertain and imprecise elements in the decision-making process. The results of the simulation show that the model can be solved more reasonably and effectively by BFO-SDC than the original bacterial foraging optimization.

Keywords: Bacterial foraging optimization · Sine function · Fuzzy portfolio optimization · Mean-semivariance-skewness

1 Introduction

The mean-variance model (MVM) originally formulated by Markowitz [1] plays a critical and vital role in modern finance theory. Inspired by the Markowitz's idea, numerous extensions have been proposed, e.g., multi-period mean-variance formulation [2], behavioral mean-variance portfolio selection [3]. Nevertheless, one of the assumptions of MVM is that the returns of assets are symmetrically distributed. Actually, in reality, there are shreds of empirical evidence [4, 5] suggesting that many security returns follow the asymmetrical distribution. People start to consider asymmetry of return distributions to overcome the inadequacy of the MVM. The one is that Ibbotson [6] and Prakash *et al.* [7] show that it can generate a higher return when skewness is considered in the decision-making process. As a result, some scholars such as Prakash *et al.* [7] extend MVM to mean-variance-skewness model (MVSM). The other directly use downside risk measure to avoid sacrificing a large amount of expected return in removing both high and low return extremes. Semivariance [8] which is clear, comparatively simple

and direct in mirroring investors' intuition about risk is one of the well-known downside risk measures.

Moreover, another basic assumption of MVM is that the asset markets in the future can accurately reflect the asset data in the past. However, it is impossible to produce correct expectations for future risks and returns based on historical data exclusively in the uncertain economic environment. Thus, the fuzzy set theory [9] is well suited to deal with vagueness and uncertainty of any stock market attributes in portfolio optimization (PO).

As a fresh comer of swarm intelligence (SI), bacterial foraging optimization (BFO) [10] has gotten much attention from different areas, and widely used in many real-world problems including supply chain optimization problem [11], flexible job-shop scheduling problem [12], and vehicle routing problem [13]. And a modified BFO with linear decreasing chemotaxis step (BFO-LDC) has been already successfully applied to solve the PO [14].

Sine function, one of the trigonometric functions, has been applied in the generation of n-scroll attractors [15], the design of ship course-keeping autopilot [16], sound encryption scheme [17], etc.

Based on the analysis of the recent literature, the problems about how to modify MVM to become more suitable for the real stock market and promote the ability of BFO to achieve better solutions in the PO attracts our attention. In this paper, we propose a novel BFO with decreasing chemotaxis step combined with sine function (BFO-SDC) and build a more complex PO model. The main contributions of this paper are listed as follows. First of all, a decreasing strategy based on sine function replaces the constant chemotaxis step size to investigate the optimization ability of BFO through combining decreasing strategy with sine function to create a nonlinear decreasing strategy. Next, use semivariance to replace variance and take skewness into account to generate asymmetry of asset distributions. And then, transaction fees, no short sales are also included. Finally, fuzzy logic [18] is used to express uncertainty to make it suitable for representing the intrinsic vagueness nature of the PO problem.

The remainder of the paper is organized as follows. Section 2 provides some introductory definitions and concepts. Section 3 reviews BFO and describes BFO-SDC. Section 4 gives a description of the modified fuzzy mean-semivariance-skewness model and related computational steps. The experimental results and analyses are shown and discussed in Sect. 5. In the end, concluding remarks are given in Sect. 6.

2 Preliminaries

The theories related to the definitions needed in the rest of the paper are briefly introduced in this section. The assistance of a membership function $\mu_A(x) \rightarrow [0,1]$, which is a single value between zero and one represents the degree of membership of the various elements.

Definition 1 [19]. A trapezoidal fuzzy number A with tolerance interval $[a, b]$, left-width $\alpha > 0$ and right-width $\beta > 0$ is fully determined by the quadruplet (a, b, α, β) .

And its membership function is given by

$$\mu_A(X) = \begin{cases} 1 - \frac{a-X}{\alpha}, & \text{if } a - \alpha \leq X \leq a \\ 1, & \text{if } a \leq X \leq b \\ 1 - \frac{X-b}{\beta}, & \text{if } b \leq X \leq b + \beta \\ 0, & \text{if otherwise} \end{cases} \quad (1)$$

Definition 2 [20]. A fuzzy number A with the γ -level set is expressed as $[A]^\gamma = [\alpha_1(\gamma), \alpha_2(\gamma)]$ for all $\gamma > 0$ and the γ -level set of A can be computed as

$$[A]^\gamma = [a - (1 - \gamma)\alpha, b + (1 - \gamma)\beta] (\gamma \in [0, 1]) \quad (2)$$

The possibilistic mean value of A is given by the following.

$$E(A) = \int_0^1 \gamma [\alpha_1(\gamma) + \alpha_2(\gamma)] d\gamma = \frac{a+b}{2} + \frac{\beta - \alpha}{6} \quad (3)$$

Definition 3 [21]. The upper and lower semivariances of A are defined as

$$Var^+(A) = \int_0^1 \gamma [E(A) - \alpha_2(\gamma)]^2 d\gamma = \left(\frac{b-a}{2} + \frac{\beta + \alpha}{6}\right)^2 + \frac{\beta^2}{18} \quad (4)$$

$$Var^-(A) = \int_0^1 \gamma [E(A) - \alpha_1(\gamma)]^2 d\gamma = \left(\frac{b-a}{2} + \frac{\beta + \alpha}{6}\right)^2 + \frac{\alpha^2}{18} \quad (5)$$

Definition 4 [22]. For any two given fuzzy numbers A with $[A]^\gamma = [\alpha_1(\gamma), \alpha_2(\gamma)]$ and B with $[B]^\gamma = [b_1(\gamma), b_2(\gamma)]$ for all $\gamma \in [0,1]$, the upper and lower possibilistic semicovariance between A and B are given as follows.

$$Cov^+(A, B) = 2 \int_0^1 \gamma [E(A) - \alpha_2(\gamma)] \cdot [E(B) - b_2(\gamma)] d\gamma \quad (6)$$

$$Cov^-(A, B) = 2 \int_0^1 \gamma [E(A) - \alpha_1(\gamma)] \cdot [E(B) - b_1(\gamma)] d\gamma \quad (7)$$

Especially, when A and B are symmetric fuzzy numbers, $Var^+(A) = Var^-(A)$, $Var^+(B) = Var^-(B)$ and $Cov^+(A, B) = Cov^-(A, B)$.

Lemma 1 [20]. Let A and B be two fuzzy numbers. For any real numbers μ and λ , the following conclusion can be drawn.

$$E(\lambda A \pm \mu B) = \lambda E(A) \pm \mu E(B) \quad (8)$$

Theorem 1 [23]. Let $A_1, A_2 \dots A_n$ be n fuzzy numbers, and let $\lambda_1, \lambda_2, \dots \lambda_n$ be n real numbers. Then

$$Var^+\left(\sum_{i=1}^n \lambda_i A_i\right) = \sum_{i=1}^n \lambda_i^2 Var^+(A_i) + 2 \sum_{i < j=1}^n \lambda_i \lambda_j Cov^+(A_i, A_j) \quad (9)$$

$$Var^-\left(\sum_{i=1}^n \lambda_i A_i\right) = \sum_{i=1}^n \lambda_i^2 Var^-(A_i) + 2 \sum_{i < j=1}^n \lambda_i \lambda_j Cov^-(A_i, A_j) \quad (10)$$

Definition 4 [24]. Let A be a fuzzy variable with finite expected value e . Then its skewness is defined as

$$Sk(A) = E[(A - e)^3] \quad (11)$$

3 BFO and BFO-SDC

3.1 Bacterial Foraging Optimization

Inspired by the social foraging behavior of the *E. coli* bacteria, BFO is an evolutionary optimization technique that has been applied to multitudinous optimization problems. Based on Liu and Passino's [10] discussion of how the control system of *E. coli* instructs and forages, bacteria migrate to nutrient-rich areas through the flagella. The cycle of optimization is divided into four steps: chemotaxis, swarming, reproduction, and elimination and dispersal. A brief introduction is as follows.

3.1.1 Chemotaxis

Bacteria "swim" or "tumble" via flagella. By switching between the two modes of motion, bacteria move randomly. Supposed $\theta^i(j, k, l)$ indicates the information of the i th bacterium including j th chemotaxis, k th reproduction, and l th elimination and dispersal step. C is the step size taken by swimming in a random direction. Δ represents the vector whose element is in the random direction of $[-1, 1]$. When calculating chemotaxis, the mathematical expression of bacterial movement is

$$\theta^i(j+1, k, l) = \theta^i(j, k, l) + C(i) \frac{\Delta(i)}{\sqrt{\Delta^T(i) \Delta(i)}} \quad (12)$$

3.1.2 Swarming

Researchers have found that *E. coli* cells use complex communication mechanisms such as quorum-sensing, chemotactic signaling, and plasmid exchange to form highly structured colonies to improve environmental fitness. Swarming of bacteria can be represented by

$$\begin{aligned} J_{cc}(\theta, P(j, k, l)) = & \sum_{i=1}^n J_{cc}^i(\theta, \theta^i(j, k, l)) + \sum_{i=1}^n [h_{replemt} \exp(-w_{replemt} \sum_{m=1}^p (\theta_m - \theta_m^i)^2)] \\ & + \sum_{i=1}^n [-d_{attract} \exp(w_{attract} \sum_{m=1}^p (\theta_m - \theta_m^i)^2)] \end{aligned} \quad (13)$$

where $J_{cc}(\theta, P(j, k, l))$ is an objective function that changes with the population distribution state, reflecting the trend of the relative distance between bacteria and the optimal individual. S represents the population size, and P indicates the dimension of the optimization problem. $d_{attract}$, $w_{attract}$, $h_{replemt}$ and $w_{replemt}$ are different coefficients chosen properly.

3.1.3 Reproduction

The bacteria are ranked by their health, with half of the less healthy bacteria dying and each of the healthier half splitting into two, leaving the total number of bacteria unchanged.

3.1.4 Elimination and Dispersal

It is well known that the life of a bacterial population can change gradually. All bacteria in an area may be killed, or a colony may disperse and emerge in a new environment. Although this change can disrupt the chemotaxis process, it is beneficial for bacteria to jump out of the local optimal domain and improve the global search ability. In this process, some bacteria die by a given probability P_{ed} and then reappear randomly throughout the search space.

3.2 BFO with Decreasing Chemotaxis Step Based on Sine Function

This part mainly introduces the application of the decreasing chemotaxis step based on sine function in algorithm improvement.

3.2.1 Sine Function in Chemotaxis Step

Sine function is a trigonometric function that is a kind basic theory of trigonometry. And the function value changes nonlinearly.

In our proposed method, a decreasing chemotaxis step length combined with sine function is used over iterations. It starts with a large value and decreases to a small value at the maximal number of iterations. The mathematical representation of the BFO method is given as

$$C(i) = C_{\max} - (C_{\max} - C_{\min}) \cdot \sin\left(\frac{\pi}{2} \cdot \frac{j}{N_{ed} \cdot N_{re} \cdot N_c}\right) \quad (14)$$

In the formula, N_c denotes the number of chemotactic. N_{ed} is the number of elimination-dispersal. N_{re} represents the number of reproduction. $C_{\min}$ and $C_{\max}$ are real numbers and j is the current number of iterations.

4 Bacterial Foraging Optimization for Portfolio Selection Problems

4.1 Portfolio Selection Model

In this section, a mean-semivariance-skewness model that considers no short sales and the transaction fee in portfolio optimization is explained. The model can be described as follows.

$$\begin{aligned} \text{Min } F(X) &= \min \{ \lambda g(X) - (1 - \lambda) [f(X) + h(X)] \} \\ \text{s.t. } &\begin{cases} \sum_{i=1}^d X_i = 1 \\ X_i > 0 \end{cases} \end{aligned} \quad (15)$$

$$f(X) = \sum_{i=1}^d r_i x_i - \sum_{i=1}^n [\mu \cdot k_i^b \cdot (x_i - x_i^0) + (1 - \mu) \cdot k_i^s \cdot (x_i^0 - x_i)]$$

$$\text{and } \mu = \begin{cases} 1 & X_i \geq X_i^0 \\ 0 & X_i < X_i^0 \end{cases} \quad (16)$$

$$g(X) = \sum_{i=1}^d \sum_{j=1}^d X_i X_j \sigma_{ij} \quad (17)$$

$$h(X) = \sum_{i=1}^d \sum_{j=1}^d \sum_{k=1}^d x_i x_j x_k s_{ijk} \quad (18)$$

Assuming that an investor will invest in d assets. λ represents the risk aversion factor. $f(X)$ represents investors' return function, $g(X)$ denotes risk function and $h(X)$ is skewness function. Let X_i be the proportion of investment of asset i , X_i^0 be the initial holding proportion of investment, r_i be the expected yields of i th asset, σ_{ij} be the covariance of r_i and r_j , s_{ijk} be the possibilistic co-skewness among r_i , r_j and r_k . k_i^s and k_i^b denote the transaction fee for selling and buying the asset i respectively.

4.2 Encoding

Encode the potentially feasible solution as ideas that individual generated in BFO-SDC. Four types of information including the value of corresponding profit, risk, and skewness, the proportion of investment are carried by each bacterium [25].

$$\theta = [X_1, X_2, X_3 \dots X_d, f(X), g(X), h(X)] \quad (19)$$

$$s = \sum_{i=1}^d X_i$$

$$\theta' = [\frac{1}{s}(X_1, X_2, X_3 \dots X_d), f(X)', g(X)', h(X)'] \quad (20)$$

Equation (19) introduces the coding of the bacteria. As is illustrated in Eq. (20), all the assets' proportions are summed up at the beginning and then each proportion of asset is divided by the s for the purpose of assuring all asset proportions' sum is equal to 1.

4.3 Constrained Boundary Control

$$\theta^i(j+1, k, l) = \begin{cases} D_{\min} + rand \cdot (D_{\max} - D_{\min}), & \text{if } P < 0 \\ \theta^i(j, k, l) + C(i) \frac{\Delta(i)}{\sqrt{\Delta^T(i) \Delta(i)}}, & \text{if } P \geq 0 \end{cases} \quad (21)$$

Boundary control assurances that every proportion is positive. Bacteria may exceed the given area since in the process of chemotaxis they change directions randomly. The rules of boundary control are shown in Eq. (21), $D_{\max}$ and $D_{\min}$ are the upper and lower boundaries. And $rand$ is a function that is used to generate a real number between 0 and 1.

5 Experiments and Discussions

For the purpose of testing the performance of the BFO-SDC algorithm, the original BFO and BFO-LDC are selected to compare. All the algorithms mentioned in this paper were coded in MATLAB R2018b.

5.1 Definition of Experiments

In the experiments, we assume that there are five assets can be selected to invest. Due to we suppose that the assets' future returns are trapezoidal fuzzy numbers, it's necessary to estimate the core interval, the right and left width of the fuzzy numbers. According to Vercher *et al.* [26], regard the historical returns as samples and use the sample percentile to approximate the spreads and core of the trapezoidal fuzzy returns. Thus, the interval of the 40th to 60th percentile is set as the core of the fuzzy return, the interval of the 5th to 40th percentile is set as the left-width and the interval of 40th to 95th percentile is set as the right-width. We consider the monthly returns over the tested period between January 2015 and December 2019. The trapezoidal fuzzy numbers are shown in Table 1.

Table 1. The trapezoidal fuzzy numbers

The asset i	1	2	3	4	5
α	-0.10516	-0.13309	-0.30344	-0.14459	-0.08656
a	-0.00244	-0.01443	0.013429	0.004195	0.020846
b	0.042983	0.026308	0.06576	0.045895	0.05327
β	0.220344	0.160453	0.543263	0.254134	0.169063

In this paper, we assume that investors are rational. Thus, the lower possibility variance is used to describe the risk. According to Theorem 1 and Definition 2, the risk of investment is shown as follows.

$$\begin{aligned}
 Var^-\left(\sum_{i=1}^d x_i r_i\right) &= \sum_{i=1}^d x_i^2 Var^-(r_i) + 2 \sum_{i<j=1}^d x_i x_j Cov^-(r_i, r_j) \\
 &= \left[\sum_{i=1}^d x_i \left(\frac{a_i - b_i}{2} + \frac{\alpha_i + \beta_i}{6}\right)\right]^2 + \frac{1}{18} \left(\sum_{i=1}^d x_i \alpha_i\right)^2 + 2 \sum_{i<j=1}^d x_i x_j Cov^-(r_i, r_j)
 \end{aligned} \tag{22}$$

$$\begin{aligned}
 Cov^-(r_i, r_j) &= \left(\frac{b_i - a_i}{2} + \frac{\beta_i - \alpha_i}{6}\right) \left(\frac{b_j - a_j}{2} + \frac{\beta_j - \alpha_j}{6}\right) + \frac{\alpha_j}{3} \left(\frac{b_i - a_i}{2} + \frac{\beta_i - \alpha_i}{6}\right) \\
 &\quad + \frac{\alpha_i}{3} \left(\frac{b_j - a_j}{2} + \frac{\beta_j - \alpha_j}{6}\right) + \frac{\alpha_i \alpha_j}{6}
 \end{aligned} \tag{23}$$

According to Liu *et al.* [27], the possibilistic skewness of the portfolio is

$$\begin{aligned}
 S\left(\sum_{i=1}^d x_i r_i\right) &= \frac{19}{1089} \left[\left(\sum_{i=1}^d x_i \beta_i \right)^3 - \left(\sum_{i=1}^d x_i \alpha_i \right)^3 \right] \\
 &+ \frac{1}{72} \left[\left(\sum_{i=1}^d x_i \alpha_i \right) \left(\sum_{i=1}^d x_i \beta_i \right)^2 - \left(\sum_{i=1}^d x_i \beta_i \right) \left(\sum_{i=1}^d x_i \alpha_i \right)^2 \right] \\
 &+ \frac{1}{24} \sum_{i=1}^d x_i (b_i - a_i) \left[\left(\sum_{i=1}^d x_i \beta_i \right)^2 - \left(\sum_{i=1}^d x_i \alpha_i \right)^2 \right] \quad (24)
 \end{aligned}$$

In the decreasing chemotaxis step length based on sine function, $C_{min} = 0.01$, $C_{max} = 0.5$. The following parameter settings are the same as in corresponding reference [28]. To show various investors based on different preference to evaluate the final fortune, λ is set as 0.15, 0.5 and 0.85 respectively. $k_i^s = 0.00075$ and $k_i^b = 0.00065$. In BFO, $N_c = 1000$, $N_{ed} = 2$, $N_{re} = 5$, N_s (the number of swimming) = 4, P_{ed} (the elimination-dispersal frequency) = 0.25. C (the swimming length) = 0.2.

5.2 Experimental Results

Numerical results with different λ and the terminal portfolio selection results obtained by the original BFO, BFO-LDC, and BFO-SDC are showed in Tables 2, 3 and 4. Figure 1 shows the convergence curves with various λ generated by the three algorithms.

Table 2. Experimental results of $\lambda = 0.15$

	BFO	BFO-LDC	BFO-SDC
Min	−2.9076E+00	−1.7248E+01	−2.1866E+01
Max	0	1	0
Mean	−2.9076E−01	−1.7248E+00	−2.1866E+00
Std.	9.1946E−01	5.4544E+00	6.9147E+00
X ₁	2.3619E−01	1.6276E−01	1.8054E−01
X ₂	1.6315E−01	2.5461E−01	2.0436E−01
X ₃	2.7044E−01	2.7919E−01	1.4639E−01
X ₄	1.3663E−01	1.4928E−01	2.6812E−01
X ₅	1.9360E−01	1.5416E−01	2.0059E−01
Return	1.0320E−01	1.0230E−01	9.1529E−02
Risk	3.3634E−03	3.3894E−03	2.7046E−03
Skewness	3.1849E−04	3.0871E−04	2.1187E−04

Table 3. Experimental results of $\lambda = 0.5$

	BFO	BFO-LDC	BFO-SDC
Min	-1.2345E+00	-3.0142E+00	-6.2287E+00
Max	0	0	0
Mean	-1.2345E-01	-3.0143E-01	-6.2287E-01
Std.	3.9039E-01	9.5319E-01	1.9697E+00
X ₁	3.2366E-01	2.4145E-01	2.7598E-01
X ₂	9.0457E-02	6.1790E-02	2.3570E-01
X ₃	1.8598E-01	2.9873E-01	9.1497E-02
X ₄	2.0407E-01	8.8298E-02	1.0163E-01
X ₅	1.9583E-01	3.0973E-01	2.9520E-01
Return	9.6784E-02	1.0777E-01	8.2556E-02
Risk	2.9941E-03	3.4405E-03	2.2346E-03
Skewness	2.6747E-04	3.4922E-04	1.5155E-04

Table 4. Experimental results of $\lambda = 0.85$

	BFO	BFO-LDC	BFO-SDC
Min	-2.5908E-02	-2.6722E-02	-2.7785E-02
Max	0	0	0
Mean	-2.5908E-03	-2.6722E-03	-2.7785E-03
Std.	8.1928E-03	8.4501E-03	8.7863E-03
X ₁	8.4933E-02	1.4236E-01	1.5807E-01
X ₂	2.0848E-01	2.4435E-01	2.1004E-01
X ₃	3.0795E-01	2.1466E-01	3.2087E-01
X ₄	2.0448E-01	2.5917E-01	9.4012E-02
X ₅	1.9415E-01	1.3945E-01	2.1701E-01
Return	1.0742E-01	9.7463E-02	1.0697E-01
Risk	3.5742E-03	3.1070E-03	3.5597E-03
Skewness	3.4445E-04	2.6401E-04	3.4545E-04

Fig. 1. The convergence curve of three algorithms

According to the tables and the figures, we can find that:

- (1) The fitness value grows up with the increase of the risk-averse factor λ , and this trend keeps the same with the structure of the fitness function.
- (2) With the different λ and different algorithm, the percentage of five assets is not the same. The results show that lower risk can be with the less profit.
- (3) Comparing the numerical results shown in Tables 2 and 3, it verifies that BFO-SDC outperforms BFO and BFO-LDC in terms of mean value, minimum value and result qualify, which clarifies the effectiveness of the BFO-SDC.
- (4) Comparing the convergence graphs shown in Fig. 1, BFO-SDC is superior to BFO-LDC and BFO for the three cases.

6 Conclusions and Future Work

In this paper, we pay attention to solving the PO problem by using BFO based methods. Based upon the standard mean-variance model, we take skewness and fuzzy set theory into consideration and use semivariance to take the place of variance. A decreasing chemotaxis step strategy based on sine function is included in the original BFO to solve

this model. The obtained results indicate that BFO-SDC outperforms than BFO and BFO-LDC.

Further works may consider some biological mechanisms to promote the performance of BFO and propose novel models containing other conditions such as dynamic objectives or multi-period to meet the demands from the real-world.

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