Functional Extensions of Knowledge Representation in General Rough Sets

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Abstract. A number of low and high-level models of general rough sets can be used to represent knowledge. Often binary relations between attributes or collections thereof have deeper properties related to decisions, inference or vision that can be expressed in ternary functional relationships (or groupoid operations) – this is investigated from a minimalist perspective in this research by the present author. General approximation spaces and reflexive up-directed versions thereof are used by her as the basic frameworks. Related semantic models are invented and an interpretation is proposed in this research. Further granular operator spaces and variants are shown to be representable as partial algebras through the method. An analogous representation for all covering spaces does not necessarily hold. Applications to education research contexts that possibly presume a distributed cognition perspective are also outlined.

Keywords: General approximation spaces · Up-directedness · Rough objects · Mereology · Groupoidal semantics · Parthood · Knowledge · AI · Higher granular operator spaces · Contamination problem · Education research

1 Introduction

In relational approach to general rough sets various granular, pointwise or abstract approximations are defined, and rough objects of various kinds are studied [1–6]. These approximations may be derived from information tables or may be abstracted from data relating to human (or machine) reasoning. A *general approximation space* is a pair of the form $S = \langle \underline{S}, R \rangle$ with $\underline{S}$ being a set and R being a binary relation (S and $\underline{S}$ will be used interchangeably throughout this paper). Approximations of subsets of $\underline{S}$ may be generated from these and studied at different levels of abstraction in theoretical approaches to rough sets. Because approximations and related semantics are of interest here, the relational system is much more than a general frame. *Often it happens that S is interpreted as a set of attributes and that any two elements of S may be associated with a third element through a mechanism of reasoning, by preference, or via decision-making*

guided by an external mechanism. The purpose of this research is to study these situations from a minimalist perspective. This approach also directly adds to the concept of knowledge in classical rough sets [7] and in general rough sets [3, 8–10] and therefore the study is referred to as an extension of the same.

Mereology, the study of parts and wholes, has been studied from philosophical, logical, algebraic, topological and applied perspectives. In the literature on mereology [9, 11, 12], it is argued that most ideas of binary *part of* relations in human reasoning are at least antisymmetric and reflexive. *A major reason for not requiring transitivity of the parthood relation is because of the functional reasons that lead to its failure* (see [11]), and to accommodate *apparent parthood* [12]. The study of mereology in the context of rough sets can be approached in at least two essentially different ways. In the approach aimed at reducing contamination by the present author [1, 2, 8, 10], the primary motivation is to avoid intrusion into the data by way of additional assumptions about the data relative to the semantic domain in question. In numeric function based approaches [13], the strategy is to base definitions of parthood on the degree of rough inclusion or membership – this differs substantially from the former approach. Rough Y-systems and granular operator spaces, introduced and studied extensively by the present author [1, 2, 8, 10, 12], are essentially higher order abstract approaches in general rough sets in which the primitives are ideas of approximations, parthood, and granularity. Part-of relations can also be the subject of considerations mentioned in the first paragraph, and the relation R in a general approximation space can be a parthood. Specific versions of parthood spaces have been investigated in a forthcoming joint work by the present author. Relative to that work new results on parthood spaces are proved, up-directedness is studied in classical approximation spaces, and the formalism on granular operator spaces and variants are improved in this research. Applications to education research contexts are also outlined.

1.1 Background

An *information table* $\mathcal{I}$, is a tuple of the form

$$\mathcal{I} = \langle S, \mathbb{A}, \{V_a : a \in \mathbb{A}\}, \{f_a : a \in \mathbb{A}\} \rangle$$

with S , $\mathbb{A}$ and V_a being sets of *objects*, *attributes* and *values* respectively. Information tables generate various types of relational or relator spaces which in turn relate to approximations of different types and form a substantial part of the problems encountered in general rough sets.

In classical rough sets [7], equivalence relations of the form R are derived by the condition $x, y \in S$ and $B \subseteq \mathbb{A}$, let $(x, y) \in R$ if and only if $(\forall a \in B) \nu(a, x) = \nu(a, y)$. $\langle S, R \rangle$ is then an *approximation space*. On the power set $\wp(S)$, lower and upper approximations of a subset $A \in \wp(S)$ operators, (apart from the usual Boolean operations), are defined as per: $A^l = \bigcup_{[x] \subseteq A} [x]$, $A^u = \bigcup_{[x] \cap A \neq \emptyset} [x]$, with $[x]$ being the equivalence class generated by $x \in S$. If $A, B \in \wp(S)$, then A is said to be *roughly included* in B ($A \sqsubseteq B$) if and only if

$A^l \subseteq B^l$ and $A^u \subseteq B^u$. A is roughly equal to B ($A \approx B$) if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$ (the classes of $\approx$ are rough objects).

The rough domain corresponds to rough objects of specific type, while the classical and hybrid one correspond to all and mixed types of objects respectively [2]. Boolean algebra with approximation operators forms a classical rough semantics. This fails to deal with the behavior of rough objects alone. The scenario remains true even when R in the approximation space is replaced by arbitrary binary relations. In general, $\wp(S)$ can be replaced by a set with a parthood relation and some approximation operators defined on it as in [2]. The associated semantic domain is the classical one for general Rough sets. The domain of discourse associated with roughly equivalent sets is a *rough semantic domain*. Hybrid domains can also be generated and have been used in the literature [1].

The problem of reducing confusion among concepts from one semantic domain in another is referred to as the contamination problem. Use of numeric functions like rough membership and inclusion maps based on cardinalities of subsets are also sources of contamination. The rationale can also be seen in the definition of operations like $\sqcup$ in pre-rough algebra (for example) that seek to define interaction between rough objects but use classical concepts that do not have any interpretation in the rough semantic domain. Details can be found in [14]. In machine learning practice, whenever inherent shortcomings in algorithmic framework being used are the source of noise then the frameworks may be said to be contaminated.

Key concepts used in the context of general rough sets (and also high granular operator spaces [1, 10]) are mentioned next.

- A *crisp object* is one that has been designated as *crisp* or is an approximation of some other object.
- A *vague object* is one whose approximations do not coincide with itself or that which has been designated as a *vague* object.
- An object that is explicitly available for computations in a rough semantic domain (in a contamination avoidance perspective) is a *discernible object*.
- Many definitions and representations are associated with the idea of *rough objects*. From the representation point of view these are usually functions of definite or crisp or approximations of objects. Objects that are invariant relative to an approximation process are said to be *definite objects*. In rough perspectives of knowledge [7, 8], algebraic combinations of definite objects (in some sense) or granules are assumed to correspond to crisp concepts, and knowledge to specific collections of crisp concepts. *It should be mentioned that non algebraic definitions are excluded in the present author's axiomatic approach [1, 2, 10].*

Definition 1. A partial algebra (see [15]) P is a tuple of the form

$$\langle \underline{P}, f_1, f_2, \dots, f_n, (r_1, \dots, r_n) \rangle$$

with $\underline{P}$ being a set, f_i 's being partial function symbols of arity r_i . The interpretation of f_i on the set $\underline{P}$ should be denoted by $f_i^{\underline{P}}$, but the superscript will be

dropped in this paper as the application contexts are simple enough. If predicate symbols enter into the signature, then P is termed a partial algebraic system.

In this paragraph the terms are not interpreted. For two terms s, t , $s \stackrel{\omega}{=} t$ shall mean, if both sides are defined then the two terms are equal (the quantification is implicit). $\stackrel{\omega}{=}$ is the same as the existence equality (also written as $\stackrel{e}{=}$) in the present paper. $s \stackrel{\omega^*}{=} t$ shall mean if either side is defined, then the other is and the two sides are equal (the quantification is implicit). $\stackrel{\omega^*}{=}$ is written as $\stackrel{s}{=}$ in [18]. Note that the latter equality can be defined in terms of the former as

$$(s \stackrel{\omega}{=} s \longrightarrow s \stackrel{\omega}{=} t) \& (t \stackrel{\omega}{=} t \longrightarrow s \stackrel{\omega}{=} t)$$

2 Relations and Groupoids

Under certain conditions, partial or total groupoid operations can correspond to binary relations on a set.

Definition 2. In a general approximation space $S = \langle \underline{S}, R \rangle$ consider the following conditions:

$$\begin{aligned} (\forall a, b)(\exists c)Rac \& Rbc & \quad (\text{up-dir}) \\ (\forall a)Raa & \quad (\text{reflexivity}) \\ (\forall a, b)(Rab \& Rba \longrightarrow a = b) & \quad (\text{anti-sym}) \end{aligned}$$

If S satisfies up-dir, then it shall said to be a up-directed approximation space. If it satisfies the last two then it shall said to be a parthood space and a up-directed parthood space when it satisfies all three.

The condition up-dir is equivalent to the set $U_R(a, b) = \{x : Rax \& Rbx\}$ being nonempty for every $a, b \in S$ and is also referred to as *directed* in the literature. It is avoided because it may cause confusion.

The problem of rewriting the semantic content of binary relations of different kinds using total or partial operations has been of much interest in algebra (for example [16, 17]). Results on using partial operations for the purpose are of more recent origin [18, 19].

Definition 3. If R is a binary relation on S , then a type-1 partial groupoid operation (1PGO) determined by R is defined as follows:

$$(\forall a, b) a \circ b = \begin{cases} b & \text{if } Rab \\ c & c \in U_R(a, b) \& \neg Rab \\ \text{undefined} & \text{otherwise} \end{cases}$$

If R is up-directed, then the operation is total. In this case, the collection of groupoids satisfying the condition will be denoted by $\mathfrak{B}(S)$ and an arbitrary element of it will be denoted by $B(S)$. If R is not up-directed, then the collection of partial groupoids associated will be denoted by $\mathfrak{B}_p(S)$. The term ' $a \circ b$ ' will be written as ' ab ' for convenience.

Theorem 1. *The partial operation $\circ$ corresponds to a binary relation R if and only if*

$$\begin{aligned} (\forall a, b)(\exists z)(ab \neq b \& az = bz = z \rightarrow a(ab) = b(ab) = ab) \\ (\forall a, b, c)(ab = c \rightarrow c = b \text{ or } (\exists z)az = bz = z) \end{aligned}$$

The following results have been proved for relational systems in [18, 19].

Theorem 2. *For a groupoid A , the following are equivalent*

- *A reflexive up-directed approximation space S corresponds to A*
- *A satisfies the equations*

$$aa = a \& a(ab) = b(ab) = ab$$

Definition 4. *If A is a groupoid, then two general approximation spaces corresponding to it are $\mathfrak{R}(A) = \langle \underline{A}, R_A \rangle$ and $\mathfrak{R}^*(A) = \langle \underline{A}, R_A^* \rangle$ with*

$$\begin{aligned} R_A &= \{(a, b) : ab = b\} \\ R_A^* &= \bigcup \{(a, ab), (b, ab)\} \end{aligned}$$

Theorem 3. – *If A is a groupoid then $\mathfrak{R}^*(A)$ is up-directed.*

- *If a groupoid $A \models a(ab) = b(ab) = ab$ then $\mathfrak{R}(A) = \mathfrak{R}^*(A)$.*
- *If S is an up-directed approximation space then $\mathfrak{R}((B)(S)) = S$.*

Theorem 4. *If $S = \langle \underline{S}, R \rangle$ is a up-directed approximation space, then*

- *R is reflexive $\Leftrightarrow \mathbf{B}(S) \models aa = a$.*
- *R is symmetric $\Leftrightarrow \mathbf{B}(S) \models (ab)a = a$.*
- *R is transitive $\Leftrightarrow \mathbf{B}(S) \models a((ab)c) = (ab)c$.*
- *If $\mathbf{B}(S) \models ab = ba$ then R is antisymmetric.*
- *If $\mathbf{B}(S) \models (ab)a = ab$ then R is antisymmetric.*
- *If $\mathbf{B}(S) \models (ab)c = a(bc)$ then R is transitive.*

Morphisms between up-directed approximation spaces are preserved by corresponding groupoids in a nice way. This is an additional reason for investigating the algebraic perspective.

3 Up-Directed General Approximation Spaces

In general, partial/quasi orders, and equivalences need not satisfy **up-dir**. When they do satisfy the condition, then the corresponding general approximation spaces will be referred to as *up-directed general approximation spaces*.

For any element $a \in S$, the neighborhood granule $[a]$ and inverse neighborhood $[a]_i$ associated with it in a general approximation space shall be given by $[a] = \{x : Rxa\}$ and $[a]_i = \{x : Rax\}$ respectively.

Definition 5. For any subset $A \subseteq S$, the following approximations can be defined:

$$\begin{aligned} A^l &= \bigcup \{[a] : [a] \subseteq A\} & (\text{lower}) \\ A^u &= \bigcup \{[a] : \exists z \in [a] \cap A\} & (\text{upper}) \end{aligned} \tag{1}$$

If inverse neighborhoods are used instead, then the corresponding approximations will be denoted by l_i and u_i respectively.

3.1 Classical Approximation Spaces

If an approximation space is up-directed, then it is essentially redundant with respect to the relation. Proof of the following theorem is not hard and can be found in a forthcoming paper due to the present author.

Theorem 5. Let S be an approximation space, then all of the following hold:

- If R is up-directed, then $S^2 = R$.
- If R is not up-directed, then the groupoid operation of Definition 3 is partial and it satisfies

$$(\forall a, b, c)(ab = c \longrightarrow b = c)$$

- For each $x \in S$, $[x]$ is closed under $\circ$ and so every equivalence class is a total groupoid that satisfies:

$$(\forall a, b, c) aa = a \ \& \ (ab)a = a \ \& \ a((ab)c) = (ab)c$$

Definition 6. On the power set $\wp(S)$, the partial operation $\circ$ induces a total operation as in Eq. 2.

$$(\forall A, B \in \wp(S)) A \odot B = \{x : (\exists a \in A)(\exists b \in B) ab = x\} \tag{2}$$

Proposition 1. If S is an approximation space then $\langle \wp(S), \cup, \cap, ^c, l, u, \circ, \perp, \top \rangle$ is a Boolean algebra with operators enhanced by a groupoid operation that satisfies all of the following (apart from the well known conditions):

$$\begin{aligned} (\forall a, b) aa \cap a &= aa \ \& \ ab \cap b = ab & (\text{pre-refl}) \\ (\forall a, b, c) ((a \cup b)c) \cap ((ac) \cup (bc)) &= (ac) \cup (bc) & (\text{pre-mo}) \\ (\forall a, b, c) (a \cup b = b \longrightarrow (ac) \cup (bc)) &= bc & (\text{mo}) \\ (\forall a, b) (ab)^l \cup b^l &= b^l & (\text{l-mo}) \\ (\forall a, b) (ab)^u \cap b^u &= (ab)^u & (\text{u-mo}) \end{aligned}$$

Proof. – Note that by the definition of the partial groupoid operation, for any two sets $a, b \in \wp(S)$ ab must be a subset of b . So the pre-refl property holds.
– pre-mo is again a consequence of pre-refl.

- If a is a subset of b , then ac must again be a subset of bc which in turn would be a subset of c . This can be verified by a purely set-theoretic argument.
- ab must be a subset of b . So $(ab)^l$ must be subset of b^l . It follows that their union must be the latter.

Because classes are closed under the groupoid operation, it follows that

Theorem 6. *On the set of definite elements $\delta(S)$ of an approximation space S , the induced operations from the algebra in Proposition 1 again forms a Boolean subalgebra with groupoid operations that satisfies reflexivity $(\forall a) aa = a$.*

It should be noted that up-directedness is not essential for a relation to be represented by groupoid operations. The following construction that differs in part from the above strategy can be used for partially ordered sets as well, and has been used by the present author in [20] in the context of knowledge generated by approximation spaces. The method relates to earlier algebraic results including [21, 22]. The groupoidal perspective can be extended for quasi ordered sets.

If $S = \langle \underline{S}, R \rangle$ is an approximation space, then define (for any $a, b \in S$)

$$a \circledast b = \begin{cases} a & \text{if } Rab \\ b & \text{if } \neg Rab \end{cases} \quad (3)$$

Relative to this operation, the following theorem (see [21]) holds:

Theorem 7. *$\langle S, \circledast \rangle$ is a groupoid that satisfies the following axioms (braces are omitted under the assumption that the binding is to the left, e.g. ‘ abc ’ is the same as ‘ $(ab)c$ ’):*

$$xx = x \quad (\text{E1})$$

$$x(az) = (xa)(xz) \quad (\text{E2})$$

$$xax = x \quad (\text{E3})$$

$$azxauz = auz \quad (\text{E4})$$

$$u(azxa)z = uaz \quad (\text{E5})$$

3.2 Parthood Spaces

Definition 7. *Let S be a parthood space, then let $S_{lu} = \{x : x = a^l \text{ or } x = a^u \text{ \& } a \in S\}$. On S_{lu} , the following operations can be defined (apart from l and u by restriction):*

$$a \mathbin{\sqcap} b = (a \cap b)^l \quad (\text{Cap})$$

$$a \mathbin{\sqcup} b = (a \cup b)^u \quad (\text{Cup})$$

$$\perp = \emptyset; \top = S^u \quad (\text{iu34})$$

The resulting algebra $S_{lu} = \langle S_{lu}, \mathbin{\sqcap}, \mathbin{\sqcup}, \cup, l, u, \perp, \top \rangle$ will be called the algebra of approximations in a up-directed space (UA algebra). If R is a up-directed

parthood relation or a reflexive up-directed relation respectively, then it shall said to be a up-directed parthood algebra of approximations (AP algebra) or a reflexive up-directed algebra of upper approximations (AR algebra) respectively.

Theorem 8. A AP algebra S_{lu} satisfies all of the following (universal quantifiers have been omitted):

$$a \sqcap a = a \ \& \ (a \sqcup a) \sqcap a = a \quad (\text{idp3})$$

$$a \sqcup a = a^u \quad (\text{qidp4})$$

$$a \sqcap b = b \sqcap a \ \& \ a \sqcup b = b \sqcup a \quad (\text{com12})$$

$$a \sqcap (b \sqcup a) = a \quad (\text{habs})$$

$$a \sqcup (b \sqcup c) = (a \sqcup b^u) \sqcup c^u \quad (\text{qas1})$$

$$(a \sqcup (b \sqcup c)) \sqcup ((a \sqcup b) \sqcup c) = ((a \sqcup a) \sqcup (b \sqcup b)) \sqcup (c \sqcup c \sqcup c) \quad (\text{qas0})$$

Proof. **idp3** $a \sqcap a = (a \cap a)^l = a^l = a$ and $a \sqcup a = a^u$ and $a^u \cap a = a$

qidp4 $a \sqcup a = (a \cup a)^u = a^u$.

com12 This follows from definition.

habs $a \sqcap (b \sqcup a) = (a \cap (b \cup a))^l = ((a \cap a^u) \cup (a \cap b^u))^l$ which is equal to $(a \cup (a \cap b^u))^l = a^l = a$

qas1 $a \sqcup (b \sqcup c) = (a \cup (b \cup c))^u = (a^u \cup b^{uu} \cup c^{uu})$ and this is $(a \cup b^u)^u \cup c^{uu} = (a \sqcup b^u) \sqcup c^u$

qas0 This can be proved by writing all terms in terms of $\cup$. In fact $(a \sqcup (b \sqcup c)) \sqcup ((a \sqcup b) \sqcup c) = a^{uuu} \cup b^{uuu} \cup c^{uuu}$. The expression on the right can be rewritten in terms of $\sqcup$ by **qidp4**.

The above two theorems in conjunction with the properties of approximations on the power set, suggest that it would be useful to enhance UA-, AP-, and AR-algebras with partial operations for defining an abstract semantics.

Definition 8. A partial algebra of the form

$$S_{lu}^* = \langle S_{lu}, \sqcap, \sqcup, \sqcap, \sqcup, \sqcap^\kappa, l, u, \perp, \top \rangle$$

will be called the algebra of approximations in a up-directed space (UA partial algebra) whenever $S_{lu} = \langle S_{lu}, \sqcap, \sqcup, \sqcap, \sqcup, l, u, \perp, \top \rangle$ is a UA algebra and $\sqcap$ and κ are defined as follows ($\cap$ and c being the intersection and complementation operations on $\wp(S)$):

$$(\forall a, b \in S_{lu}) a \sqcap b = \begin{cases} a \cap b & \text{if } a \cap b \in S_{lu} \\ \text{undefined} & \text{otherwise} \end{cases} \quad (4)$$

$$(\forall a \in S_{lu}) a^\kappa = \begin{cases} a^c & \text{if } a^c \in S_{lu} \\ \text{undefined} & \text{otherwise} \end{cases} \quad (5)$$

If R is an up-directed parthood relation or a reflexive up-directed relation respectively, then it shall said to be a up-directed parthood partial algebra of approximations (AP partial algebra) or a reflexive algebra of upper approximations (AR partial algebra) respectively.

Theorem 9. *If S is an up-directed approximation space, then its associated enhanced up-directed parthood partial algebra $S_{lu}^* = \langle \underline{S}_{lu}, \mathbb{M}, \mathbb{U}, \cup, \cap, \circ, \kappa, l, u, \perp, \top \rangle$ satisfies all of the following:*

$$\langle \underline{S}_{lu}, \mathbb{M}, \mathbb{U}, \cup, l, u, \perp, \top \rangle \text{ is a AP algebra} \quad (\text{app1})$$

$$a \cap a = a \ \& \ a \cap \perp = \perp \ \& \ a \cap \top = a \quad (\text{app2})$$

$$a \cap b \stackrel{\omega}{=} b \cap a \ \& \ a \cap (b \cap c) \stackrel{\omega}{=} (a \cap b) \cap c \quad (\text{app3})$$

$$a \cap a^u = a = a \cap a^l \ \& \ a^{\kappa\kappa} \stackrel{\omega}{=} a \quad (\text{app4})$$

$$a \cap (b \cup c) \stackrel{\omega}{=} (a \cap b) \cup (a \cap c) \quad (\text{app5.0})$$

$$a \cup (b \cap c) \stackrel{\omega}{=} (a \cup b) \cap (a \cup c) \quad (\text{app5.1})$$

$$(a \cap b)^{\kappa} \stackrel{\omega}{=} a^{\kappa} \cup b^{\kappa} \ \& \ (a \cup b)^{\kappa} \stackrel{\omega}{=} a^{\kappa} \cap b^{\kappa} \quad (\text{app6})$$

$\circ$ is the partial groupoid operation induced from its power set.

Proof. The theorem follows from the previous theorems in this section.

If the parthood relation is both up-directed and also transitive, then it is possible to have an induced groupoid operation on the set of definite elements $(\delta_{lu}(S) = \{x : x_i^l = x_i^u \ \& \ x \in \wp(S)\})$. If $A, B \in \delta_{lu}(S)$, then let $A \cdot B = \{aba \in A, \ \& \ b \in B\}$. $\delta_{lu}(S)$ is closed under set union and intersection, and the pseudo-complementation $^+$ is defined from $[x]_i^+ = \bigcup \{A : A \in \delta_{lu}(S) \ \& \ A \cap [x]_i = \emptyset \text{ for any } x \in S$.

Theorem 10. *If S is an up-directed parthood space in which R is transitive, then $\langle \delta_{lu}(S), \cdot, \cap, \cup, ^+, 0, 1 \rangle$ is a Heyting algebra with an extra groupoid operation induced by the groupoid operation on S .*

Proof. The proof that $\langle \delta_{lu}(Q), \cap, \cup, ^+, \perp, \top \rangle$ is a Heyting algebra is analogous to the proof in [23].

If $a, b \in [x]_i$, for an inverse neighborhood granule, then there exists a c such that Rac and Rbc , but by the definition of $[x]$, $c \in [x]$ follows. Therefore $[x]_i$ is a subgroupoid of S for each $x \in S$.

Suppose $A, B \in \delta_{lu}(S)$, then (as any element in these sets must be in some granules) for any $a \in [x]_i \subseteq A$ and $b \in [z]_i \subseteq B$, ab is in the order filter generated by $[x] \cup [z]$. So AB must be an element of $\delta_{lu}(S)$. This essentially proves the theorem.

3.3 Examples, Meaning and Interpretation

Abstract examples are easy to construct for the situations covered and many are available in other papers [11, 12, 18] by the present author and others. So an application strategy to student-centric learning (a constructive teaching method in which students learn by explorative open-ended activities) is proposed. It should be noted that education researchers adhere to various ideas of distributed

cognition (that the environment has a key role in cognitive process that are inherently personal [24]) and so the basic assumptions of formal concept analysis may be limiting [25]. Suppose a student has access to a set K of concepts and is likely to arrive at a another set of potentially vague concepts H . Teachers typically play the role of facilitators, are not required to be the sole source of knowledge, and would need to direct the activity to an improved set of concepts H^+ . In the construction of these sets, groupoidal operations can play a crucial role. Equations of the form $ab = c$ can be read in terms of concepts – c can be a better relevant concept for the activity in comparison to the a and b . Note that no additional order structure on the set of concepts is presumed. *This is important also because concepts may not be structured as in lattice-theoretic perspective of formal concept analysis or classical rough sets.*

In classical rough sets, definite concepts correspond to approximations (definite objects). From the present study, it can be seen that the induced total groupoid operation on the set of definite objects is the part of a concept b that can be read from another concept a . This interpretation is primarily due to the relation R being symmetric and reflexive. When the approximation space is up-directed, then it happens that every object is indiscernible from every other object. So the property of up-directedness is not of much interest in the classical context. The $\oplus$ operation concerns choice between two things and so is relevant for pairwise comparisons [26].

In parthood and other up-directed general approximation spaces, a groupoid operation typically corresponds to answering the question *which attribute or object is preferable to two given attributes or objects?* Therefore collections of all possible definable groupoid operations correspond to all answers. Ideas of vision then must be about choices of subsets or subclasses among possible definable operations. Formally,

Definition 9. A vision for an up-directed approximation space, S is a subset $\mathcal{V}(S)$ of $\mathfrak{B}(S)$.

4 Formalism of Higher Granular Operator Spaces

Granular operator spaces and variants [1, 8, 10, 27] are abstract frameworks for extending granularity and parthood in the context of general rough sets, and are also variants of rough Y-systems studied by the present author [2]. In this section, it will be shown that all types of granular operator spaces and variants can be transformed into partial algebras that satisfy additional conditions. This is also nontrivial because all covering approximation spaces cannot be transformed in the same way.

Definition 10. A High General Granular Operator Space (GGS) $\mathbb{S}$ shall be a partial algebraic system of the form $\mathbb{S} = \langle \underline{\mathbb{S}}, \gamma, l, u, \mathbf{P}, \leq, \vee, \wedge, \perp, \top \rangle$ with $\underline{\mathbb{S}}$ being a set, γ being a unary predicate that determines $\mathcal{G}$ (by the condition γx if and only if $x \in \mathcal{G}$) an admissible granulation (defined below) for $\mathbb{S}$ and l, u being operators : $\underline{\mathbb{S}} \mapsto \underline{\mathbb{S}}$ satisfying the following ($\underline{\mathbb{S}}$ is replaced with $\mathbb{S}$ if clear from the

context. $\vee$ and $\wedge$ are idempotent partial operations and $\mathbf{P}$ is a binary predicate. Further γx will be replaced by $x \in \mathcal{G}$ for convenience.):

$$(\forall x)\mathbf{P}xx \quad (\text{PT1})$$

$$(\forall x, b)(\mathbf{P}xb \ \& \ \mathbf{P}bx \longrightarrow x = b) \quad (\text{PT2})$$

$$(\forall a, b)a \vee b \stackrel{\omega}{=} b \vee a ; \ (\forall a, b)a \wedge b \stackrel{\omega}{=} b \wedge a \quad (\text{G1})$$

$$(\forall a, b)(a \vee b) \wedge a \stackrel{\omega}{=} a ; \ (\forall a, b)(a \wedge b) \vee a \stackrel{\omega}{=} a \quad (\text{G2})$$

$$(\forall a, b, c)(a \wedge b) \vee c \stackrel{\omega}{=} (a \vee c) \wedge (b \vee c) \quad (\text{G3})$$

$$(\forall a, b, c)(a \vee b) \wedge c \stackrel{\omega}{=} (a \wedge c) \vee (b \wedge c) \quad (\text{G4})$$

$$(\forall a, b)(a \leq b \leftrightarrow a \vee b = b \leftrightarrow a \wedge b = a) \quad (\text{G5})$$

$$(\forall a \in \mathbb{S}) \mathbf{P}a^l a \ \& \ a^{ll} = a^l \ \& \ \mathbf{P}a^u a^{uu} \quad (\text{UL1})$$

$$(\forall a, b \in \mathbb{S})(\mathbf{P}ab \longrightarrow \mathbf{P}a^l b^l \ \& \ \mathbf{P}a^u b^u) \quad (\text{UL2})$$

$$\perp^l = \perp \ \& \ \perp^u = \perp \ \& \ \mathbf{P}\top^l \top \ \& \ \mathbf{P}\top^u \top \quad (\text{UL3})$$

$$(\forall a \in \mathbb{S}) \mathbf{P}\perp a \ \& \ \mathbf{P}a \top \quad (\text{TB})$$

Let $\mathbb{P}$ stand for proper parthood, defined via $\mathbf{P}ab$ if and only if $\mathbf{P}ab \ \& \ \neg \mathbf{P}ba$. A granulation is said to be admissible if there exists a term operation t formed from the weak lattice operations such that the following three conditions hold:

$$(\forall x \exists x_1, \dots, x_r \in \mathcal{G}) t(x_1, x_2, \dots, x_r) = x^l$$

$$\text{and } (\forall x)(\exists x_1, \dots, x_r \in \mathcal{G}) t(x_1, x_2, \dots, x_r) = x^u, \quad (\text{Weak RA, WRA})$$

$$(\forall a \in \mathcal{G})(\forall x \in \mathbb{S})(\mathbf{P}ax \longrightarrow \mathbf{P}ax^l), \quad (\text{Lower Stability, LS})$$

$$(\forall x, a \in \mathcal{G})(\exists z \in \mathbb{S}) \mathbb{P}xz, \ \& \ \mathbb{P}az \ \& \ z^l = z^u = z, \quad (\text{Full Underlap, FU})$$

The conditions defining admissible granulations mean that every approximation is somehow representable by granules in an algebraic way, that every granule coincides with its lower approximation (granules are lower definite), and that all pairs of distinct granules are part of definite objects (those that coincide with their own lower and upper approximations). Special cases of the above are defined next.

Definition 11. – In a GGS, if the parthood is defined by $\mathbf{P}ab$ if and only if $a \leq b$ then the GGS is said to be a high granular operator space $\mathcal{GS}$.

- A higher granular operator space ($\mathcal{HGOS}$) $\mathbb{S}$ is a $\mathcal{GS}$ in which the lattice operations are total.
- In a higher granular operator space, if the lattice operations are set theoretic union and intersection, then the $\mathcal{HGOS}$ will be said to be a set $\mathcal{HGOS}$.

Theorem 11. In the context of Definition 10, the binary predicates $\mathbf{P}$ can be replaced by partial two-place operations $1\mathcal{PGO} \odot$ and γ is replaceable by a total unary operation h defined as follows:

$$hx = \begin{cases} x & \text{if } \gamma x \\ \perp & \text{if } \neg \gamma x \end{cases} \quad (6)$$

Consequently $\mathbb{S}^+ = \langle \mathbb{S}, h, l, u, \odot, \vee, \wedge, \perp, \top \rangle$ is a partial algebra that is semantically (and also in a category-theoretic sense) equivalent to the original GGS $\mathbb{S}$.

Proof. Because of the restriction UL3 on $\perp$ and the redundancy of $\leq$ (because of G5), the result follows.

Definition 12. The partial algebra formed in the above theorem will be referred to a high granular operator partial algebra (GGSp).

Problem 1. All covering approximation spaces considered in the rough set literature actually assume partial Boolean or partial lattice theoretical operations. Some authors (especially in modal logic perspectives) [3, 5, 28] presume that all Boolean operations are admissible – this view can be argued against. A natural question is *Are the modal logic semantics themselves only a possible interpretation of the actuality?* All this suggests the problem of finding minimal operations involved in the context.

Because all covering approximation spaces do not use granular approximations in the sense mentioned above, it follows that they do not form GGSo always. In the next example, the applicability of the above to activity based mathematics teaching is considered.

Example 1. In constructivist activity based learning, teachers almost always set learning goals ahead of initiating activities. Therefore knowledge constructed in such contexts are constrained by concept maps (typically directed) accessible to the teacher in question [29–31]. As a consequence desired concept granules (and explicit or implicit ontology) can be specified by teachers. But students and teachers are likely to make use of a number of additional vague or exact concepts in any specific activity. In addition, general ideas of parthood as specified in the definition of GGS can be interpreted over the collection of vague and exact concepts. It may even make sense to define additional groupoid operations apart from the ones induced by the relations. [30, 31] do not make room for vague concepts and presume a transitive parthood that operates over the teacher’s goals.

For example, in [32], the goal of the game activity is to understand and apply Pythagoras theorem in few situations. The board game (see Fig. 1) involves students throwing a pair of die, form the square root of the sum of the squares of the values obtained and round off the result to a whole number and advance that many squares on the board. The goal of the game (for students) is to reach the finish block. It can be seen that concepts such as sample space, events, floor and ceiling functions, and vague variants thereof, incorrect concepts of biased dies are all part of the potential learning space. All these can be approximated (irrespective of consequence) relative to the teacher’s specification of granules. Moreover they may improperly specify the relation between concepts that are of lesser interest to the lesson plan.

It is not hard to see that the generalized scenario described in the last two paragraphs can be modeled by a GGSp.

An expanded version of the last example will appear separately.

Fig. 1. Board game

5 Further Directions and Remarks

In this research methods of representing important ideas of decisions or preferences inherent in information tables (related to data including those relating to human reasoning) have been invented and the semantics considered in two types of rough sets. A representation theorem is proved for transitive parthood spaces. Further the formalism of higher granular operator spaces and variants are shown to be representable as partial algebras. Examples illustrating key aspects of the research in education research contexts have also been constructed.

Among the many directions of research motivated by this paper, the following are more important: a finer algebraic classification of the derived groupoids and partial groupoids, representation of derived partial algebras as quasi varieties, detailed application to education research contexts (especially in the direction indicated in Example 1), and in self-organizing systems.

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