A Numerical Approach for Computing Euler Characteristics of Affine Varieties

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Abstract. We develop a numerical nonlinear algebra approach for computing the Euler characteristic of an affine variety. Our approach is to relate Euler characteristics of a smooth affine variety with the number of critical points using Morse theory. In general, we stratify a variety into the union of smooth affine varieties to obtain results on singular varieties.

Keywords: Euler characteristic · Numerical algebraic geometry · Homotopy continuation

1 Introduction

The Euler characteristic is one of the most fundamental topological invariants. In the past decade, a series of work appeared which relate Euler characteristics of complex algebraic varieties with the complexity of algebraic optimization problems [1, 5, 15, 20, 21]. There are several existing approaches to compute the Euler characteristics of complex algebraic varieties [3, 6, 16, 19], each having their own benefits. Our new approach has the following advantages.

1. Our methods directly compute the Euler characteristic of an affine variety without involving any compactification. This is useful because the closure of a smooth affine variety can have singularities along infinity.
2. We stratify and compute the Euler characteristics of smooth affine varieties. In theory, any d -dimensional affine variety can be stratified into the union of at most $d + 1$ smooth affine varieties. In contrast to the inclusion-exclusion principle, our method does not involve too many varieties.
3. We can tailor the stratification to reduce the degree of each stratum.

A standard method to compute Euler characteristics of complex algebraic varieties is to reduce to the projective hypersurface case. The drawback of this method is that to compute the Euler characteristic of a projective variety, the number of involved hypersurfaces grows exponentially in the codimension.

In contrast, we compute the Euler characteristic of a smooth equidimensional affine variety X by counting the critical points of $\dim(X) + 1$ algebraic functions.

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Given a singular complex affine variety, we stratify it into smooth affine varieties to reduce to the smooth case. In theory, we can always stratify a d -dimensional affine variety into $d + 1$ smooth (possibly not connected) equidimensional affine varieties of dimension $d, d - 1, \dots, 1, 0$. So we need to compute the number of critical points of at most $(d + 1)(d + 2)/2$ algebraic functions. Our algorithms also have the practical feature of minimizing the degree of the algebraic functions at the expense of increasing the number of functions to consider.

This work is organized follows. In Sect. 2, we recall a theorem to determine the Euler characteristic of a smooth equidimensional variety with a general hyperplane removed by counting critical points of a function. In Sect. 3, we provide some key definitions from numerical algebraic geometry. In Sects. 4–5 we present algorithms for computing Euler characteristics.

2 Euler Characteristics and Critical Points

Let X be a topological space that is homotopy equivalent to a finite CW-complex. The Euler characteristic of X , denoted by $\chi(X)$, is the alternating sum of the Betti numbers of X [9, Page 146]. We are only interested in the situation where X is a (complex) affine algebraic variety [8, Corollary 6.10]. In this case, the Euler characteristic of X is an alternating sum of cardinalities of several sets of critical points as shown in Theorem 1.

Given a singular affine algebraic variety, it admits a stratification into locally closed smooth subvarieties. We can always refine the stratification into locally closed smooth affine subvarieties. Since the Euler characteristic is additive for a stratification of locally closed subvarieties, it is enough to compute the Euler characteristic of smooth affine varieties. In this paper, we discuss the following two objectives.

1. Compute the Euler characteristics of smooth affine varieties.
2. Find algorithms to stratify singular affine varieties into locally closed smooth subvarieties.

Let X be a smooth subvariety of $\mathbb{C}^n$. Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a regular function. If X is defined by polynomials $g_1, \dots, g_l \in \mathbb{C}[x_1, \dots, x_n]$, then the critical points of $f|_X$ for a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ are the points $P \in X$ such that the vector $(\frac{df}{dx_1}, \dots, \frac{df}{dx_n})|_P$ is in the linear span of $\{(\frac{dg_i}{dx_1}, \dots, \frac{dg_i}{dx_n})|_P : i = 1, \dots, l\}$.

This theorem relates the number of critical points to the Euler characteristic.

Theorem 1 ([20]). *Let ℓ denote a general affine linear function $\ell : \mathbb{C}^n \rightarrow \mathbb{C}$ and let X denote a smooth equidimensional affine subvariety of $\mathbb{C}^n$. Then*

$$(-1)^{\dim(X)} \chi(X \setminus V(\ell)) = \#\{\text{critical points of } \ell|_X\}.$$

As a corollary we are able to determine the Euler characteristic of X itself.

Corollary 1. *Let X be a smooth subvariety of $\mathbb{C}^n$. For $i = 1, \dots, \dim(X)$, let h_i denote a general affine linear function $\mathbb{C}^n \rightarrow \mathbb{C}$. Then we have the equality*

$$\chi(X) = p + \sum_{i=1}^{\dim(X)} (-1)^{\dim(X)-i+1} \eta_i$$

where η_i is the number of critical points of $h_i|_{X \cap V(h_1, \dots, h_{i-1})}$ and p is the cardinality of $X \cap V(h_1, \dots, h_{\dim(X)})$.

Proof. The additive property of Euler characteristic implies the equality

$$\chi(X) = \sum_{i=1}^{\dim(X)} \chi(X \cap V(h_1, \dots, h_{i-1}) \setminus V(h_i)) + \chi(X \cap V(h_1, \dots, h_{\dim(X)})).$$

It follows from Bertini's theorem that each intersection $X \cap V(h_1, \dots, h_{i-1})$ is smooth and $X \cap V(h_1, \dots, h_{\dim(X)})$ is a set of p points. By Theorem 1, we have $\eta_i = (-1)^{\dim(X)-i+1} \chi(X \cap V(h_1, \dots, h_{i-1}) \setminus V(h_i))$. $\square$

3 Numerical Algebraic Geometry Basics

In this section we recall a witness set [4, 23], which is a fundamental concept in numerical algebraic geometry. A witness set is used to analyze algebraic varieties and is manipulated using homotopy continuation [2], as seen in Sects. 3.2–3.3.

3.1 Witness Sets and Numerical Irreducible Decomposition

Let X be an equidimensional subvariety of affine space $\mathbb{C}^n$. As a consequence of Bertini's Theorem, there are two invariants, dimension and degree, of X that can be understood by intersecting X with a general linear space. The dimension $\dim(X)$ of a subvariety X of $\mathbb{C}^n$ is the codimension of a general affine linear space $\mathcal{L} \subseteq \mathbb{C}^n$ such that $X \cap \mathcal{L}$ is finite and nonempty. The degree $\deg(X)$ of X is the number of points in $X \cap \mathcal{L}$.

Definition 1 (Witness set). Suppose X is an equidimensional subvariety of $\mathbb{C}^n$. A witness set for X is a triple (F, L, W) , where F is a finite set of polynomials with each irreducible component of X being an irreducible component of $V(F)$, L is a set of $\dim(X)$ general¹ affine linear functions, and W is the set of points $X \cap V(L)$.

In numerical algebraic geometry, W is called a *witness point set* for X . Since L consists of general affine linear functions, the affine linear space $V(L)$ is general and the cardinality of the set W is $\deg(X)$. Throughout, we assume the ideal generated by F in Definition 1 defines a reduced scheme by using *deflation* [13, 18].

Given a (not necessarily equidimensional) subvariety X of $\mathbb{C}^n$, we denote by X_i the union of i -dimensional irreducible components of X . We call a set of witness sets of X_i for $i = 0, 1, \dots, k$ a *numerical equidimensional decomposition* of X . This decomposition can be refined to a *numerical irreducible decomposition* of X by providing a witness set for each irreducible component of X (see [4]).

¹ By “general” here, we mean the intersection $X \cap V(L)$ is transverse and has cardinality $\deg(X)$.

Example 1 (Embedding). Suppose X is an equidimensional subvariety of $\mathbb{C}^n$ and h is a linear function $\mathbb{C}^n \rightarrow \mathbb{C}$. Let $\widehat{X}$ be the image of X under the closed embedding $\mathbb{C}^n \rightarrow \mathbb{C}^{n+1}$ given by $x \mapsto (x, h(x))$. Given a witness set (F, L, W) for X , we construct a witness set for $\widehat{X}$ as $(F \cup \{h - x_{n+1}\}, L, \{(x, h(x)) : x \in W\})$.

Example 2. We can also easily construct a witness set for the Cartesian product of two varieties. Suppose X_i is a subvariety of $\mathbb{C}^{n_i}$ for $i = 1, 2$. If (F_i, L_i, W_i) is a witness set for X_i , then $(F_1 \cup F_2, L_1 \cup L_2, W_1 \times W_2)$ is a witness set for $X_1 \times X_2 \subset \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$.

3.2 Witness Collections of Subvarieties in $\mathbb{C}^n \times \mathbb{C}^n$

A witness collection is a generalization of a witness set and is used to study varieties that are defined by polynomials with a natural multi-variable group structure. For a complete description of witness collections see [10, 11, 17]. For our purposes, it suffices to study the following special case.

A *witness collection* for a d -dimensional irreducible subvariety Z of $\mathbb{C}^n \times \mathbb{C}^n$ is the following collection of triples, which we call *multi-affine witness sets*:

$$(F, L^i \cup M^{d-i}, Z \cap (L^i \times M^{d-i})) \text{ for } i = 0, 1, \dots, d,$$

where F is a set of polynomials such that $V(F)$ contains Z as an irreducible component; L^i and M^i are general codimension i affine linear spaces in $\mathbb{C}^n$ defined the sets of general linear functions L^i and M^i respectively.

Witness collections are used to understand the intersection of Z with a Cartesian product of linear spaces. Let $\mathcal{A}^i$ and $\mathcal{B}^i$ denote general codimension i affine linear spaces in $\mathbb{C}^n$. With homotopy continuation, we determine the isolated points in the intersection $Z \cap (\mathcal{A}^i \times \mathcal{B}^{d-i})$. These points are contained in the set of endpoints of the homotopy $H^i : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \rightarrow \mathbb{C}^{N+d}$ with

$$(x, y, t) \mapsto (F(x, y), tL^i(x) + (1-t)A^i(x), tM^{d-i}(y) + (1-t)B^{d-i}(y)) \quad (1)$$

where $L^i, A^i, M^{d-i}, B^{d-i}$ are sets of affine linear functions defining $\mathcal{L}^i, \mathcal{A}^i, \mathcal{M}^{d-i}$ and $\mathcal{B}^{d-i}$ respectively. For more details see [10, Remark 1.3]. To conveniently denote linear functions, for $a, x \in \mathbb{C}^n$, we take $a \circ x$ to be the usual inner product.

Example 3 (Conormal variety). Let $X = V(f_1, \dots, f_k)$ be a smooth equidimensional variety in $\mathbb{C}^n$ with $(f_1, \dots, f_k)$ generating a radical ideal. The (affine) conormal variety of X is a subvariety $\mathcal{C}(X)$ in $\mathbb{C}^n \times \mathbb{C}^n$ with an ideal

$$\langle f_1, \dots, f_k \rangle + \langle (1 + \text{codim}(X))\text{-minors of } \text{Jac}_x(x \circ y, f_1, \dots, f_k) \rangle \subset \mathbb{C}[x, y],$$

where $\text{Jac}_x(x \circ y, f_1, \dots, f_k)$ is a $(k+1) \times n$ matrix of partial derivatives with respect to x . The dimension of $\mathcal{C}(X)$ is n . For a projective formulation, see [22].

A witness collection for the conormal variety of X is given by

$$(F, L^i \cup M^{n-i}, W_i) \text{ for } i = 0, \dots, n, \quad (2)$$

where $V(F)$ contains $\mathcal{C}(X)$ as an irreducible component and

$$W_i := \mathcal{C}(X) \cap (\mathcal{L}^i \times \mathcal{M}^{n-i}).$$

Each multi-affine witness set $(F, L^i \cup M^{n-i}, W_i)$ has information about the variety X . For example, the dimension of X is the maximal i such that $W_i \neq \emptyset$ and the degree of X is the cardinality of $W_{\dim(X)}$. Moreover, for $i = 0$, the linear space $V(M^n) \subset \mathbb{C}^n$ contains a unique point, say c . The set of points (x, y) in W_0 such that $x \circ c \neq 0$ is the set of critical points of the general linear function $x \circ c$ on $X \setminus V(x \circ c)$. The cardinality of the set $W_0 \setminus V(x \circ c)$ is the Euler characteristic of $X \setminus V(x \circ c)$ up to a sign. In general, the number of points in W_i is an upperbound for the η_i appearing in Corollary 1.

Fig. 1. We illustrate W_0 (left) and W_1 (right) for the circle $X = V(x_1^2 + x_2^2 - 1)$ in $\mathbb{C}^2$.

3.3 Regeneration and Removing a Hypersurface

Given a witness collection for an irreducible variety X and a polynomial g , regeneration determines a witness set for $X \cap V(g)$. One of two situations can occur. First, if X is contained in $V(g)$, then $X = X \cap V(g)$ and we are done. Second, if X is not contained in $V(g)$, then for $i = 1, \dots, \deg(g)$ compute a witness set $(F \cup \{\ell_i\}, L, W_i)$ for $X \cap V(\ell_i)$ where $\ell_i : \mathbb{C}^n \rightarrow \mathbb{C}$ is a general affine linear function. This is easy to do using standard homotopy continuation methods when given the witness set $(F, L \cup \{\ell\}, W)$ for X . This produces a witness set $(F \cup \{\ell_1 \cdots \ell_{\deg(g)}\}, L, \cup_{i=1}^{\deg(g)} W_i)$ for $X \cap V(\ell_1 \cdots \ell_{\deg(g)})$. Finally, the homotopy $H(x, t) = (F(x), t\ell_1(x) \cdots \ell_{\deg(g)}(x) + (1-t)g(x), L(x))$ provides a witness set for $X \cap V(g)$ when $t = 0$. This procedure for computing a witness set for $X \cap V(g)$ from a witness set for X is called *regeneration* [11, 12, 14].

Example 4. For $X \subset \mathbb{C}^n$, consider the embedding $\hat{X} \subset \mathbb{C}^{n+1}$ as in Example 1. Fix a polynomial $g \in \mathbb{C}[x_1, \dots, x_n]$. The affine variety $\hat{X} \cap V(gx_{n+1} - 1)$ is isomorphic to $X \setminus V(g)$. A useful application of regeneration computes a witness set for $\hat{X} \cap V(gx_{n+1} - 1)$. From a witness set for $\hat{X} \subset \mathbb{C}^{n+1}$ and a polynomial $gx_{n+1} - 1$, regeneration produces a witness set for $\hat{X} \cap V(gx_{n+1} - 1)$.

4 Euler Characteristics of Smooth Varieties

Theorem 1 leads to an algorithm that outputs the the Euler characteristic of an algebraic variety by computing critical points. The proof of correctness of the following algorithm is easily derived from Corollary 1.

Algorithm 1. Smooth X

```

1: Input
2:    $X$            A smooth equidimensional affine variety in  $\mathbb{C}^n$ 
3: Output
4:    $\chi(X)$         Euler characteristic of the variety  $X$ 
5: procedure
6:   for  $i = 1, \dots, \dim X$  do
7:      $h_i \leftarrow$  general affine linear functions  $\mathbb{C}^n \rightarrow \mathbb{C}$ 
8:      $\eta_i \leftarrow$  the number of critical points of  $h_i$  on  $X \cap V(h_1, \dots, h_{i-1}) \setminus V(h_i)$ 
9:   end for
10:   $p \leftarrow$  number of points in  $X \cap V(h_1, \dots, h_{\dim(X)})$ 
11:   $\chi(X) \leftarrow \sum_{i=1}^{\dim(X)} (-1)^{\dim(X)-i+1} \eta_i + p$ 
12: return  $\chi(X)$ 

```

We can compute these critical points from a witness set as follows.

Algorithm 2. Numerical Smooth X

```

1: Input
2:    $W_X$            A witness set for a smooth equidimensional affine variety  $X \subset \mathbb{C}^n$ 
3: Output
4:    $\chi(X)$         Euler characteristic of the variety  $X$ 
5: procedure
6:    $W_{X \times \mathbb{C}^n} \leftarrow$  Witness set for  $X \times \mathbb{C}^n$ , computed by using  $W_X$  and methods
    described in example 2
7:    $W_{\mathcal{C}(X)} \leftarrow$  Witness collection for the conormal variety  $\mathcal{C}(X) \subset \mathbb{C}^n \times \mathbb{C}^n$ , by
    regeneration  $\triangleright$  Recall a witness collection consists of multi-affine witness sets
     $(F, L^i \cup M^{n-i}, W_i)$  as described in equation (2).
8:   for  $i = 1, \dots, \dim X$  do
9:      $p_1, \dots, p_i \leftarrow$  general points in  $\mathbb{C}^n$ 
10:     $q_1, \dots, q_{n-i} \leftarrow$  basis for the perpendicular complement of the linear span of
     $p_1, \dots, p_i$ 
11:     $a, b \leftarrow$  general points in  $\mathbb{C}^n$ 
12:     $A^i \leftarrow \{p_1 \circ (x - a), \dots, p_i \circ (x - a)\}$   $\triangleright$  set of  $i$  affine linear functions
13:     $B^{n-i} \leftarrow \{q_1 \circ (y - b), \dots, q_{n-i} \circ (y - b)\}$   $\triangleright$  set of  $n - i$  affine linear functions
14:     $S_i \leftarrow$  the endpoints of the homotopy (1) with  $L^i, M^{n-i}$  and  $A^i, B^{n-i}$  as
    above and start points  $W_i$ .
15:     $\zeta_i \leftarrow$  the number of nonsingular isolated points  $(x, y)$  in  $S_i$  such that  $x \circ b \neq 0$ 
16:  end for
17:   $\chi(X) \leftarrow \sum_{i=0}^{\dim(X)} (-1)^{\dim(X)-i} \zeta_i$ 
18: return  $\chi(X)$ 

```

Proof. (Correctness sketch). Example 3 explains how $\zeta_{\dim(X)} = p$ and $\zeta_0 = \eta_{\dim(X)}$. With some substitutions and algebra one shows $\zeta_i = \eta_{\dim(X)-i}$ for $i = 1, \dots, \dim(X)$, and then the result follows. $\square$

Example 5. Consider the smooth curve $X = V(x_1^2 + x_2^2 - 1) \subset \mathbb{C}^2$ and its conormal variety $\mathcal{C}(X) = V(x_1^2 + x_2^2 - 1, x_2y_1 - x_1y_2) \subset \mathbb{C}^2 \times \mathbb{C}^2$. The algorithms find $(p, \eta_1, \eta_2) = (\zeta_2, \zeta_1, \zeta_0) = (2, 2, 0)$ and both output $\chi(X) = 0$. The points corresponding to p and η are illustrated in Fig. 1 by plotting points on X with the respective normal vectors. The homotopy used to determine ζ_i is given by $H^i : \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C} \rightarrow \mathbb{C}^4$. Concretely, for H^1 , with general affine linear functions $\ell, m : \mathbb{C}^2 \rightarrow \mathbb{C}$, we have $H^1(x, y, t)$ is

$$(x_1^2 + x_2^2 - 1, x_2y_1 - x_1y_2, (1-t)\ell(x) + tp_1 \circ (x-a), (1-t)m(y) + tq_1 \circ (y-b)).$$

5 Euler Characteristics of Singular Varieties

In this section, an excision-restriction method to compute $\chi(X)$ is presented.

Algorithm 3. Excision-restriction method

```

1: Input
2:    $X$            An affine variety in  $\mathbb{C}^n$ 
3: Output
4:    $\chi(X)$        Euler characteristic of the variety  $X$ 
5: procedure
6:   for  $i = 1, \dots, \dim X$  do
7:      $X_i \leftarrow i$ -dimensional irreducible components of  $X$ 
8:   end for
9:    $S \leftarrow |X_0|$ , the cardinality of  $X_0$ 
10:   $Y \leftarrow \bigcup_{i=1}^{\dim(X)} X_i$ 
11:  if  $Y$  is smooth then
12:    for  $i = 1, \dots, \dim X$  do
13:      if  $X_i \neq \emptyset$  then
14:         $S \leftarrow S + \chi(X_i)$  (Algorithm 1)
15:      end if
16:    end for
17:  else
18:     $C \leftarrow$  a union of irreducible components of  $\text{Sing}(Y) \cup \bigcup_{i=1}^{\dim(X)-1} X_i$ 
19:    Choose a polynomial  $g$  such that  $X_{\dim(X)} \subsetneq V(g)$  and  $V(g)$  contains  $C$ .
20:     $S_1 \leftarrow \chi(Y \cap V(g))$   $\triangleright$  Algorithm 3, recursion
21:     $S_2 \leftarrow \chi(Y \setminus V(g))$   $\triangleright$  Algorithm 3, recursion  $\triangleright$  Here,  $Y \setminus V(g)$  is considered as
       a subvariety of  $\mathbb{C}^{n+1}$  with coordinate functions  $x_1, \dots, x_n, 1/g$  as in Example 4.
22:  end if
23:   $\chi(X) \leftarrow S + S_1 + S_2$ 
24:  return  $\chi(X)$ 

```

The freedom of choosing g in step 19 is a feature of this algorithm. One way to find such a g is by taking a generic linear combination of minimal generators

of the ideal of $\text{Sing}(Y) \cup \bigcup_{i=1}^{\dim(X)-1} X_i$. An alternative numerical approach is to sample a set of points from an irreducible component of $\text{Sing}(Y) \cup \bigcup_{i=1}^{\dim(X)-1} X_i$, to use numerical implicitization, which has been implemented in [7]. To minimize the number of recursions, choose a g vanishing on $\text{Sing}(Y) \cup \bigcup_{i=1}^{\dim(X)-1} X_i$. Alternatively, we could choose g so that the degree is small as possible. This heuristic works well for a numerical approach.

Next, we tailor the previous algorithm for regeneration, which was described in Sect. 3.3.

Algorithm 4. Numerical Excision-Restriction

```

1: Input
2:    $\cup_i (F_i, L_i, W_i)$  A numerical equidimensional decomposition for an affine variety
    $X \subset \mathbb{C}^n$  where  $(F_i, L_i, W_i)$  is a witness set for  $X_i$ , the union of  $i$ -dimensional
   irreducible components of  $X$ 
3: Output
4:    $\chi(X)$  Euler characteristic of the variety  $X$ 
5: procedure
6:    $S \leftarrow |W_0|$ 
7:    $Y \leftarrow \bigcup_{i=1}^{\dim(X)} X_i$ 
8:   if  $Y$  is smooth then
9:     for  $i = 1, \dots, \dim X$  do
10:       $S \leftarrow S + \chi(X_i)$  ▷ Input  $(F_i, L_i, W_i)$  to Algorithm 2.
11:     end for
12:   else
13:      $C \leftarrow$  a union of irreducible components of  $\text{Sing}(Y) \cup \bigcup_{i=1}^{\dim(X)-1} X_i$ 
14:      $g \leftarrow$  a polynomial such that  $X_{\dim(X)} \subsetneq V(g)$  and  $V(g)$  contains  $C$ 
15:      $\mathcal{D}_1 \leftarrow$  a numerical decomposition for  $Y \cap V(g)$ , using methods in Section 3.3
16:      $S_1 \leftarrow \chi(Y \cap V(g))$  ▷ Input  $\mathcal{D}_1$  to Algorithm 4, recursion
17:      $\mathcal{D}_2 \leftarrow$  a numerical decomposition for  $Y \setminus V(g)$ , using methods in Example 4
18:      $S_2 \leftarrow \chi(Y \setminus V(g))$  ▷ Input  $\mathcal{D}_2$  to Algorithm 4, recursion
19:   end if
20:    $\chi(X) \leftarrow S + S_1 + S_2$ 
21: return  $\chi(X)$ 

```

Example 6. Consider the Whitney umbrella $X = V(x^2y - z^2) \subset \mathbb{C}^3$. The variety X is two dimensional with a singular locus given by the y -axis. The output of the algorithm to compute $\chi(X)$ is summarized by the following equations,

$$\begin{aligned}
\chi(X) &= \chi(X \cap V(g)) + \chi(X \setminus V(g)) \\
&= \chi(X \cap V(g) \cap V(g')) + \chi(X \cap V(g) \setminus V(g')) + \chi(X \setminus V(g)) \\
&= 1 + 0 + 0,
\end{aligned}$$

where we have made the following choices: $g = 2x + 5z$ and $g' = y - x - 4/25$ such that $V(g) \supset \text{Sing}(X)$ and $V(g') \supset \text{Sing}(X \cap V(g))$.

We input a numerical equidimensional decomposition for X . Since X is itself irreducible, we have $Y = X$ and $S = |W_0| = 0$. Using regeneration we compute a numerical decomposition $\mathcal{D}_1$ for $X \cap V(g)$, which is a union of two lines that intersect at the point $(0, 4/25, 0)$.

In step 16, we apply Algorithm 4 to $\mathcal{D}_1$, and in this recursive step we let g' play the role of g . The variety $X \cap V(g) \cap V(g')$ is the point $(0, 4/25, 0)$ so we have $\chi(X \cap V(g) \cap V(g')) = 1$. On the other hand, $X \cap V(g) \setminus V(g')$ is smooth, so by using Algorithm 2 we get $\chi(X \cap V(g) \setminus V(g')) = 0$.

Lastly, in step 17, we find a numerical decomposition $\mathcal{D}_2$ for $X \setminus V(g)$ to do step 18. Since $X \setminus V(g)$ is smooth, we use Algorithm 2 to find $\chi(X \setminus V(g)) = 0$.

In this paper we focussed on algorithm development, and we are working towards implementing Algorithm 4. In addition, we are also working to implement Algorithm 3 using Grobner basis to compare with the numerical version and previous techniques.

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