Combinatorial Properties of Degree Sequences of 3-Uniform Hypergraphs Arising from Saind Arrays

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Abstract. The characterization of k -uniform hypergraphs by their degree sequences, say k -sequences, has been a longstanding open problem for $k \geq 3$. Very recently its decision version was proved to be NP -complete in [3]. In this paper, we consider Saind arrays S_n of length $3n - 1$, i.e. arrays $(n, n - 1, n - 2, \dots, 2 - 2n)$, and we compute the related 3-uniform hypergraphs incidence matrices M_n as in [3], where, for any M_n , the array of column sums, $\pi(n)$ turns out to be the degree sequence of the corresponding 3-uniform hypergraph. We show that, for a generic $n \geq 2$, $\pi(n)$ and $\pi(n + 1)$ share the same entries starting from an index on. Furthermore, increasing n , these common entries give rise to the integer sequence A002620 in [15]. We prove this statement introducing the notion of *queue-triad* of size n and pointer k . Sequence A002620 is known to enumerate several combinatorial structures, including symmetric Dyck paths with three peaks, some families of integers partitions in two parts, bracelets with beads in three colours satisfying certain constraints, and special kind of genotype frequency vectors. We define bijections between queue triads and the above mentioned combinatorial families, thus showing an innovative approach to the study of 3-hypergraphic sequences which should provide subclasses of 3-uniform hypergraphs polynomially reconstructable from their degree sequences.

1 Introduction

A fundamental and widely investigated notion related both to graphs and to hypergraphs is the characterization of their degree sequences (i.e. the array of their vertex degrees), indeed they are of relevance to model and gather information about a wide range of statistics of complex systems (see the book by Berge [14]). The characterization of the degree sequences of k -uniform simple hypergraphs, i.e., those hypergraphs whose hyperedges have the same cardinality k and such that no loops and no parallel hyperedges are present, called k -hypergraphic sequences, has been a long-standing open problem for the case

$k > 2$, until very recently has been proved to be *NP*-complete [3]. Formally, that is: Given $\pi = (d_1, d_2, \dots, d_n)$ a sequence of positive integers, can π be the degree sequence of a k -uniform simple hypergraph?

The degree sequences for $k = 2$, that is for simple graphs, have been studied by many authors, including the celebrated work of Erdős and Gallai [2], which effectively characterizes them. A polynomial time algorithm to reconstruct the adjacency matrix of a graph G having π as degree sequence (if G exists) has been defined by Havel and Hakimi [16].

In this article, we push further the study of degree sequences of simple k -uniform hypergraphs, with $k \geq 3$: the NP-completeness result of their characterization led our interest to find some subclasses that are polynomially tractable in order to restrict the NP-hard core of the problem. Some literature on recent developments on this subject can be found in [3, 5, 7, 10, 12]. In particular, the present study aims at studying degree sequences that generalize those used as gadget for the NP-completeness proof in [3] and that can be computed starting from a generic integer vector, as shown in the next section. We mainly focus on vectors of the form $(n, n-1, n-2, \dots, 2-2n)$, that we call *Saind arrays*, and we analyze the combinatorial properties of the derived degree sequences in order to characterize them and to gather information about the associated 3-uniform hypergraphs. The results are obtained by borrowing some mathematical tools from recent research areas involving Discrete Mathematics: Discrete Tomography, Enumerative Combinatorics and Combinatorics on words. The next section is devoted to definitions and results about graphs and hypergraphs that are useful for our study.

In Sect. 3 we first introduce the notion of Saind array and we restrict our investigation to 3-uniform hypergraphs and Saind sequences. We compute the related incidence matrices M_n as in [3], where, for any M_n , the array of column sums, $\pi(n)$ turns out to be the degree sequences of the corresponding 3-uniform hypergraph. We show that, for a generic $n \geq 2$, $\pi(n)$ and $\pi(n+1)$ share the same entries starting from an index on. Furthermore, increasing n , these common entries give rise to the integer sequence A002620 in [15], that we call the *Saind sequence*. Then, in Sect. 4 we analyze the combinatorial properties of the computed degree sequences and then we are able to describe the Saind sequences by introducing the notion of queue triads of given size and fixed pointer.

Then, we show their connections with other families of combinatorial structures known in the literature. Precisely, we show bijections between queue triads and integer partitions in two parts; queue triads and symmetric Dyck paths with three peaks.

2 Definitions and Previous Results

The seminal books by Berge [14] will give to the reader the formal definitions and vocabulary, some results with the related proofs, and more about applications of hypergraphs. In the following we recall the main concepts.

The notion of hypergraph generalizes that of graph, in the sense that each hyperedge is a non-void subset of the set of vertices, without constraints on its

cardinality. Formally, a *hypergraph* $\mathcal{H}$ is a pair (V, E) , where $V = \{v_1, \dots, v_n\}$ is a finite set of vertices, and $E \subset 2^V \setminus \{\emptyset\}$ is a set of hyperedges, i.e. a collection of subsets of V . A hypergraph is *simple* if none of its hyperedges is a singleton and there are no two hyperedges one included in (or equal to) another. From now on we will only consider simple hypergraphs.

The *degree* of a vertex is the number of hyperedges that contain it. The degree sequence $\pi = (d_1, d_2, \dots, d_n)$ of a simple hypergraph $\mathcal{H}$ is the sequence of the degrees of its vertices, usually arranged in non increasing order. When $\mathcal{H}$ is k -uniform (i.e. each hyperedge contains exactly k vertices) the sequence π is called k -hypergraphic.

The study of k -hypergraphic sequences started with the simplest case of $k = 2$, i.e. the case of *graphs*. A 2-graphic sequence is simply called *graphic*. Observe that a simple graph is then a graph without loops or parallel edges. The problem of characterizing graphic sequences of simple graphs was solved by Erdős and Gallai [2]:

Theorem 1. *A sequence $\pi = (d_1, d_2, \dots, d_n)$, where $d_1 \geq d_2 \geq \dots \geq d_n$ is graphic if and only if $\sum_{i=1}^n d_i$ is even and*

$$\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min\{k, d_i\}, 1 \leq k \leq n.$$

Let us denote with k -Seq the problem of deciding if an integer sequence π is a k -sequence. The problem k -Seq for $k \geq 3$ was raised by Colbourn et al. in [9] and only recently proved to be *NP*-complete by Deza et al. in [3]. The proof consists in a reduction of the *NP*-complete problem 3-Partition into 3-Seq and it is based, in an intermediate step, on the construction of a 3-uniform hypergraph $\mathcal{H}_S$ from an integer sequence S related to an instance of 3-Partition.

We provide a generalized version of this construction: let $S = (s_1, \dots, s_k)$ be an array of integers. We define a binary matrix M_S of dimension $k' \times k$ collecting all the distinct rows (arranged in lexicographical order) that satisfy the following constraint: for every index i , the i -th row of M_S has all elements equal to zero except three entries in positions j_1, j_2 and j_3 such that $s_{j_1} + s_{j_2} + s_{j_3} > 0$. The number of rows k' is bounded by $\binom{k}{3}$. For instance, the matrix M_S with $S = (5, 2, 2, -1, -4, -4)$ is

	5	2	2	-1	-4	-4
1	1	1	0	0	0	
1	1	0	1	0	0	
1	1	0	0	1	0	
1	1	0	0	0	1	
1	0	1	1	0	0	
1	0	1	0	1	0	
1	0	1	0	0	1	
0	1	1	1	0	0	
	7	5	5	3	2	2

where S is depicted in red, and π_S in blue.

The matrix M_S is **incidence matrix** of a (simple) 3-uniform hypergraph $\mathcal{H}_S = (V, E)$ such that the element $M_S(i, j) = 1$ if and only if the hyperedge $e_i \in E$ contains the vertex v_j . Let $\pi_S = (p_1, \dots, p_k)$ denote the degree sequence

of $\mathcal{H}_S$. It holds $\sum_{i=1}^{k'} M_S(i, j) = p_j$. In [3], the authors underline the remarkable property that $\mathcal{H}_S$ is the only 3-uniform hypergraph (up to isomorphism) having degree sequence π_S . Moving the spot on M_S , it is the only binary matrix having distinct rows, 3-constant row sums and π_S column sums. The following problems can be addressed:

Problem 1: determine the computational complexity of 3-Seq restricted to the class of the instances π_S .

Problem 2: provide a combinatorial characterization of the 3-sequences of 3-uniform hypergraphs which are unique up to isomorphism. Determine the computational complexity of 3-Seq restricted to that class of instances.

The present study constitutes a step ahead in the solution of Problem 1: in the next section we consider a family of non decreasing integer sequences and we establish a connection between the 3-uniform hypergraphs constructed from these sequences to several different combinatorial objects in order to find some common properties that will provide a useful starting point for their characterization and for the reconstruction of the associated hypergraphs.

3 Saind Arrays and Their Incidence Matrices

In our analysis of the k -sequences, we restrict the investigation to those π_{S_n} obtained when S_n is a *Saind arrays*, i.e. a sequence defined, for any $n \geq 2$, as $S_n = (n, n-1, n-2, \dots, 2-2n)$. For the sake of simplicity we will often refer to the elements of the array S_n as $(s_1, \dots, s_{3n-1})$, where $s_i = n - i + 1$, and to the related degree sequence π_{S_n} as $\pi(n)$. For every $n \geq 2$, according to [3] we associate to S_n its (unique) incidence matrix M_{S_n} (briefly, M_n), obtained as described in the previous section.

So for example, $S_2 = (2, 1, 0, -1, -2)$, $S_3 = (3, 2, 1, 0, -1, -2, -3, -4)$, and their incidence matrices, M_2 and M_3 , respectively, are depicted in Fig. 1.

By definition, for any $n \geq 2$, $\pi(n) = (\pi_1, \dots, \pi_{3n-1})$ is such that $\pi_1 \geq \pi_2 \geq \dots \geq \pi_{3n-1} = 1$. For small values of $n \geq 2$, the vectors $\pi(n)$ are reported below:

$$\begin{aligned}\pi(2) &= (4, 3, 2, \mathbf{2}, \mathbf{1}) \\ \pi(3) &= (12, 10, 8, 6, 5, \mathbf{4}, \mathbf{2}, \mathbf{1}) \\ \pi(4) &= (25, 21, 18, 15, 12, 10, \mathbf{9}, \mathbf{6}, \mathbf{4}, \mathbf{2}, \mathbf{1}) \\ \pi(5) &= (42, 37, 32, 28, 24, 20, 17, 15, \mathbf{12}, \mathbf{9}, \mathbf{6}, \mathbf{4}, \mathbf{2}, \mathbf{1}).\end{aligned}$$

We observe that, for every $n \geq 2$ a final sequence of elements at the end of a vector $\pi(n)$ is repeated at the end of the vector $\pi(n+1)$ (in the list above, these elements are in boldface). We refer to this array of elements as the *queue* $\tilde{Q}(n)$ of $\pi(n)$. To avoid problems with the indices we often consider the *reverse* $\tilde{Q}(n)$ of $Q(n)$, i.e. the vector obtained reading the entries of $Q(n)$ from right to left. So, we have for instance:

$$\tilde{Q}(2) = (\mathbf{1}, \mathbf{2}); \quad \tilde{Q}(3) = (\mathbf{1}, \mathbf{2}, \mathbf{4}); \quad \tilde{Q}(4) = (\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{6}, \mathbf{9}); \quad \tilde{Q}(5) = (\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{6}, \mathbf{9}, \mathbf{12}).$$

$$\begin{array}{c}
 \left[\begin{array}{ccccc}
 2 & 1 & 0 & -1 & -2 \\
 1 & 1 & 1 & 0 & 0 \\
 1 & 1 & 0 & \boxed{1} & 0 \\
 1 & 1 & 0 & 0 & \boxed{1} \\
 1 & 0 & 1 & \boxed{1} & 0
 \end{array} \right]
 \end{array}
 \quad
 \begin{array}{c}
 \left[\begin{array}{ccccccccc}
 3 & 2 & 1 & 0 & -1 & -2 & -3 & -4 \\
 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
 \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & \boxed{1} & 0 & 0 \\
 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
 1 & 0 & 1 & 0 & 0 & \boxed{1} & 0 & 0 \\
 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
 \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 & \boxed{1} & 0 & 0 \\
 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & \boxed{1} & 0 & 0 \\
 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0
 \end{array} \right]
 \end{array}$$

Fig. 1. The matrices M_2 (left) and M_3 (right). Queue triads are in boldface and the pointer of each triad has been put in box.

An element that belongs to $\tilde{Q}(n+1)$ but not to $\tilde{Q}(n)$ is said a *last entry* of $\tilde{Q}(n+1)$ (they are the boldface elements in the above list). By extension, we speak of *last entry* of $Q(n+1)$ (and of $\pi(n)$).

Remark 1. A neat inspection shows that the queue of $\pi(n)$ has two last entries if n is even, and one last entry otherwise (red elements in $\pi(n)$). The reason for this fact will be made clear in the sequel.

As n increases, the entries of $\tilde{Q}(n)$ (and of the queue of $\pi(n)$) give rise to an infinite sequence, that we call the *Saind sequence* $(w_n)_{n \geq 1}$. The first few terms of w_n are:

$$1, 2, 4, 6, 9, 12, 16, 20, 25, 30, 36, 42, 49, 56, 64, 72, 81, 90, 100, 110, 121, 132, \dots$$

In the following part of this section we prove some properties of the rows of M_n . First of all, given $n \geq 2$, the generic row r of the incidence matrix M_n associated with S_n is uniquely described by the triad of indices $t_r = (i_r, j_r, k_r)$ of the entries in r that are equal to 1. Each of these triads contributes to increase by one three entries of $\pi(n)$, precisely the entries in the position specified by the indices i , j , and k . By abuse of notation, with n fixed, we will sometimes refer to a generic triad (i, j, k) of M_n by means of the corresponding elements of the Saind array, i.e. (s_i, s_j, s_k) , where clearly, for any h , the two triples are related by $s_h = n - h + 1$.

Let us introduce further notation. With $1 \leq i < j$, we denote by $B_n(i)$ (briefly, $B(i)$) the submatrix of M_n comprising rows that have the leftmost 1 in position i , and by $B_n(i, j)$ (briefly, $B(i, j)$) the submatrix of $B(i)$ where the rows

have the second occurrence of 1 in position j . For instance, the matrix $B_6(2, 6)$ is the following

$$\begin{bmatrix} 6 & 5 & 4 & 3 & 2 & 1 & 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We observe that $B_n(n-1)$ is the bottom block of M_n , and its last row, for any n , is obtained by considering $s_1 = 2, s_2 = 0, s_3 = -1$. Therefore, $M_n = \cup_{i=1}^{n-1} B_n(i)$.

Lemma 1. *Let $n \geq 2$ and $1 \leq i < j \leq 3n - 1$. Then we have:*

1. $|B_n(i, j)| > 0$ if and only if $j < \frac{3n-i+2}{2}$;
2. For any $j < \frac{3n-i+2}{2}$, we have $|B_n(i, j)| = 3n - i - 2j + 2$;
3. $|B_n(i)| = \lfloor \frac{n-i+1}{2} \rfloor \cdot \lceil \frac{n-i+1}{2} \rceil$;
4. $|M_n| = \sum_{i=1}^{n-1} \lfloor \frac{n-i+1}{2} \rfloor \cdot \lceil \frac{n-i+1}{2} \rceil$.

Proof. 1. $B_n(i, j)$ contains at least a row if and only if $s_i + s_j + s_{j+1} > 0$. This holds if and only if $j < \frac{3n-i+2}{2}$.
 2. With i, j fixed, $B_n(i, j)$ contains all the triads of the form (i, j, k) , so that $s_i + s_j + s_k > 0$. This is satisfied by the values of $k = j+1, \dots, 3n - (i+j) + 2$. So the number of rows of $B_n(i, j)$ is $(3n - (i+j) + 2) - (j+1) + 1 = 3n - i - 2j + 2$.
 3. It is clear that for any i, n , $|B_n(i)| = |B_{n-i+1}(1)|$. So it is sufficient to compute the cardinality of $|B_m(1)|$, for a generic $m \geq 2$. So, using the formulas in 1. and 2., and omitting the computation, we obtain that:

$$|B_m(1)| = \begin{cases} 1 + 3 + 5 + \dots + 3(m-1) & \text{with } m \text{ even} \\ 2 + 4 + 6 + \dots + 3(m-1) & \text{with } m \text{ odd.} \end{cases}$$

Using standard methods, both the expressions above can be written in a more compact way, i.e.

$$|B_m(1)| = \left\lfloor \frac{m}{2} \right\rfloor \cdot \left\lceil \frac{m}{2} \right\rceil.$$

Then, we have that:

$$|M_n| = |B_n(1)| + |B_{n-1}(2)| + \dots + |B_2(n-1)| = \sum_{i=1}^{n-1} \left\lfloor \frac{n-i+1}{2} \right\rfloor \cdot \left\lceil \frac{n-i+1}{2} \right\rceil.$$

□

4 Queue Triads and the Saind Sequence

As mentioned above, for every n , there are triads of indices $(i_k, j_k, k(n))$ (briefly, (i, j, k)) that contribute to the appearance of the new entry(entries) of the queue, in correspondence with the index k , and these triads are called *queue triads* of size n and *pointer* k . These triads are depicted in bold in the matrices in Fig. 1, whereas, for any triad, the pointer is put in a box. The following gives account of the existence of the queue of the arrays $\pi(n)$ (hence of the Saind sequence).

Theorem 2. *For any given $h \geq 1$ there is a positive integer $\tilde{n}(h)$ such that, with $n \geq \tilde{n}(h)$, the h -th entry of $\tilde{\pi}(n)$ (the reverse of $\pi(n)$) is equal to h -th entry of $\tilde{\pi}(n+1)$.*

Proof. We observe that the h -th element of $\tilde{\pi}(n)$ corresponds to the entry $2 - 2n + h - 1$ in S_n . A generic triad that contributes to the value of the h -th element of the sequence $\tilde{\pi}(n)$ will contain two other nonnegative distinct integers k_1, k_2 , such that the corresponding entries in S_n are $n - k_1$ and $n - k_2$, with and $k_1 < k_2$. It holds, by construction, $1 - 2n + h > 0$, so $k_1 + k_2 < h + 1$. Hence, setting $\tilde{n}(h) = \sum_{i=1}^{h-2} p(i)$, with $p(i) = (\lceil i/2 \rceil - 1)$ being the number of pairs of different elements that sum to i (see also Lemma 1, 3.), we get the result. $\square$

Using the simple argument above we provide a characterization of queue triads of size n .

Proposition 1. *Queue triads (i, j, k) of size n can be generated as follows:*

Step 1: We determine the pointers which can be associated with n :

$$\begin{cases} k_o = 3 \cdot \frac{n+1}{2} & \text{if } n \text{ is odd;} \\ k_e = \frac{3n+2}{2} + 1, k'_e = \frac{3n+2}{2} & \text{if } n \text{ is even.} \end{cases}$$

Step 2: We calculate the values of i for the pointers determined in Step 1:

– n odd:

$$\begin{cases} 1 \leq i \leq \frac{3 \cdot n - k_o}{2} & \text{if } k_o \text{ odd;} \\ 1 \leq i \leq \frac{3 \cdot n - k_o + 1}{2} & \text{if } k_o \text{ even.} \end{cases}$$

– n even, and $k \in \{k_e, k'_e\}$: $\begin{cases} 1 \leq i \leq \frac{3 \cdot n - k + 1}{2} & k \text{ odd;} \\ 1 \leq i \leq \frac{3 \cdot n - k}{2} & k \text{ even.} \end{cases}$

Step 3: We calculate j for each of the values of i , $k \in \{k_o, k_e, k'_e\}$ obtained in Steps 1,2: $i + 1 \leq j \leq 3 \cdot n - k - (i - 2)$.

Proof. The proof is obtained using technical arguments similar to those used for Theorem 2. $\square$

Let us denote by $\mathcal{Q}_n$ the set of queue triads of size n generated using Proposition 1. As an example, we determine the queue triads for $n = 4$:

Step 1: $k'_e = 7, k_e = 8$;

Step 2: if $k_e = 8$, then $1 \leq i \leq 2$; if $k'_e = 7$, then $1 \leq i \leq 3$;

Step 3: If $k_e = 8$: with $i = 1$, $2 \leq j \leq 6$; with $i = 2$, $3 \leq j \leq 5$; with $i = 3$, $4 \leq j \leq 4$. Otherwise, if $k'_e = 7$: with $i = 1$: $2 \leq j \leq 5$; with $i = 2$: $3 \leq j \leq 4$. Therefore, the queue triads of size 4 are:

$$(1, 2, 8), (1, 3, 8), (1, 4, 8), (1, 5, 8), (2, 3, 8), (2, 4, 8)$$

with pointer 8, and

$$(1, 2, 7), (1, 3, 7), (1, 4, 7), (1, 5, 7), (1, 6, 7), (2, 3, 7), (2, 4, 7), (2, 5, 7), (3, 4, 7)$$

with pointer 7, giving $w_4 = 6$, and $w_5 = 9$.

We would like to point out that the relation between queue triads of given size and the Saind sequence can be described as follows:

1. if n is odd then the pointer k_o is unique and the number of queue triads of size n gives the term w_{3n-k_o} ;
2. if n is even, then we have two pointers: k_e and k'_e , then the number of queue triads of size n and pointer k_e (resp. k'_e) gives the term w_{3n-k_e} (resp. $w_{3n-k'_e}$).

Summarizing, the number of queue triads of size n and pointer k is the $m = (3n - k)$ -th term of the Saind sequence w_m . This clearly explains the observation in Remark 1. Moreover, using the arguments above, we can obtain a closed formula for m -th entry of the Saind sequence:

Theorem 3. *For any $m \geq 1$, we have $w_m = \lfloor \frac{m+1}{2} \rfloor \cdot \lceil \frac{m+1}{2} \rceil$.*

We point out that the n th term of the Saind sequence w_n coincides with the $(n + 1)$ th term of sequence A002620 in the On-line Encyclopedia of Integer Sequences, [15]. This sequence has several combinatorial interpretations. Therefore, rather than giving an analytical proof of Theorem 3, we will prove it bijectively, in the next section, by establishing bijections between the queue triads of a given size and pointer, and other combinatorial objects counted by sequence A002620.

5 Bijections Between Queue Triads and Other Combinatorial Objects

In this section we provide bijections between queue triads and other combinatorial objects counted by A002620, precisely: symmetric Dyck paths with 3 peaks and integer partitions in two parts. These bijections provide a combinatorial proof for the formula of Theorem 3.

Queue Triads and Integer Partitions in Two Parts. A *partition* of a positive integer n in k parts is a sequence of positive integers $(\lambda_1, \lambda_2, \dots, \lambda_k)$, such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ and $\lambda_1 + \lambda_2 + \dots + \lambda_k = n$. Let $P(i, 2)$ denote the number of integer partitions of i into 2 parts, it is known that $P(i, 2)$ is $\lfloor \frac{i}{2} \rfloor$, and $a_n = \sum_{i=2}^n P(i, 2)$. It is known that a_n is the n -th entry of sequence A002620 (see [15]).

Proposition 2. *For any $n \geq 2$, there is a bijection between queue triads of size n and pointer k and integer partitions in two parts of the integers $2, 3, \dots, k - 2$.*

Proof. Using the characterization of the pointers associated with a given n , provided in Proposition 1, we have that $k - 2$ is equal to:

- i) $3 \left(\frac{n-1}{2} \right) + 1$, if n is odd;
- ii) $3 \left(\frac{n-2}{2} \right) + 2$ or $3 \left(\frac{n-2}{2} \right) + 3$, if n is even.

We define the function f as follows: given a queue triad $t = (x, y, k)$, the corresponding integer partition $f(t) = (g, p)$ is obtained by setting $g = y - 1$ and $p = x$. The sum of the partition is then $x + y - 1$, and this value runs from 2 to $k - 2$.

Conversely, given an integer partitions of $2, 3, \dots, i$ in two parts, we can find the corresponding queue triads in the following way: we calculate the size n of the Saind array, that is the only integer element of the set $N = \left\{ \frac{2 \cdot i + 1}{3}, \frac{2 \cdot (i + 1)}{3}, \frac{2 \cdot i}{3} \right\}$. Then, we define $f^{-1}(g, p)$ as the queue triad $(p, g + 1, k)$ of S_n , where, if n is odd, then $k = 3 \cdot \frac{n+1}{2}$, else if $n = \frac{2 \cdot (i+1)}{3}$ (resp. $n = \frac{2 \cdot i}{3}$), then $k = \frac{3 \cdot n + 2}{2} + 1$ (resp. $k = \frac{3 \cdot n + 2}{2}$). $\square$

Let us see an example with $n = 3$. The pointer associated with $n = 3$ is $k = 6$, and the queue triads are $(1, 2, 6)$, $(1, 3, 6)$, $(1, 4, 6)$ and $(2, 3, 6)$. They are in correspondence with the partitions of the integers $2, 3, \dots, 3 \cdot \frac{3-1}{2} + 1 = 4$, precisely: $(1, 1)$, $(2, 1)$, $(3, 1)$, $(2, 2)$. According to Proposition 2 we have: $f(1, 2, 6) = (1, 1)$, $f(1, 3, 6) = (2, 1)$, $f(1, 4, 6) = (3, 1)$, and $f(2, 3, 6) = (2, 2)$.

Queue Triads and Symmetric Dyck Paths with Three Peaks. Recall that a Dyck path of semi-length n is a lattice path using up $U = (1, 1)$ and down $D = (1, -1)$ unit steps, running from $(0, 0)$ to $(2n, 0)$ and remaining weakly above the x -axis. Any occurrence of a UD factor in a Dyck path is called a *peak* of the path. Here, we consider Dyck paths that are *symmetric* with respect to the line which passes through the upper end of the n -th step and it is parallel to the y -axis (see Fig. 2). Deutsch showed that the number of symmetric Dyck paths with three peaks and semi-length n is given by the $(n - 1)$ th term of sequence A002620, [15].

Fig. 2. A symmetric Dyck path of length 8 and its axis of symmetry.

Proposition 3. *For any $n \geq 1$, the family of queue triads with size n and pointer k is in bijection with symmetric Dyck paths with exactly three peaks and semi-length $\ell = (3n - 1) - k + 3$.*

Proof. We observe that a symmetric Dyck path with 3 peaks of semi-length 2ℓ has the central peak lying on the symmetry line and is uniquely determined by its prefix of length ℓ , which ends with an U step. We denote this family of prefixes by $\mathcal{D}(\ell)$. The function g maps queue triads with size n and pointer k onto the paths of $\mathcal{D}(\ell)$, where $\ell = (3n - 1) - k + 3$. Precisely, the queue triad (x, y, k) with size n is mapped onto the unique path of $\mathcal{D}(\ell)$ obtained as follows: y gives

the number of steps between the vertex of the peak and the axis of symmetry that passes through the upper end of the last step of the prefix, and x gives the number of D steps between the first peak and the first valley.

Conversely, given a path $G \in \mathcal{D}(\ell)$, first we calculate the size n of the queue triad, as the only integer element of the set: $N = \left\{ \frac{2 \cdot \ell - 1}{3}, \frac{2 \cdot \ell}{3}, \frac{2 \cdot (\ell - 1)}{3} \right\}$.

Then the queue triad $g^{-1}(G) = (x, y, k)$, is obtained as follows: x is equal to the number of D steps immediately after the vertex of the first peak, y is equal to the number of steps between the vertex of the first peak and the axis of symmetry and k is equal to $k = 3 \cdot \frac{n+1}{2}$ if n is odd, or to $k = \ell + 2$ (resp. $k = \ell$) if $n = \frac{2 \cdot \ell}{3}$ (resp. $n = \frac{2 \cdot (\ell - 1)}{3}$). $\square$

For example, the queue triads of size 3 and pointer 6 (i.e. $(1, 2, 6)$, $(1, 3, 6)$, $(1, 4, 6)$, $(2, 3, 6)$) are mapped onto the paths of $\mathcal{D}(5)$. The correspondence is shown in Fig. 3.

Fig. 3. The 4 paths in $\mathcal{D}(5)$ and the corresponding queue triads of size 3 and pointer 6.

6 Future Developments

Other than the bijections we showed in the previous section, in our study we also established bijections (not presented in this paper) between queue triads of size n and other families of objects as: (a) bracelets with $n + 3$ beads, two of which are red and one of which is blue; (b) distinct genotype frequency vectors possible for a sample of n diploid individuals at a biallelic genetic locus with a specified major allele [15].

The study concerning Saind arrays led us to consider significant the investigation of other arrays having regular shapes. The following table shows some examples that we have obtained experimentally with an approach similar to that for the Saind sequence. We point out that each term of the sequence shown in the first (resp. second) row is repeated twice (resp. three times). The second column provides the sequence reference according to [15].

Array	Number sequence	First terms
$(n, n, n-1, n-1, \dots, 1-2n, 1-2n)$	A035608	1, 5, 10, 18, 27, 39, 52, 68, 85, ...
$(n, n, n, \dots, -n, -n, -n)$	A079079	3, 6, 12, 24, 42, 63, 90, 120, ...

This study seems extremely relevant both from a combinatorial point of view, since it would lead to new combinatorial interpretations of the computed sequences, and from a graph theoretical perspective, due to their close connection with hypergraphic degree sequences.

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