Automorphism Groups of Finite BL-Algebras

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Abstract. Using a category dual to finite $\mathbb{BL}$ -algebras and their homomorphisms, in this paper we characterise the structure of the automorphism group of any given finite $\mathbb{BL}$ -algebra. Further, we specialise our result to the case of the variety generated by the k -element $\mathbf{MV}$ -algebra, for each $k > 1$.

Keywords: $\mathbb{BL}$ -algebras, Automorphism group, Substitutions

1 Introduction

The variety of $\mathbb{BL}$ -algebras constitutes the algebraic semantics of Hájek's Basic Fuzzy Logic BL , which in turn is the logic of all continuous t -norms and their residua. The characterisation of the automorphism group of an algebraic structure is a typical problem in algebra. When the algebraic structure is associated with a logical system, BL in our case, then automorphisms are related with substitutions. Indeed, a substitution acting on the first n propositional letters can be conceived as an endomorphism of the free n -generated $\mathbb{BL}$ -algebra, and then automorphisms of the same algebra coincide with invertible substitutions. As any $\mathbb{BL}$ -algebra is a quotient of some free $\mathbb{BL}$ -algebra, an automorphism of a $\mathbb{BL}$ -algebra is a substitution which is invertible over the equivalence classes determined by the quotient.

In this work we use a category dually equivalent to finite $\mathbb{BL}$ -algebras with their homomorphisms, namely the category of *finite weighted forests*, to characterise the structure of the automorphism group of any given finite $\mathbb{BL}$ -algebra.

We exploit the finite combinatorial description of finite weighted forests to decompose any such automorphism group by means of direct and semidirect products of symmetric groups. Our results constitute a generalisation of [6], where the same approach has been used to characterise the automorphism groups of finite Gödel algebras, and they are related with the forthcoming paper [1], where these combinatorial techniques are applied to several subvarieties of $\mathbf{MTL}$ -algebras. The paper is structured as follows. In the first section we introduce the logic BL and $\mathbb{BL}$ -algebras; further we recall the notion of automorphism group

of an algebra, and its relations with invertible substitutions in the associated logic. Then, as an example we apply our approach based on dual categorical equivalence to characterise the automorphism groups of finite Boolean algebras. We recall also the notion of semidirect product of groups. In the third section we introduce the category of *finite weighted forests* and the dual equivalence with finite $\mathbb{BL}$ -algebras. In the fourth section we prove our main result about automorphism groups of finite $\mathbb{BL}$ -algebras. Finally, we specialise our result to the case of the variety generated by the k -element $\mathbb{MV}$ -algebra, for each $k > 1$.

Hájek's Basic Fuzzy Logic BL [15] is proven in [9] to be the logic of all continuous t -norms and their residua. We recall that a t -norm is an operator $*$: $[0, 1]^2 \rightarrow [0, 1]$ that is associative, commutative, monotonically non-decreasing in both arguments, having 0 as absorbent element, and 1 as unit. The t -norm $*$ is continuous if $*$ it is so in the standard euclidean topology of $[0, 1]^2$. BL is axiomatised as follows.

- (A1) $(\phi \rightarrow \chi) \rightarrow ((\chi \rightarrow \psi) \rightarrow (\phi \rightarrow \psi))$
- (A2) $(\phi \odot \chi) \rightarrow \phi$
- (A3) $(\phi \odot \chi) \rightarrow (\chi \odot \phi)$
- (A4) $(\phi \odot (\phi \rightarrow \chi)) \rightarrow (\chi \odot (\chi \rightarrow \phi))$
- (A5a) $(\phi \rightarrow (\chi \rightarrow \psi)) \rightarrow ((\phi \odot \chi) \rightarrow \psi)$
- (A5b) $((\phi \odot \chi) \rightarrow \psi) \rightarrow (\phi \rightarrow (\chi \rightarrow \psi))$
- (A6) $((\phi \rightarrow \chi) \rightarrow \psi) \rightarrow (((\chi \rightarrow \phi) \rightarrow \psi) \rightarrow \psi)$
- (A7) $\perp \rightarrow \phi$

with modus ponens

$$\frac{\varphi \quad \varphi \rightarrow \psi}{\psi}$$

as the only inference rule.

The equivalent algebraic semantics of BL is given by the variety of $\mathbb{BL}$ -algebras, as follows: A $\mathbb{BL}$ -algebra is an algebra $(A, \odot, \rightarrow, \perp)$ of type $(2, 2, 0)$ such that upon defining $x \wedge y = x \odot (x \rightarrow y)$ (*divisibility*), $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$, and $\top = a \rightarrow a$ for some arbitrarily fixed $a \in A$, the following holds:

- (i) $(A, \odot, \top)$ is a commutative monoid;
- (ii) $(A, \vee, \wedge, \top, \perp)$ is a bounded lattice;
- (iii) *residuation* holds, that is, $x \odot z \leq y$ if and only if $z \leq x \rightarrow y$.

Gödel logic G is obtained by extending BL via the idempotency axiom $\varphi \rightarrow (\varphi \&\varphi)$. Łukasiewicz logic L is the extension of BL via the involutiveness of negation axiom $\neg\neg\varphi \rightarrow \varphi$ (where $\neg\varphi$ stands for $\varphi \rightarrow \perp$).

The equivalent algebraic semantics of G is given by the variety of Gödel algebras $\mathbb{G}$, that is the subvariety of $\mathbb{BL}$ formed by those algebras satisfying the identity $x = x \odot x$. $\mathbb{MV}$ -algebras are those $\mathbb{BL}$ -algebras satisfying $x = (x \rightarrow \perp) \rightarrow \perp$. The variety $\mathbb{MV}$ of $\mathbb{MV}$ -algebras constitutes the equivalent algebraic semantics of L .

A variety $\mathbb{V}$ is locally finite if its n -generated free algebra $\mathbf{F}_n(\mathbb{V})$ is finite, for each $n \geq 0$.

We define $x^0 = \top$ and for all $n \geq 0$, $x^{n+1} = x^n \odot x$, and say that a $\mathbb{BL}$ -algebra is n -contractive if it satisfies the identity $x^{n+1} = x^n$. While $\mathbb{G}$ is locally finite (and 1-contractive), $\mathbb{BL}$ and $\mathbb{MV}$ are not (see [2, 13, 19] for a description of finitely generated free algebras in these varieties), but their n -contractive subvarieties are. As a matter of fact, only the n -contractive subvarieties of $\mathbb{BL}$ (or $\mathbb{MV}$) are locally finite.

Let L be a schematic extension of BL having the variety $\mathbb{L}$ as equivalent algebraic semantics. Then its Lindenbaum algebra is the $\mathbb{L}$ -algebra of the classes $\varphi/ \equiv$ of logically equivalent formulas, that is $\varphi \equiv \psi$ iff L proves $\varphi \rightarrow \psi$ and $\psi \rightarrow \varphi$. The operations are defined through the connectives: for all binary connectives $*$, $(\varphi/ \equiv) * (\psi/ \equiv) := (\varphi * \psi)/ \equiv$, the bottom element is $\perp/ \equiv$.

The Lindenbaum algebra of L is isomorphic with the free $\mathbb{L}$ -algebra over ω generators $\mathbf{F}_\omega(\mathbb{L})$. Analogously, for each $n \geq 0$, the Lindenbaum algebra of the formulas of L built using only the first n propositional letters $x_1, x_2, \dots, x_n$ is isomorphic with the free $\mathbb{L}$ -algebra over n many generators $\mathbf{F}_n(\mathbb{L})$.

A *substitution* σ over $\{x_1, \dots, x_n\}$ is displayed as

$$x_1 \mapsto \varphi_1, \dots, x_n \mapsto \varphi_n$$

for $\varphi_1, \dots, \varphi_n$ formulas built over $\{x_1, \dots, x_n\}$, with the obvious meaning that $\sigma(x_i) = \varphi_i$. The substitution σ extends naturally to each formula over $\{x_1, \dots, x_n\}$, via the inductive definition $\sigma(*(\psi_1, \dots, \psi_k)) = *(\sigma(\psi_1), \dots, \sigma(\psi_k))$, for each k -ary connective $*$ and k -tuple of formulas $(\psi_1, \dots, \psi_k)$. As it is clear that if $\varphi \equiv \psi$ then $\sigma(\varphi) \equiv \sigma(\psi)$, then the substitution σ can be identified with an endomorphism of the n -generated free algebra:

$$\sigma: \mathbf{F}_n(\mathbb{L}) \rightarrow \mathbf{F}_n(\mathbb{L}).$$

The set of all substitutions over $\{x_1, \dots, x_n\}$, equipped with functional composition, forms the *monoid of endomorphisms* $\mathbf{End}(\mathbf{F}_n(\mathbb{L}))$ of $\mathbf{F}_n(\mathbb{L})$, having the identity $id: x_i \mapsto x_i$ as neutral element. The bijective endomorphisms in $\mathbf{End}(\mathbf{F}_n(\mathbb{L}))$ are clearly the same as isomorphisms of $\mathbf{F}_n(\mathbb{L})$ onto itself, and form the *group of automorphisms* $\mathbf{Aut}(\mathbf{F}_n(\mathbb{L}))$ of $\mathbf{F}_n(\mathbb{L})$. In terms of substitutions, $\mathbf{Aut}(\mathbf{F}_n(\mathbb{L}))$ is the group of invertible substitutions over $\{x_1, \dots, x_n\}$, that is, those σ such that there exists a substitution σ^{-1} such that $\sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = id$.

In an earlier work [6] we have characterised the automorphism group of finite Gödel algebras—the algebraic semantics of propositional Gödel logic—by means of a dual categorical equivalence. In this paper we shall apply and generalise those techniques to finite $\mathbb{BL}$ -algebras (and hence, to locally finite subvarieties of $\mathbb{BL}$, when the technique is applied to their free algebras). We take here the opportunity to correct a nasty mistake in the introduction of [6]: for a quirk of carelessness, there we erroneously declared that automorphisms preserve logical equivalence, which is clearly not the case (this mistake does not invalidate any technical result in the paper). In that paper we had in mind the more algebraic

notion of equivalence given in the following Proposition, stressing the fact that automorphisms preserve all relevant algebraic information of logical equivalence between the formulas.

Proposition 1. *Let $\mathbb{L}$ be a locally finite variety constituting the algebraic semantics of a schematic extension L of BL . Then σ is an automorphism of $\mathbf{F}_n(\mathbb{L})$ if and only if for all pairs of formulas φ, ψ :*

$$\varphi \equiv \psi \quad \text{if and only if} \quad \sigma(\varphi) \equiv \sigma(\psi).$$

Proof. Clearly the property holds for automorphisms. Pick then $\sigma \in \mathbf{End}(\mathbf{F}_n(\mathbb{L})) \setminus \mathbf{Aut}(\mathbf{F}_n(\mathbb{L}))$. Since $\mathbb{L}$ is locally finite, we assume σ is not injective. Then, there are $\varphi \neq \psi$ such that $\sigma(\varphi) \equiv \sigma(\psi)$.

Clearly, every algebra in a variety comes with its automorphism group. Each algebra $\mathbf{A} \in \mathbb{L}$ is a quotient of some free algebra in $\mathbb{L}$. If $\mathbf{A} = \mathbf{F}_n(\mathbb{L})/\Theta$ for some congruence Θ of $\mathbf{F}_n(\mathbb{L})$, then we can interpret elements of $\mathbf{Aut}(\mathbf{A})$ as substitutions. As a matter of fact, $\sigma \in \mathbf{Aut}(\mathbf{A})$ if there is a substitution σ' defined over $\{x_1, \dots, x_n\}$ such that $\sigma: \varphi/\Theta \mapsto (\sigma'(\varphi))/\Theta$ is bijective (and hence invertible). Whence σ is conceived as an invertible substitution over $\{x_1/\Theta, \dots, x_n/\Theta\}$. In logical terms we have an analogous of Proposition 1. Let $\varphi \leftrightarrow \psi$ be a shortening of $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

Proposition 2. *Let $\mathbb{L}$ be a locally finite variety constituting the algebraic semantics of a schematic extension L of BL . Let $\mathbf{A} = \mathbf{F}_n(\mathbb{L})/\Theta$ for some congruence Θ . Let Γ_Θ be the theory formed by all formulas φ such that $\varphi\Theta\top$. Then σ is an automorphism of $\mathbf{A}$ if and only if for all pair of formulas φ, ψ :*

$$\Gamma_\Theta \models \varphi \leftrightarrow \psi \quad \text{if and only if} \quad \Gamma_\Theta \models \sigma(\varphi) \leftrightarrow \sigma(\psi).$$

Proof. The same proof of Proposition 1, *mutatis mutandis*.

In the paper [1], the same technique used in [6] is adapted to determine the structure of the group of automorphisms of finitely generated free algebras in several locally finite subvarieties of $\mathbf{MTL}$ -algebras. We recall that $\mathbf{MTL}$ forms a supervariety of $\mathbf{BL}$ [12], which constitutes the equivalent algebraic semantics of the logic $\mathbf{MTL}$, which in turns is the logic of all left-continuous t -norms and their residua. In [1] the algebra of automorphism invariant (equivalence classes of) formulas is introduced and studied (just for the case of Gödel logic).

2 Automorphism Groups

With each algebraic structure $\mathbf{A}$ we can associate its *monoid of endomorphisms* $\mathbf{End}(\mathbf{A}) = (\{f: \mathbf{A} \rightarrow \mathbf{A}\}, \circ, id)$, having as universe the set of all homomorphisms of $\mathbf{A}$ into itself, where $\circ$ is functional composition, and $id: a \mapsto a$ for each $a \in \mathbf{A}$ is the identity. The invertible elements of $\mathbf{End}(\mathbf{A})$, that is, those f such that there exists $f^{-1} \in \mathbf{End}(\mathbf{A})$ with the property $f \circ f^{-1} = id = f^{-1} \circ f$, constitute the universe of the *group of automorphisms* $\mathbf{Aut}(\mathbf{A})$ of $\mathbf{A}$.

Let $\mathbf{Sym}(n)$ denote the symmetric group over n elements, that is, the group of all permutations of an n -element set.

Let $\mathbb{B}$ denote the variety of Boolean algebras. We prove the well-known fact stated in Proposition 4 by means of a dual categorical equivalence, since the same approach is used in [6] and [1], and we shall use it for the case of finite $\mathbb{B}\mathbb{L}$ -algebras.

We recall that two categories $\mathbf{C}$ and $\mathbf{D}$ are dually equivalent iff there exists a pair of contravariant functors $F: \mathbf{C} \rightarrow \mathbf{D}$ and $G: \mathbf{D} \rightarrow \mathbf{C}$ whose compositions FG and GF are naturally isomorphic with the identities in $\mathbf{D}$ and $\mathbf{C}$.

Proposition 3. *The category $\mathbb{B}_{fin}$ of finite Boolean algebras and their homomorphisms is dually equivalent to the category $\mathbf{Set}_{fin}$ of finite sets and functions between them.*

Proof. This is just the restriction to finite objects of the well known Stone's duality between Boolean algebras and Stone spaces.

Let us call $\mathbf{Sub}: \mathbf{Set}_{fin} \rightarrow \mathbb{B}_{fin}$ and $\mathbf{Spec}: \mathbb{B}_{fin} \rightarrow \mathbf{Set}_{fin}$ the functors implementing the equivalence. It is folklore that $\mathbf{Sub} S$ is the Boolean algebra of the subsets of S , and $\mathbf{Spec} \mathbf{A}$ is the set of maximal filters of $\mathbf{A}$. On arrows, $\mathbf{Sub}$ and $\mathbf{Spec}$ are defined by taking preimages.

Clearly, for each Boolean algebra $\mathbf{A}$, $\mathbf{Aut}(\mathbf{A}) \cong \mathbf{Aut}(\mathbf{Spec} \mathbf{A})$.

Proposition 4. $\mathbf{Aut}(\mathbf{F}_n(\mathbf{B})) \cong \mathbf{Sym}(2^n)$.

Proof. Just recall that $\mathbf{Spec} \mathbf{F}_n(\mathbf{B})$ is the set of 2^n elements, and an automorphism of a finite set is just a permutation of its elements.

To deal with the structure of the automorphism groups of finite $\mathbb{B}\mathbb{L}$ -algebras we shall introduce some constructions from group theory. We refer to [20] for background.

Definition 1. *Given two groups $\mathbf{H}$ and $\mathbf{K}$ and a group homomorphism $f: k \in \mathbf{K} \mapsto f_k \in \mathbf{Aut}(\mathbf{H})$, the semidirect product $\mathbf{H} \rtimes_f \mathbf{K}$ is the group obtained equipping $H \times K$ with the operation $(h, k) * (h', k') = (hf_k(h'), kk')$.*

Theorem 1. *Let $\mathbf{G}$ be a group with identity e and let $\mathbf{H}, \mathbf{K}$ be two subgroups of $\mathbf{G}$. If the following hold:*

- $\mathbf{K} \triangleleft \mathbf{G}$ ($\mathbf{K}$ is a normal subgroup of $\mathbf{G}$);
- $G = H \times K$;
- $H \cap K = \{e\}$,

then $\mathbf{G}$ is isomorphic to the semidirect product of $\mathbf{H}$ and $\mathbf{K}$ with respect to the homomorphism $f: k \in \mathbf{K} \rightarrow f_k \in \mathbf{Aut}(\mathbf{H})$ where for each $h \in \mathbf{H}$, $f_k(h) = khk^{-1}$. Hence $|\mathbf{G}| = |\mathbf{H}| \cdot |\mathbf{K}|$.

In the following, we shall simply write $\mathbf{H} \rtimes \mathbf{K}$ instead of $\mathbf{H} \rtimes_f \mathbf{K}$, as in any usage we assume f is as in Theorem 1.

3 Finite BL-Algebras and Finite Weighted Forests

In this Section we shall present the relevant facts for our purposes about the dual categorical equivalence between finite BL-algebras (with homomorphisms) and the category of *finite weighted forests*. The duality is introduced in [4].

A *forest* is a poset such that the collection of lower bounds $\downarrow x = \{y \mid y \leq x\}$ of any given element x is totally ordered. A morphism of forests $f: F \rightarrow G$ is an order-preserving map that is *open*, that is whenever $z \leq f(x)$ for $z \in G$ and $x \in F$, then there is $y \leq x$ in F such that $f(y) = z$.

Finite forests and their morphisms form a category, denoted $\mathbf{F}_{fin}$.

Let $\mathbb{N}^+$ denote the set of positive natural numbers. A *weighted forest* is a function $w: F \rightarrow \mathbb{N}^+$, where F is a forest, called the *underlying forest* of w . A weighted forest is *finite* if its underlying forest is. Consider two finite weighted forests $w: F \rightarrow \mathbb{N}^+$, $w': F' \rightarrow \mathbb{N}^+$. By a *morphism* $g: w \rightarrow w'$ we mean an order-preserving map $g: F \rightarrow F'$ that is:

- (M1) *open* (or is a *p-morphism*), that is, whenever $x' \leq g(x)$ for $x' \in F'$ and $x \in F$, then there is $y \leq x$ in F such that $g(y) = x'$, and
- (M2) *respects weights*, meaning that for each $x \in F$, there exists $y \leq x$ in F such that $g(y) = g(x)$ and $w'(g(y))$ divides $w(y)$.

Contemplation of these definitions shows that finite weighted forests and their morphisms form a category. Let us write $\mathbf{WF}_{fin}$ for the latter category, and $\mathbf{BL}_{fin}$ for the category of finite BL-algebras and their homomorphisms.

In the following we shall often manipulate the underlying forest F of a finite weighted forest $w: F \rightarrow \mathbb{N}^+$. Henceforth, in order to simplify exposition, we shall write (F, w) for such a finite weighted forest.

Theorem 2. *The category of finite BL-algebras and their homomorphisms is dually equivalent to the category of weighted forests and their morphisms. That is, there are functors*

$$\mathbf{wSpec}: \mathbf{BL}_{fin} \rightarrow \mathbf{WF}_{fin} \quad \text{and} \quad \mathbf{Sub}: \mathbf{WF}_{fin} \rightarrow \mathbf{BL}_{fin}$$

such that the composite functors $\mathbf{wSpec} \circ \mathbf{Sub}$ and $\mathbf{Sub} \circ \mathbf{wSpec}$ are naturally isomorphic to the identity functors on $\mathbf{WF}_{fin}$ and $\mathbf{BL}_{fin}$, respectively.

Proof. See [4] for the definition of the functors $\mathbf{wSpec}$ and $\mathbf{Sub}$.

In particular, by [18, Thm. IV.4.1] the functor $\mathbf{wSpec}$ is essentially surjective, and this yields the following representation theorem for finite BL-algebras.

Corollary 1. *Any finite BL-algebra is isomorphic to $\mathbf{Sub}(F, w)$ for a weighted forest (F, w) that is unique to within an isomorphism of weighted forests.*

While the previous corollary has been already proved in [11, Sect. 5] and, as a special case of a more general construction, in [17, Sect. 6], the finite duality theorem is first introduced in [4].

4 The Automorphism Group of a Finite Weighted Forest

In this Section we shall study the automorphism group of a finite weighted forest (F, w) since by Theorem 2 and Corollary 1 this group is the same as the automorphism group of the dual finite $\mathbb{BL}$ -algebra $\text{Sub}(F, w)$.

An *order-preserving permutation* of F is a bijective map $\pi: F \rightarrow F$ such that if $x \leq y$ then $\pi(x) \leq \pi(y)$.

Theorem 3. *The automorphism group $\mathbf{Aut}(F)$ of a finite forest F is isomorphic with the group of order-preserving permutations of F .*

Proof. It is clear that $\pi: F \rightarrow F$ is an isomorphism iff π is a bijective morphism in $\mathbf{F}_{fin}$. Since morphisms of finite forests do preserve order, then if $x \leq y$ it must be $\pi(x) \leq \pi(y)$.

Let (F, w) be a weighted forest. For each $x \in F$ we denote $\uparrow x = \{y \in F \mid x \leq y\}$ the *upset* of x and $\downarrow x = \{y \in F \mid y \leq x\}$ the *downset* of x .

If x is not minimal in F we denote by $p(x)$ its unique predecessor.

For any $x \in F$, let $h(x) = |\downarrow x|$ be the *height* of x . Let $h(F) = \max\{h(x) \mid x \in F\}$.

Let $f: F \rightarrow F$ be an order-preserving permutation of F . Then we say that f *preserves the weights* of the map $w: F \rightarrow \mathbb{N}^+$ when $w(f(x)) = w(x)$ for every $x \in F$.

Proposition 5. *An automorphism of a weighted forest (F, w) is an order-preserving bijection f from F to F preserving the weights of w .*

Proof. An automorphism $f: (F, W) \rightarrow (F, W)$ is a bijective morphism in $\mathbf{WF}_{fin}$. Perusal of the definitions of morphism in $\mathbf{WF}_{fin}$ and in $\mathbf{F}_{fin}$, shows that f is a map $f: F \rightarrow F$ which is a bijective morphism in $\mathbf{F}_{fin}$, that is f is an order-preserving permutation of F , by Theorem 3. As $f: F \rightarrow F$ is invertible, condition (M2) holds iff $w(f(x)) = w(x)$ for any $x \in F$.

We denote by $\mathbf{Aut}(F, w)$ the group of all automorphisms of (F, w) .

For every $i = 1, \dots, h(F) - 1$, we consider the partition of F given by its *levels*:

$$A_i = \{x \in F \mid h(x) = i\}$$

and the relation $\mathcal{R}$ on F such that $x\mathcal{R}y$ if and only if $\uparrow x \cong \uparrow y$ as finite weighted forests, $w(x) = w(y)$ and either x and y are both minimal or $p(x) = p(y)$.

The relation $\mathcal{R}$ is an equivalence relation and we denote by $[x]$ its equivalence classes. Note that for every $x \in F$, if $x \in A_i$ then $[x] \subseteq A_i$ and the set $[A_i] = \{[x] \mid x \in A_i\}$ is a partition of A_i . Further, if $x \neq y$ and $x\mathcal{R}y$ then for every $x_1 \in \uparrow x$ there is no $y_1 \in \uparrow y$ such that $x_1\mathcal{R}y_1$.

If $\varphi \in \mathbf{Aut}(F, w)$ and $A \subseteq F$, by $\varphi \upharpoonright A$ we mean the restriction of φ to A .

Lemma 1. *The following hold:*

- (i) *For every $\varphi \in \mathbf{Aut}(F, w)$ and $x \in F$, $\varphi([x]) = [\varphi(x)]$;*

(ii) if $\varphi, \psi \in \mathbf{Aut}(F, w)$ are such that $\varphi(p(x)) = \psi(p(x))$, then $\varphi([x]) = \psi([x])$.

In the following, we fix a strict order relation $\prec$ on F such that $\prec$ is total on each equivalence class and for any $x, y \in F$, x, y are incomparable if $[x] \neq [y]$. Hence $\prec$ is an order relation that compares just elements in the same class (hence at the same level, with the same weight and with an isomorphic upset).

Definition 2. Let (F, w) be a weighted forest with $h(F) = h$. Then for $i = 1, \dots, h$, an i -permutation respecting $\prec$ on (F, w) is an order-preserving bijection $\pi : F \rightarrow F$ also preserving the weight w and such that:

- (i) For every $j < i$ and $x \in A_j$, $\pi(x) = x$;
- (ii) For every $k > i$ and $x, y \in A_k$, if $x \prec y$ then $\pi(x) \prec \pi(y)$;

Note that this definition is well given since if $x \prec y$ then $[x] = [y]$ and, being $\pi \in \mathbf{Aut}(F, w)$, by Lemma 1 we have $[\pi(x)] = [\pi(y)]$. If π is an i -permutation and $x \in A_i$ (with $i > 1$) then $\pi(p(x)) = p(x)$ hence $\pi(x) \in [x]$ and $\pi([x]) = [x]$.

By definition, i -permutations fix everything below the level i and permute (in such a way to respect order and weights) elements in the levels above i by keeping the same order $\prec$ in the classes. So, the order $\prec$ is needed to fix a canonical i -permutation π once we have defined π on A_i , as explained in the following lemma.

Lemma 2. Let $\varphi \in \mathbf{Aut}(F, w)$ and $i \in \{1, \dots, h(F)\}$. If $\varphi([x]) = [x]$ for every $x \in A_i$ then there is a unique i -permutation $(\varphi)_i$ respecting $\prec$ such that

$$(\varphi)_i \upharpoonright A_i = \varphi \upharpoonright A_i.$$

Proof. For every $j < i$ and $x \in A_j$, set $(\varphi)_i(x) = x$ and for every $x \in A_i$ set $(\varphi)_i(x) = \varphi(x)$. Since $\varphi([x]) = [x]$, then $p(x) = p(\varphi(x))$ and for every $x \in A_i$,

$$\downarrow (\varphi)_i(x) = \varphi(x) \cup \{y \mid y \leq x\} = (\varphi)_i(\downarrow x).$$

Then $(\varphi)_i$ is an order-preserving permutation of $\downarrow A_i$. Further, for every $x \in A_i$ and $y \in (\varphi)_i([x])$, $\uparrow x \cong \uparrow y$.

Let $x \in A_{i+1}$. Then φ bijectively maps $[x]$ onto $\varphi([x])$. Display $[x]$ as $\{x_1 \prec \dots \prec x_s\}$ and $\varphi([x])$ as $\{y_1 \prec \dots \prec y_s\}$. Then we set $(\varphi)_i(x_j) = y_j$ for $j = 1, \dots, s$. Note that $(\varphi)_i$ is an order-preserving permutation of $\downarrow A_{i+1}$ and $\uparrow x \cong \uparrow y$ for every $y \in (\varphi)_i([x])$.

In general, let $k > i$ and suppose to have extended $(\varphi)_i$ to an order-preserving permutation of $\downarrow A_{k-1}$ such that $\uparrow x \cong \uparrow y$ for every $y \in (\varphi)_i([x])$. Let $x \in A_k$. Since $p(x) \in A_{k-1}$ then $\uparrow p(x) \cong \uparrow (\varphi)_i(p(x))$. Then there exists $y \in \uparrow (\varphi)_i(p(x)) \cap A_k$ such that $\uparrow [x] \cong \uparrow [y]$. We can hence display $[x]$ as $\{x_1 \prec \dots \prec x_s\}$ and $[y]$ as $\{y_1 \prec \dots \prec y_s\}$. Then we set $(\varphi)_i(x_j) = y_j$ for $j = 1, \dots, s$. Clearly, $(\varphi)_i$ is an order-preserving permutation of A_k and $\uparrow x \cong \uparrow y$ for every $y \in (\varphi)_i([x])$.

We can hence define $(\varphi)_i$ on the whole F and it follows from the construction that $(\varphi)_i$ is the unique i -permutation respecting $\prec$ and coinciding with φ on A_i .

Fig. 1. Weighted forest (F, w) of Example 1

Example 1. Let (F, w) be as in Fig. 1, where each node is labeled with a letter and with its weight.

We have $h(F) = 3$ and levels are given by

$$\begin{aligned} A_1 &= \{a, n\} \\ A_2 &= \{b, c, d, o, p, q, r\} \\ A_3 &= \{e, f, g, h, i, j, k, l, m\}. \end{aligned}$$

Non-singleton classes are given by

$$\begin{aligned} [b] &= [c] = \{b, c\} \subseteq A_2 \\ [e] &= [f] = \{e, f\} \subseteq A_3 \\ [h] &= [i] = \{h, i\} \subseteq A_3 \\ [k] &= [l] = \{k, l\} \subseteq A_3 \\ [o] &= [p] = [q] = \{o, p, q\} \subseteq A_2 \end{aligned}$$

while $[x] = \{x\}$ for every $x \in \{a, d, g, j, m, n, r\}$. Let us fix the order $b \prec c$, $e \prec f$, $h \prec i$, $k \prec l$ and $o \prec p \prec q$. The map φ_{21} defined by

$$\frac{x}{\varphi_{21}(x)} \left| \begin{array}{cccccccccccccccc} a & b & c & d & e & f & g & h & i & j & k & l & m & n & o & p & q & r \\ a & c & b & d & h & i & j & e & f & g & k & l & m & n & o & p & q & r \end{array} \right|$$

maps b to c and viceversa and it is defined in accordance with $\prec$ on A_3 , so it is a 2-permutation. Also φ_{22} defined by

$$\frac{x}{\varphi_{22}(x)} \left| \begin{array}{cccccccccccccccc} a & b & c & d & e & f & g & h & i & j & k & l & m & n & o & p & q & r \\ a & b & c & d & e & f & g & h & i & j & k & l & m & n & p & q & o & r \end{array} \right|$$

is a 2-permutation since it fixes all elements but $\{o, p, q\}$. The composition $\varphi_2 = \varphi_{21}\varphi_{22} = \varphi_{22}\varphi_{21}$ is again a 2-permutation. Consider the automorphism φ defined by

$$\frac{x}{\varphi(x)} \left| \begin{array}{cccccccccccccccc} a & b & c & d & e & f & g & h & i & j & k & l & m & n & o & p & q & r \\ a & c & b & d & i & h & j & f & e & g & k & l & m & n & p & q & o & r \end{array} \right|$$

Note that φ is not a 2-permutation since $e \prec f$ but $\varphi(e) = i \not\prec \varphi(f) = h$. Nevertheless $\varphi \upharpoonright A_2 = \varphi_2 \upharpoonright A_2$. Note also that $\varphi \upharpoonright A_3 \neq \varphi_2 \upharpoonright A_3$ and we cannot apply Lemma 2 to level 3 since $\varphi([e]) \neq [e]$.

In the following, we shall omit the reference to the order $\prec$, and it will be tacitly assumed that all i -permutations are i -permutations respecting $\prec$.

The set P_i of all i -permutations is the domain of a subgroup $\mathbf{P}_i$ of $\mathbf{Aut}(F, w)$. We are going to describe it in terms of the symmetric groups $\mathbf{Sym}(n)$.

Definition 3. Given $x \in A_i$, an $[x]$ -permutation is an i -permutation π such that $\pi \upharpoonright [x]$ is a permutation of $[x]$ and $\pi(y) = y$ for every $y \in A_i \setminus [x]$.

Lemma 3. Every i -permutation respecting $\prec$ is the composition of $[x]$ -permutations for $[x] \in [A_i]$. Further,

$$\mathbf{P}_i \cong \prod_{[x] \in [A_i]} \mathbf{Sym}([x]).$$

Proof. Note that, since $[A_i] = \{[x] \mid x \in A_i\}$ is a partition of A_i , if π and π' are respectively an $[x]$ -permutation and a $[y]$ -permutation with $[x] \neq [y]$, then $\pi\pi' = \pi'\pi$. The claim then follows by the definition of i -permutation and by Lemma 2, noticing that for every i -permutation φ and $x \in A_i$, $\varphi([x]) = [x]$.

Example 2. Consider the weighted forest (F, w) from Example 1. We have

$$\begin{aligned} [A_1] &= \{\{a\}, \{n\}\} \\ [A_2] &= \{\{b, c\}, \{d\}, \{o, p, q\}, \{r\}\} \\ [A_3] &= \{\{e, f\}, \{h, i\}, \{k, l\}, \{g\}, \{j\}, \{m\}\}. \end{aligned}$$

If $x_1, \dots, x_u \in A_i$, we use the notation $(x_1 x_2 \dots x_u)$ to denote the unique i -permutation (in $\mathbf{Aut}(F, w)$) that maps $x_1 \mapsto x_2 \mapsto \dots \mapsto x_u \mapsto x_1$ and fixes all the other elements of $\downarrow A_i$. Then we get

$$\mathbf{P}_3 = \{\text{id}_F, (ef), (hi), (kl), (ef)(hi), (ef)(kl), (hi)(kl), (ef)(hi)(kl)\} \cong \mathbf{Sym}^3(2)$$

while

$$\begin{aligned} \mathbf{P}_2 &= \{\text{id}_F, (bc), (op), (oq), (pq), (opq), (oqp), (bc)(op), (bc)(oq), \\ &\quad (bc)(pq), (bc)(opq), (bc)(oqp)\} \cong \mathbf{Sym}(2) \times \mathbf{Sym}(3) \end{aligned}$$

and

$$\mathbf{P}_1 = \{\text{id}_F\} \cong \mathbf{Sym}(1).$$

Let $\mathbf{N}_0 = \mathbf{Aut}(F, w)$ and for any $i = 1, \dots, h(F)$,

$$\mathbf{N}_i = \{\varphi \in \mathbf{Aut}(F, w) \mid \varphi \upharpoonright (\downarrow A_i) = \text{id}_{\downarrow A_i}\}.$$

Theorem 4. Let $h = h(F)$. Then $\mathbf{N}_h = \{\text{id}_F\}$, $\mathbf{N}_{h-1} = \mathbf{P}_h$ and for any $i = 0, \dots, h-2$

$$\mathbf{N}_i \cong \mathbf{P}_{i+1} \rtimes \mathbf{N}_{i+1},$$

Proof. By definition, $\mathbf{N}_h = \{id_F\}$ and $\mathbf{N}_{h-1}$ is the set of order-preserving permutations that are distinct from the identity only on the level A_h . Since such a map π must satisfy $\pi([x]) = [x]$ for every $x \in A_h$, we have $\mathbf{N}_{h-1} = \mathbf{P}_h$.

It is easy to check that for every $i = 0, \dots, h-2$, $\mathbf{P}_{i+1}$ and $\mathbf{N}_{i+1}$ are subgroups of $\mathbf{N}_i$. We first prove that $N_i = P_{i+1} \times N_{i+1}$ (as sets), that is for every element $\varphi \in \mathbf{N}_i$ there exist $\pi \in \mathbf{P}_{i+1}$ and $\psi \in \mathbf{N}_{i+1}$ such that $\varphi = \pi\psi$. Since for every $x \in A_{i+1}$, $\varphi([x]) = [x]$, then we can apply Lemma 2 and find $\pi = (\varphi)_{i+1} \in \mathbf{P}_{i+1}$ coinciding with φ over A_{i+1} . Let $\psi = (\varphi)_{i+1}^{-1}\varphi$. For every $x \in A_{i+1}$, $\varphi(x) \in A_{i+1}$ hence $\psi(x) = (\varphi)_i^{-1}(\varphi(x)) = \varphi^{-1}\varphi(x) = x$. For $x \in (\downarrow A_i)$, $\psi(x) = id_{A_i}id_{A_i}(x) = x$, hence $\psi \in N_{i+1}$. Clearly, $\varphi = \pi\psi$.

In order to prove that $\mathbf{N}_{i+1} \triangleleft \mathbf{N}_i$, let $\varphi \in \mathbf{N}_i$ and $\psi \in \mathbf{N}_{i+1}$. Then φ and ψ are both the identity function when restricted to $\downarrow A_i$ and further for every $x \in A_{i+1}$, $\varphi(\psi(\varphi^{-1}))(x) = \varphi\varphi^{-1}(x) = x$. Then $\varphi\psi\varphi^{-1}$ is the identity over $\downarrow A_{i+1}$ hence $\varphi\psi\varphi^{-1} \in N_{i+1}$.

Finally, we have to prove that $\mathbf{P}_{i+1} \cap \mathbf{N}_{i+1} = \{id_F\}$, but this is trivial since the only element in $\mathbf{P}_{i+1}$ coinciding with the identity over A_{i+1} is id_F .

The claim follows by Theorem 1.

Note that, by Theorem 1, the operation of the group $\mathbf{P}_{i+1} \rtimes \mathbf{N}_{i+1}$ is given by $(\pi, \psi) * (\pi', \psi') = (\pi\psi\pi'\psi^{-1}, \psi\psi')$.

Example 3. Consider the weighted forest (F, w) from Example 1 and the maps φ and φ_2 . By definition, $\mathbf{N}_1 = \mathbf{Aut}(F, w) = \mathbf{N}_0$ since all the maps in $\mathbf{Aut}(F, w)$ are equal to the identity over the roots a and n . Further $\mathbf{N}_2 = \mathbf{P}_3$ and $\mathbf{N}_3 = \{id_F\}$. By Theorem 4 we have $\mathbf{N}_1 \cong \mathbf{P}_2 \rtimes \mathbf{N}_2$. Consider for example the maps $\varphi \in \mathbf{Aut}(F, w)$ and $\varphi_2 \in \mathbf{P}_2$ from Example 1. Then $\varphi_2 = (bc)(opq) \in \mathbf{P}_2$ and $\varphi = \varphi_2 \circ (ef)(hi)(kl)$ where $(ef)(hi)(kl) \in \mathbf{P}_3 = \mathbf{N}_2$.

We are ready to state the structure of the automorphism group of a weighted forest (F, w) .

Corollary 2. *Let (F, w) be a finite weighted forest. Let $h(F) = h$ and let $\mathbf{P}_i$ be the group of i -permutations of (F, w) . Then*

$$\mathbf{Aut}(F, w) \cong \mathbf{P}_1 \rtimes (\mathbf{P}_2 \rtimes (\cdots \rtimes \mathbf{P}_h))$$

and

$$|\mathbf{Aut}(F, w)| = \prod_{i=1}^h \prod_{[x] \in [A_i]} |[x]|!$$

Proof. The claim follows by Theorem 4 and Lemma 3.

Example 4. Let (F, w) be the weighted forest from Example 1. Then

$$\begin{aligned} \mathbf{Aut}(F, w) &\cong \mathbf{Sym}(1) \rtimes ((\mathbf{Sym}(2) \times \mathbf{Sym}(3)) \rtimes (\mathbf{Sym}^3(2))) \\ &\cong (\mathbf{Sym}(2) \times \mathbf{Sym}(3)) \rtimes (\mathbf{Sym}^3(2)) \end{aligned}$$

and

$$|\mathbf{Aut}(F, w)| = \prod_{i=1}^3 \prod_{[x] \in [A_i]} |[x]|! = 2 \cdot 3! \cdot 2 \cdot 2 \cdot 2 = 96.$$

We finally get,

Theorem 5. *Let $\mathbf{A}$ be a finite $\mathbb{BL}$ -algebra. Let $(F, w) = \text{wSpec } \mathbf{A}$, with $h(F) = h$. Let $\mathbf{P}_i$ be the group of i -permutations of (F, w) . Then*

$$\text{Aut}(\mathbf{A}) \cong \mathbf{P}_1 \rtimes (\mathbf{P}_2 \rtimes (\cdots \rtimes \mathbf{P}_h))$$

and

$$|\text{Aut}(\mathbf{A})| = \prod_{i=1}^h \prod_{[x] \in [A_i]} |[x]|!$$

Proof. Immediate, from Corollary 2, and the dual equivalence between WF_{fin} and $\mathbb{BL}_{fin}$.

5 Finite MV-Algebras and Finite Multisets of Natural Numbers

The following result is well-known [3, 10, 16].

Theorem 6. *The category of finite forests and order-preserving open maps is dually equivalent to the category of finite Gödel algebras and their homomorphisms.*

Notice that a finite forest F can be readily considered as a weighted forest $w: F \rightarrow \mathbb{N}^+$ such that w is a constant function. The most natural choice is $w(x) = 1$ for each $x \in F$, as the actual dual in $\mathbb{BL}_{fin}$ of a finite $\mathbb{BL}$ -algebra that happens to be a Gödel algebra is a weighted forest where all nodes have weight 1. Whence, the automorphism group of a finite Gödel algebra is the same as the automorphism group of a weighted forest $w: F \rightarrow \mathbb{N}^+$, such that w is constant. The characterisation of these groups is precisely the subject of [6].

Another interesting case is when the weighted forest $w: F \rightarrow \mathbb{N}^+$ is such that all elements of F are uncomparable, that is F can be conceived as a set. Then (F, w) can be thought of as a multiset of positive natural numbers. This is precisely the case when (F, w) is the dual of a finite MV -algebra.

The variety MV of MV -algebras constitutes the algebraic semantics of propositional Łukasiewicz logic [8]. MV is not locally finite, but the k -contractive MV -algebras form a locally finite subvariety of MV . Here we consider the subvariety MV_k generated by the MV -chain with k elements, which constitutes the algebraic semantics of k -valued Łukasiewicz logic. MV_k is axiomatised by imposing $(k-1)$ -contractivity: $x^k = x^{k-1}$, and Grigolia's axioms [14] $k(x^h) = (h(x^{h-1}))^k$ for every integer $2 \leq h \leq k-2$ that does not divide $k-1$.

For any integer $d > 1$ let $\text{Div}(d)$ be the set of coatoms in the lattice of divisors of d , and for any finite set of natural numbers X , let $\text{gcd}(X)$ be the greatest common divisor of the numbers in X . Then let $\alpha(0, 1) = 1$, $\alpha(0, d) = 0$ for all $d > 1$, and for all $n \geq 1$,

$$\alpha(n, d) = (d+1)^n + \sum_{\emptyset \neq X \subseteq \text{Div}(d)} (-1)^{|X|} (\text{gcd}(X) + 1)^n.$$

Then $\alpha(n, d)$ counts the number of points in $[0, 1]^n$ whose denominator is d . It is known that

$$\mathbf{F}_n(\mathbb{MV}_k) \cong \prod_{d|(k-1)} \mathbf{L}_{d+1}^{\alpha(n, d)},$$

where $\mathbf{L}_m$ is the $\mathbb{MV}$ -chain of cardinality m . Let $\mathbf{MN}_{kfin}$ be the category whose objects are finite multisets of natural numbers dividing $k-1$ and whose arrows $f: M \rightarrow N$ are functions from M to N such that $f(x)$ divides x for any $x \in M$. Then $\mathbf{MN}_{kfin}$ is dually equivalent to $\mathbb{MV}_k$. In particular, denoted $\mathbf{Spec}: \mathbb{MV}_k \rightarrow \mathbf{MN}_{kfin}$ one of the pair of functors implementing the duality,

$$\mathbf{Spec} \mathbf{F}_n(\mathbb{MV}_k) \cong \bigcup_{d|(k-1)} \biguplus_{i=1}^{\alpha(n, d)} \{d\},$$

where $\biguplus_{i=1}^m \{t\}$ denotes the multiset formed by m copies of t .

It is clear that an automorphism $f: M \rightarrow M$ in $\mathbf{MN}_{kfin}$ must be a bijection such that each copy of $x \in M$ is mapped to a copy of $x \in M$. Then

Theorem 7.

$$\mathbf{Aut}(\mathbf{F}_n(\mathbb{MV}_k)) \cong \prod_{d|(k-1)} \mathbf{Sym}(\alpha(n, d)),$$

and

$$|\mathbf{Aut}(\mathbf{F}_n(\mathbb{MV}_k))| = \prod_{d|(k-1)} (\alpha(n, d))!.$$

6 Conclusion

One can apply Theorem 5 to determine the automorphism group of all finitely generated free algebras in any given locally finite subvariety of $\mathbb{BL}$. Indeed, let $\mathbb{V}$ be such a subvariety. Then, by universal algebra, and by the dual equivalence between $\mathbb{BL}_{fin}$ and $\mathbf{WF}_{fin}$, it holds that $\mathbf{wSpec}(\mathbf{F}_n(\mathbb{V})) \cong (\mathbf{wSpec}(\mathbf{F}_1(\mathbb{V})))^n$, where the power is computed using the recurrences for computing the products of finite forests (notice that in $\mathbf{WF}_{fin}$ and in $\mathbf{F}_{fin}$ the underlying set of the product is not the cartesian product of the underlying sets of the factors), see [4] and [5] for details. Once $\mathbf{wSpec}(\mathbf{F}_n(\mathbb{V}))$ is computed, Theorem 5 yields its automorphism group, which coincides with the automorphism group of $\mathbf{F}_n(\mathbb{V})$. Studying the structure of the automorphism group in subvarieties of $\mathbb{BL}$ that are not locally finite is much harder, as the combinatorial approach alone is not sufficient. The paper [7] provides a characterisation of the automorphism groups for a class of $\mathbb{MV}$ -algebras containing members of infinite cardinality.

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