Orders Preserving Convexity Under Intersections for Interval-Valued Fuzzy Sets

Pedro Huidobro¹, Pedro Alonso² , Vladimir Janiš³ ,
and Susana Montes¹

¹ Department of Statistics and O.R., University of Oviedo, Oviedo, Spain
{huidobropedro,montes}@uniovi.es

² Department of Mathematics, University of Oviedo, Oviedo, Spain
palonso@uniovi.es

³ Department of Mathematics, Matej Bel University, Banská Bystrica, Slovakia
vladimir.janis@umb.sk

Abstract. Convexity is a very important property in many areas and the studies of this property are frequent. In this paper, we have extended the notion of convexity for interval-valued fuzzy sets based on different order between intervals. The considered orders are related and their behavior analyzed. In particular, we study the preservation of the convexity under intersections, where again the chosen order is essential. After this study, we can conclude the appropriate behavior of the admissible orders for this purpose.

Keywords: Interval-valued fuzzy sets · Order between intervals · Intersection · Convexity

1 Introduction

Convexity is a basic mathematical concept that has been used as a tool in many different problems. It has important applications in many areas, like optimization [15], image processing [22], robotics [14] or geometry [13].

In real problems, the information we have to deal with is, in most of the cases, approximate. By this reason, the study of the convexity of a fuzzy set has been a very studied topic (see, for instance, Ammar and Metz [1], Diaz et al. [7], Ramik and Vlach [16], Sarkar [18], Syau and Lee [21] and Yang [25]).

Taking into account several real world problems, several extensions of the fuzzy sets have been introduced and studied in the last years. In particular, we

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are interested in interval-valued fuzzy sets. They were introduced independently by Zadeh [26], Grattan-Guinness [10], Jahn [12], Sambuc [17] in the seventies. From then, several concepts related to this extension have to be studied. Taking into account the previous comments, we are especially interested in the concept of a convex interval-valued fuzzy set. Since convexity is based on an order over the membership degrees and how the membership values are not numbers but intervals, we will obtain a different definition of convexity for each interval order on the set of intervals. The main aim of this paper is to introduce this general definition and study its dependence on the interval order considered. In particular, we are going to study in deep the preservation of the convexity under intersections, since it is a necessary property in many applications, as optimization (see [15]).

This paper is organized as follows. In Sect. 2, some basic concepts are introduced and the notation is fixed. Section 3 is devoted to the study of the different definitions we can consider for the intersection of two interval-valued fuzzy sets depending on the chosen order. In Sect. 4 we propose a definition of convexity for interval-valued fuzzy sets and we study the cases when the intersection of two convex sets remains convex. Finally, some conclusions and open problems are drawn in Sect. 5.

2 Basic Concepts

Let X denote the universe of discourse. An interval-valued fuzzy subset of X is a mapping $A : X \rightarrow L([0, 1])$ such that $A(x) = [\underline{A}(x), \overline{A}(x)]$, where $L([0, 1])$ denotes the family of closed intervals included in the unit interval $[0, 1]$. Thus, an interval-valued fuzzy set A is totally characterized by two mapping, $\underline{A}$ and $\overline{A}$, from X into $[0, 1]$ such that $\underline{A}(x) \leq \overline{A}(x), \forall x \in X$. These maps represent the lower and upper bound of the corresponding intervals. Let us notice that if $\underline{A}(x) = \overline{A}(x), \forall x \in X$, then A is a classical fuzzy sets. The collection of all the interval-valued fuzzy sets in X is denoted by $IVFS(X)$ and the subset formed by all the fuzzy sets in X is denoted by $FS(X)$.

For any pair of IVFS, it is usually considered that A is a subset of B if, and only if, $A(x)$ is lower than or equal to $B(x)$ for any $x \in X$. This definition is clear when we are dealing with fuzzy sets, since the usual order $\leq$ for the real number is used to define the inclusion. However, there is not a usual total order in $L([0, 1])$ and so, several definitions of inclusion could be considered in accordance with the order considered in $L([0, 1])$. The different usual orders for intervals are based on the specific points within the intervals which are considered as representatives.

Thus, if $a = [\underline{a}, \overline{a}]$ and $b = [\underline{b}, \overline{b}]$ are any two intervals in $L([0, 1])$, we say that a is lower than or equal to b for the most usual orders between intervals if:

- Interval dominance [8]: $a \preceq_{ID} b$ if $\overline{a} \leq \underline{b}$.
- Lattice order [9]: $a \preceq_{Lo} b$ if $\underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$, which is induced by the usual partial order in $\mathbb{R}^2$.
- Lexicographical order type 1 [5]: $a \preceq_{Lex1} b$ if $\underline{a} < \underline{b}$ or $\underline{a} = \underline{b}$ and $\overline{a} \leq \overline{b}$.

- Lexicographical order type 2 [5]: $a \preceq_{Lex2} b$ if $\bar{a} < \bar{b}$ or $\bar{a} = \bar{b}$ and $\underline{a} \leq \underline{b}$.
- The Xu and Yager order [24]: $a \preceq_{YX} b$ if $\underline{a} + \bar{a} < \underline{b} + \bar{b}$ or $\underline{a} + \bar{a} = \underline{b} + \bar{b}$ and $\bar{a} - \underline{a} < \bar{b} - \underline{b}$.
- Maximax order [19]: $a \preceq_{MM} b$ if $\bar{a} \leq \bar{b}$.
- Maximin order [20, 23]: $a \preceq_{Mm} b$ if $\underline{a} \leq \underline{b}$.
- Hurwicz order [11]: $a \preceq_{H(\alpha)} b$ if $\alpha \cdot \underline{a} + (1 - \alpha) \cdot \bar{a} \leq \alpha \cdot \underline{b} + (1 - \alpha) \cdot \bar{b}$ with $\alpha \in [0, 1]$.
- Weak order [3]: $a \preceq_{wo} b$ if $\underline{a} \leq \bar{b}$.

Some of these orders are clearly related. Thus, it is well-known that if one interval a is lower than or equal to another interval b w.r.t. the order ID , a is also lower than or equal to b w.r.t. the lattice order. This also implies the same relation w.r.t. the lexicographical order type 1, which implies the same w.r.t. the maximax order and this implies that a is lower than or equal to b w.r.t. the weak order. All these implications and some other similar ones are summarized at the following figure.

The implications represented here are fulfilled, but it is also known that the converse implications are not fulfilled in general.

In order to provide a total order that extends the usual orders between intervals, Bustince et al. introduced in [5] the concept of admissible order.

Definition 1. [5] Let $(L([0, 1]), \preceq)$ be a poset. The order $\preceq$ is called an *admissible order* if

- $\preceq$ is a linear order on $L([0, 1])$,
- for all $[a, b], [c, d] \in L([0, 1])$, $[a, b] \preceq [c, d]$ whenever $[a, b] \leq_{Lo} [c, d]$.

Once they introduced this definition, they also proposed a method to build these admissible orders in terms of two aggregation functions.

Definition 2. [2] Let $\mathcal{A} : \bigcup_{i=1}^n [0, 1]^i \rightarrow [0, 1]$ such that

- $\mathcal{A}(0, 0, \dots, 0) = 0, \mathcal{A}(1, 1, \dots, 1) = 1$,
- $\mathcal{A}(x) = x$ for all $x \in [0, 1]$,

– $\mathcal{A}$ is monotone in each variable,

then $\mathcal{A}$ is an aggregation function.

Notice that, there is a natural bijection between $L([0, 1])$ and $K([0, 1]) = \{(u, v) \in [0, 1]^2 \mid u \leq v\}$ such that it identifies any interval $[\underline{a}, \bar{a}]$ to the point in $\mathbb{R}^2$ formed by its extremes, that is, $(\underline{a}, \bar{a})$ (see [5]). Thus, we can use aggregation functions to summarize the information given by an interval. Taking into account this idea, they obtained the following method to build admissible orders.

Proposition 1. [5] *Let $\mathcal{A}, \mathcal{B} : [0, 1]^2 \rightarrow [0, 1]$ be two continuous aggregation functions, such that for all $(u, v), (u', v') \in K([0, 1])$, the equalities $\mathcal{A}(u, v) = \mathcal{A}(u', v')$ and $\mathcal{B}(u, v) = \mathcal{B}(u', v')$ can only hold if $(u, v) = (u', v')$. Define the relation $\preceq_{\mathcal{A}, \mathcal{B}}$ on $L([0, 1])$ by $a \preceq_{\mathcal{A}, \mathcal{B}} b$ if and only if*

$$\mathcal{A}(\underline{a}, \bar{a}) < \mathcal{A}(\underline{b}, \bar{b})$$

or

$$\mathcal{A}(\underline{a}, \bar{a}) = \mathcal{A}(\underline{b}, \bar{b}) \text{ and } \mathcal{B}(\underline{a}, \bar{a}) \leq \mathcal{B}(\underline{b}, \bar{b}).$$

Then $\preceq_{\mathcal{A}, \mathcal{B}}$ is an admissible order on $L([0, 1])$.

A particular way of obtaining admissible orders on $L([0, 1])$ is defining them by means of the weighted mean (see [4]):

$$K_\alpha(u, v) = (1 - \alpha) \cdot u + \alpha \cdot v \text{ with } \alpha \in [0, 1].$$

This function can be seen as the α -quantile of a probability distribution uniformly distributed over the interval $[u, v]$. By applying Proposition 1 to the aggregation functions K_α and K_β with $\alpha \neq \beta$, we obtain the admissible order $\preceq_{K_\alpha, K_\beta}$, which is denoted, by simplicity, as $\preceq_{\alpha, \beta}$.

The lexicographical orders with respect to the first and the second coordinate and the Xu and Yager order are particular cases of these admissible orders. Thus, $\preceq_{Lex1} \equiv \preceq_{0,1}$, $\preceq_{Lex2} \equiv \preceq_{1,0}$ and $\preceq_{YX} \equiv \preceq_{1/2, \beta}$ for any $\beta \in (1/2, 1]$ (see [5]).

3 Intersection of Interval-Valued Fuzzy Sets

From any order $\preceq_x$ in $L([0, 1])$, we can deduce an order in $IVFS(X)$ given by the content relation obtained from this order. This relation in $IVFS(X)$ will be denoted by $\subseteq_x$. Thus, for instance,

$$A \subseteq_{YX} B \text{ iff } A(x) \preceq_{YX} B(x), \forall x \in X.$$

Let us notice that the inclusion between two interval-orders fuzzy sets does not imply the inclusion of the intervals which represent the membership degrees at any point in the referential, but that the interval associated to A is lower than or equal to w.r.t. the corresponding order to the interval associated to B .

Example 1. Let $X = \{x\}$ be the universe and let be $A, B, C \in IVFS(X)$ such that $A(x) = [0.4, 0.8]$, $B(x) = [0.2, 0.6]$ and $C(x) = [0.3, 0.9]$. We have that $B \subseteq_{Lo} A$ and $B \subseteq_{Lo} C$, since $[0.2, 0.6] \preceq_{Lo} [0.4, 0.8]$ and $[0.2, 0.6] \preceq_{Lo} [0.3, 0.9]$. However, $[0.2, 0.6] \not\subseteq [0.4, 0.8]$ and $[0.2, 0.6] \not\subseteq [0.3, 0.9]$. In fact, this is totally logical, since the membership degree of x to B is “lower” than the membership degree to A or C . In fact, we are using the same criteria as the usual one considered for fuzzy sets.

If the intersection of two sets is defined as the greatest set that is contained in both sets, then we have a different definition of intersection for each order we are considering in $IVFS(X)$.

Definition 3. Let A, B be two sets in $IVFS(X)$ and let $\preceq_x$ an order in $L([0, 1])$. We define the x -intersection of A and B , and we denote $A \cap_x B$ as the greatest interval-valued fuzzy set such that $A \cap_x B \subseteq_x A$ and $A \cap_x B \subseteq_x B$.

So each order would have its own way to construct intersections between IVFS. To better understand this definition, we will see some examples after the general result of each order.

For any two interval orders $\preceq_x$ and $\preceq_y$ in $IVFS(X)$ such that $a \preceq_x b$ implies that $a \preceq_y b$, $\forall a, b \in L([0, 1])$, we have that $A \cap_x B \subseteq_y A \cap_y B$ for any $A, B \in IVFS(X)$. Thus,

$$A \cap_{ID} B \subseteq_{Lo} A \cap_{Lo} B \subseteq_{Lex1} A \cap_{Lex1} B \subseteq_{MM} A \cap_{MM} B \subseteq_{wo} A \cap_{wo} B,$$

$$A \cap_{Lo} B \subseteq_{Lex2} A \cap_{Lex2} B \subseteq_{Mm} A \cap_{Mm} B \subseteq_{wo} A \cap_{wo} B,$$

$$A \cap_{Lo} B \subseteq_{YX} A \cap_{YX} B \subseteq_{H(1/2)} A \cap_{H(1/2)} B \subseteq_{wo} A \cap_{wo} B,$$

and

$$A \cap_{Lo} B \subseteq_{H(\alpha)} A \cap_{H(\alpha)} B \subseteq_{wo} A \cap_{wo} B.$$

Taking into account the relationship among the considered orders, we will study the definition that we obtain for the intersection for any of them, by considering some general behavior in those cases which is possible.

For the interval dominance we have that the intersection of two IVFS is a fuzzy set:

Proposition 2. Let A, B be two sets in $IVFS(X)$. The ID -intersection of A and B is the interval-valued fuzzy set defined by

$$A \cap_{ID} B(x) = \min\{\underline{A(x)}, \underline{B(x)}\}$$

for any $x \in X$.

Next, an example of the intersection of two IVFS is presented.

Example 2. In the same conditions of Example 1.

- The ID -intersection of A and B is the interval-valued fuzzy set $A \cap_{ID} B(x) = 0.2$.

- The ID -intersection of A and C is the interval-valued fuzzy set $A \cap_{ID} C(x) = 0.3$.

It is clear that $A \cap_{ID} B \subseteq_{ID} A$ since $0.2 \leq 0.4$ and $0.2 \leq 0.2$. Similarly, we can see that $A \cap_{ID} B \subseteq_{ID} B$. Both expressions are immediate, since the definition of intersection was given exactly as the set such that it is included in both of them and any other set included in them is a subset of the intersection. Of course, when we deal with the intersection and the content, the same order between intervals is considered.

However, for the less restrictive lattice ordering the intersection is not just a fuzzy set in general. In fact, with this order, we will obtain the usual way to define the intersection for two IVFS.

Proposition 3. *Let A, B be two sets in $IVFS(X)$. The Lo -intersection of A and B is the interval-valued fuzzy set defined by*

$$A \cap_{Lo} B(x) = [\min\{\underline{A}(x), \underline{B}(x)\}, \min\{\overline{A}(x), \overline{B}(x)\}]$$

for any $x \in X$.

This proposition coincides with the usual way to define the intersection between two interval-valued fuzzy sets, as it can be seen in [6], where intersection is studied in general for convolution lattices.

Following example shows what happens when we intersect two IVFS.

Example 3. In the same conditions of Example 1.

- The Lo -intersection of A and B is the interval-valued fuzzy set $A \cap_{Lo} B(x) = [0.2, 0.6]$.
- The Lo -intersection of A and C is the interval-valued fuzzy set $A \cap_{Lo} C(x) = [0.3, 0.8]$.

We can also use this example to remark that the intersection we are done is between two interval-valued fuzzy set, but it is not an intersection between intervals. Thus, since x has a membership degree in A given by the interval $[0.4, 0.8]$ and a membership degree in C given by the interval $[0.3, 0.9]$, we can say that the degree in which x is in their intersection is between 0.3 and 0.8. A value greater than 0.8 is impossible, since we have not this degree for A , but a value between 0.3 and 0.4 is possible, since x belongs to A at least in degree 0.4 and to B at least in degree 0.3.

The case of the lexicographical order and the Xu and Yager order, can be considered in a general way, since all of them are particular cases of admissible orders, as we commented previously.

Proposition 4. *Let $\mathcal{A}, \mathcal{B} : [0, 1]^2 \rightarrow [0, 1]$ be two continuous aggregation functions, such that for all $(u, v), (u', v') \in K([0, 1])$, the equalities $\mathcal{A}(u, v) = \mathcal{A}(u', v')$ and $\mathcal{B}(u, v) = \mathcal{B}(u', v')$ can only hold if $(u, v) = (u', v')$. Let $\preceq_{\mathcal{A}, \mathcal{B}}$ be the admissible order on $L([0, 1])$ induced by them. For any $A, B \in IVFS(X)$, the $\mathcal{A}, \mathcal{B}$ -intersection of A and B is the interval-valued fuzzy set defined by:*

$$A \cap_{\mathcal{A}, \mathcal{B}} B(x) = \begin{cases} [\underline{A(x)}, \overline{A(x)}] & \text{if } \mathcal{A}([\underline{A(x)}, \overline{A(x)}]) < \mathcal{A}([\underline{B(x)}, \overline{B(x)}]) \\ & \text{or } \left\{ \mathcal{A}([\underline{A(x)}, \overline{A(x)}]) = \mathcal{A}([\underline{B(x)}, \overline{B(x)}]) \right. \\ & \left. \text{and } \mathcal{B}([\underline{A(x)}, \overline{A(x)}]) \leq \mathcal{B}([\underline{B(x)}, \overline{B(x)}]) \right\}, \\ [\underline{B(x)}, \overline{B(x)}] & \text{if } \left\{ \mathcal{A}([\underline{A(x)}, \overline{A(x)}]) = \mathcal{A}([\underline{B(x)}, \overline{B(x)}]) \right. \\ & \left. \text{and } \mathcal{B}([\underline{B(x)}, \overline{B(x)}]) < \mathcal{B}([\underline{A(x)}, \overline{A(x)}]) \right\} \\ & \text{or } \mathcal{A}([\underline{B(x)}, \overline{B(x)}]) < \mathcal{A}([\underline{A(x)}, \overline{A(x)}]). \end{cases}$$

By applying this result to the specific admissible orders, we obtain that

Proposition 5. *Let A, B be two sets in $IVFS(X)$.*

- *The Lex1-intersection of A and B is the interval-valued fuzzy set whose membership function for any x in X is:*

$$A \cap_{Lex1} B(x) = \begin{cases} [\underline{A(x)}, \min\{\overline{A(x)}, \overline{B(x)}\}] & \text{if } \underline{A(x)} = \underline{B(x)}, \\ [\underline{A(x)}, \overline{A(x)}] & \text{if } \underline{A(x)} < \underline{B(x)}, \\ [\underline{B(x)}, \overline{B(x)}] & \text{if } \underline{B(x)} < \underline{A(x)}. \end{cases}$$

- The *Lex2*-intersection of A and B is the interval-valued fuzzy set whose membership function for any x in X is:

$$A \cap_{Lex2} B(x) = \begin{cases} [\min\{A(x), B(x)\}, \overline{A(x)}] & \text{if } \overline{A(x)} = \overline{B(x)}, \\ [A(x), \overline{A(x)}] & \text{if } A(x) < B(x), \\ [\overline{B(x)}, \overline{B(x)}] & \text{if } B(x) < A(x). \end{cases}$$

- The *YX*-intersection of A and B is the interval-valued fuzzy set whose membership function for any x in X is:

$$A \cap_{YX} B(x) = \begin{cases} [\overline{A(x)}, \overline{A(x)}] & \text{if } A(x) + \overline{A(x)} < B(x) + \overline{B(x)} \\ & \text{or } \{A(x) + \overline{A(x)} = B(x) + \overline{B(x)} \\ & \text{and } \overline{A(x)} - A(x) \leq \overline{B(x)} - B(x)\}, \\ [\overline{B(x)}, \overline{B(x)}] & \text{if } \{A(x) + \overline{A(x)} = B(x) + \overline{B(x)}, \\ & \text{and } \overline{B(x)} - B(x) < \overline{A(x)} - A(x)\} \\ & \text{or } B(x) + \overline{B(x)} < A(x) + \overline{A(x)}. \end{cases}$$

In order to clarify this result, let us show some examples.

Example 4. In the same conditions of Example 1.

- The *Lex1*-intersection of A and B is the interval-valued fuzzy set $A \cap_{Lex1} B(x) = [0.2, 0.6]$.
- The *Lex1*-intersection of A and C is the interval-valued fuzzy set $A \cap_{Lex1} C(x) = [0.3, 0.9]$.

- The *Lex2*-intersection of A and B is the interval-valued fuzzy set $A \cap_{Lex2} B(x) = [0.2, 0.6]$.
- The *Lex2*-intersection of A and C is the interval-valued fuzzy set $A \cap_{Lex2} C(x) = [0.4, 0.8]$.

- The YX -intersection of A and B is the interval-valued fuzzy set $A \cap_{YX} B(x) = [0.2, 0.6]$.
- The YX -intersection of A and C is the interval-valued fuzzy set $A \cap_{YX} C(x) = [0.4, 0.8]$.
- On the other hand, if $D(x) = [0.2, 0.9]$, the YX -intersection of A and D is the interval-valued fuzzy set $A \cap_{YX} D(x) = [0.2, 0.9]$.

We are not going to consider the maximax, the maximin, the Hurwicz and the weak orders to define the intersection since it is not unique. Using the order $\preceq_{MM}$, we obtain that the intersection between two IVFS A and B is

$$A \cap_{MM} B(x) = [u, \min\{\overline{A(x)}, \overline{B(x)}\}], \forall x \in X,$$

where u could be any number in the interval $[0, \min\{\underline{A(x)}, \underline{B(x)}\}]$.

Analogously, using the order $\preceq_{Mm}$, we obtain that the intersection between two IVFS A and B is

$$A \cap_{Mm} B(x) = [\min\{\underline{A(x)}, \underline{B(x)}\}, v], \forall x \in X,$$

where v could be any number in the interval $[\min\{\underline{A(x)}, \underline{B(x)}\}, 1]$.

If we consider the order $\preceq_{H(\alpha)}$, we obtain that the intersection between two IVFS A and B is

$$A \cap_{H(\alpha)} B(x) = [t - k, t + k], \forall x \in X,$$

where $t = \min\{\alpha \cdot \underline{A(x)} + (1 - \alpha) \cdot \overline{A(x)}, \alpha \cdot \underline{B(x)} + (1 - \alpha) \cdot \overline{B(x)}\}$ and k could be any number in the interval $[0, \min\{t, 1 - t\}]$.

For the weak order, the intersection is

$$A \cap_{MM} B(x) = [u, v], \forall x \in X,$$

where u could be any number in the interval $[0, \min\{\overline{A(x)}, \overline{B(x)}\}]$ and v could be any number in the interval $[\min\{\underline{A(x)}, \underline{B(x)}\}, 1]$.

4 Preservation on the Convexity Under Intersections

Taking into the previous comments, we are not going to consider the intersection based on the maximax, the maximin, the Hurwicz or the weak order. For the remaining intersections, we will study if the convexity of two convex IVFS is still a convex set. First of all, let us introduce the definition of convexity we are going to consider in this work.

Definition 4. Let X be an ordered space and let $\preceq_x$ be an order in $L([0, 1])$. An interval-valued fuzzy set A on X is said to be x -convex, if for each $x < y < z$ in X the following inequalities are fulfilled:

$$A(x) \preceq_x A(y) \text{ or } A(z) \preceq_x A(y).$$

It is a natural definition, based on the usual idea of convexity. It is immediate that a convex fuzzy set considered as an interval-valued fuzzy set with singleton as membership values is convex. It is also immediate that this definition coincides with the usual one of convexity for crisp sets.

If we deal with the particular orders considered in the previous section, we obtain that ID -convexity implies Lo -convexity and this implies $Lex1$ -convexity, $Lex2$ -convexity and YX -convexity.

About the important property of the preservation of the convexity under intersections, we have obtained the following results.

Proposition 6. *Let X be an ordered space, if $A, B \in IVFS(X)$ are ID -convex, then $A \cap_{ID} B$ is also ID -convex, whenever it is not empty.*

Unfortunately, the Lo -intersection of two IVFS which are Lo -convex is not always Lo -convex, as we can see at the following counterexample.

Example 5. Let $X = \{x, y, z\}$ with $x < y < z$. If we consider the IVFS A and B defined as follows

X	x	y	z
A	$[0.1, 0.7]$	$[0.2, 0.8]$	$[0.3, 0.5]$
B	$[0.1, 0.7]$	$[0.4, 0.6]$	$[0.3, 0.5]$
$A \cap_{Lo} B$	$[0.1, 0.7]$	$[0.2, 0.6]$	$[0.3, 0.5]$

Then A is Lo -convex, since $[0.1, 0.7] \preceq_{Lo} [0.2, 0.8]$ and B is Lo -convex since $[0.3, 0.5] \preceq_{Lo} [0.4, 0.6]$. However, $A \cap_{Lo} B$ is not Lo -convex since $[0.2, 0.6]$ is not related with $[0.1, 0.7]$ or $[0.3, 0.5]$ by means of the order relation $\preceq_{Lo}$. This is the typical problem we can find any time we use partial orders. Thus, it is clear that convexity only makes sense for total orders as, for instance, admissible orders.

In fact, for admissible orders, we have been able to obtain a general results where we prove the good behaviour of them with respect to the convexity. Thus,

Proposition 7. *Let X be an ordered space and let $\preceq_{\mathcal{A}, \mathcal{B}}$ an admissible order based on two aggregations functions $\mathcal{A}$ and $\mathcal{B}$. If $A, B \in IVFS(X)$ are $\mathcal{A}, \mathcal{B}$ -convex, then $A \cap_{\mathcal{A}, \mathcal{B}} B$ is also $\mathcal{A}, \mathcal{B}$ -convex, whenever it is not empty.*

Thus, the case of the lexicographical orders and the Xu and Yager order are automatically solved.

Corollary 1. *Lex1-convexity, Lex2-convexity and YX -convexity are preserved under intersections.*

5 Concluding Remarks

In this paper we have proposed a definition of convexity for IVFS and we characterized the cases where it is preserved under intersections. The intersection of two IVFS is based on the chosen order between intervals and so several definitions of intersection are considered. It is not surprising that not all of the orders between intervals are appropriate for defining the intersection and that the lattice ordering defines the usual definition of intersection considered in the literature. However, this order has not a good behavior about preservation of convexity under intersections and admissible orders seem to be better for this purpose. An immediate pending work is the study of the cutworthy approach for this concept of convexity.

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