



On Integral Transforms for Residuated Lattice-Valued Functions

Michal Holčapek^(✉) and Viec Bui

CE IT4I - IRAFM, University of Ostrava, 30. dubna 22,
701 03 Ostrava 1, Czech Republic
michal.holcapek@osu.cz, bqviiec@gmail.com
<http://irafm.osu.cz>

Abstract. The article aims to introduce four types of integral transforms for functions whose function values belong to a complete residuated lattice. The integral transforms are defined using so-called qualitative residuum based fuzzy integrals and integral kernels in the form of binary fuzzy relations. We present some of the basic properties of proposed integral transforms including a linearity property that is satisfied under specific conditions for comonotonic functions.

Keywords: Integral transform · Fuzzy transform · Residuated lattice · Integral kernel · Fuzzy integral

1 Introduction

Mathematical operators known as integral transforms produce a new function $g(y)$ by integrating the product of an existing function $f(x)$ and an integral kernel function $K(x, y)$ between suitable limits. The Fourier and Laplace transforms belong among the most popular integral transforms and are applied for real or complex-valued functions. The importance of the integral transforms is mainly in solving (partial) differential equations, algebraic equations, signal and image processing, spectral analysis of stochastic processes (see, e.g., [2, 21, 23]).

In fuzzy set theory we often deal with functions whose function values belong to an appropriate algebra of truth values as a residuated lattice and its special variants as the BL-algebra, MV-algebra, IMTL-algebra (see, e.g. [1, 5, 18]). In [20], Perfilieva introduced lattice-valued upper and lower fuzzy transforms that are, among others, used for an approximation of functions. A deeper investigation of fuzzy transforms properties can be found in [13–17, 19, 22]. In a recent article [9], we demonstrated that the lower and upper fuzzy transforms can be introduced as two type of integral transforms, where the multiplication based fuzzy integral is applied [3, 4]. Namely, for a fuzzy measure space (X, \mathcal{F}, μ) , an integral kernel $K : X \times Y \rightarrow L$ and a function $f : X \rightarrow L$, where L is a complete

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residuated lattice, we proposed the integral transforms given by the following formulas:

$$F_{(K, \otimes)}^{\otimes}(f)(y) = \int^{\otimes} K(x, y) \otimes f(x) \, d\mu,$$

$$F_{(K, \otimes)}^{\rightarrow}(f)(y) = \int^{\otimes} K(x, y) \rightarrow f(x) \, d\mu,$$

where $F_{(K, \otimes)}^{\otimes}$ becomes the upper fuzzy transform if $\mu(A) = 1$ for any $A \in \mathcal{F} = \mathcal{P}(X)$ such that $A \neq \emptyset$ and $F_{(K, \otimes)}^{\rightarrow}(f)$ becomes the lower fuzzy transform if $\mu(A) = 0$ for any $A \in \mathcal{F} = \mathcal{P}(X)$ such that $A \neq X$.¹ Moreover, to get the exact definitions of lower and upper fuzzy transforms, the family of fuzzy sets $\{K(\cdot, y) \mid y \in Y\}$ has to form a fuzzy partition of X (see [20]).

The aim of this article is to introduce further integral transforms for residuated lattice-valued functions and analyze their basic properties for which we consider the residuum based fuzzy integrals that were proposed by Dvořák and Holčápek in [4] and Dubois, Prade and Rico in [3]. Together with the integral transform with the multiplication based fuzzy integral introduced in [9] we get a class of nonstandard integral transforms for the residuated lattice-valued functions based on fuzzy (or also qualitative) integrals that are often used in data processing. Note that the fuzzy integrals aggregate data and, in this way, provide summary information that is not directly visible from data. Obviously, the proposed integral transforms also provide an aggregation of function values, mainly, if the set Y has a significantly smaller size than the set X . This can be used, for example, in hierarchical decision making, classification problem or signal and image processing, where kernels can express relationships between different levels of criteria, object attributes and classes or introduce windows for some kind of filtering, respectively.

The article is structured as follows. In the next section, we recall the definition of complete residuated lattices and the basic concepts of fuzzy set theory and the theory of fuzzy measure spaces. The third section introduces two types of the residuum based fuzzy integrals and shows their basic properties. The integral transforms for residuated lattice-valued functions are established in the fourth section. We present their elementary properties and demonstrate the linearity property under the restriction to comonotonic functions. The last section is a conclusion.

Because of the space limitation almost all proofs are omitted in this article.

2 Preliminary

2.1 Truth Value Structures

We assume that the structure of truth values is a *complete residuated lattice*, i.e., an algebra $L = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$ with four binary operations and two

¹ Note that we use here the denotation of the integral transforms employed in this article which is slightly different from [9].

constants such that $\langle L, \wedge, \vee, 0, 1 \rangle$ is a complete lattice, where 0 is the least element and 1 is the greatest element of L , $\langle L, \otimes, 1 \rangle$ is a commutative monoid (i.e., \otimes is associative, commutative and the identity $a \otimes 1 = a$ holds for any $a \in L$) and the adjointness property is satisfied, i.e.,

$$a \leq b \rightarrow c \quad \text{iff} \quad a \otimes b \leq c \tag{1}$$

holds for each $a, b, c \in L$, where \leq denotes the corresponding lattice ordering, i.e., $a \leq b$ if $a \wedge b = a$ for $a, b \in L$. A residuated lattice L is said to be *divisible* if $a \otimes (a \rightarrow b) = a \wedge b$ holds for arbitrary $a, b \in L$. The operation of *negation* on L is defined as $\neg a = a \rightarrow 0$ for $a \in L$. A residuated lattice L satisfies the *law of double negation* if $\neg\neg a = a$ holds for any $a \in L$. A divisible residuated lattice satisfying the law of double negation is called an *MV-algebra*. A residuated lattice is said to be *linearly ordered* if the corresponding lattice ordering is linear, i.e., $a \leq b$ or $b \leq a$ holds for any $a, b \in L$.

Theorem 1. *Let $\{b_i \mid i \in I\}$ be a non-empty set of elements from L , and let $a \in L$. Then*

- (a) $a \otimes (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \otimes b_i)$,
- (b) $a \rightarrow \bigwedge_{i \in I} b_i = \bigwedge_{i \in I} (a \rightarrow b_i)$,
- (c) $(\bigvee_{i \in I} b_i) \rightarrow a = \bigwedge_{i \in I} (b_i \rightarrow a)$,
- (d) $a \otimes \bigwedge_{i \in I} b_i \leq \bigwedge_{i \in I} (a \otimes b_i)$,
- (e) $\bigvee_{i \in I} (a \rightarrow b_i) \leq a \rightarrow \bigvee_{i \in I} b_i$,
- (f) $\bigvee_{i \in I} (b_i \rightarrow a) \leq \bigwedge_{i \in I} b_i \rightarrow a$.

If L is a complete MV-algebra the above inequalities may be replaced by equalities.

For more information about residuated lattices, we refer to [1, 18]. In what follows, we present two examples of linearly ordered lattice.

Example 1. It is easy to prove that the algebra

$$L_T = \langle [0, 1], \min, \max, T, \rightarrow_T, 0, 1 \rangle,$$

where T is a left continuous t -norm (see, e.g., [11]) and $a \rightarrow_T b = \bigvee \{c \in [0, 1] \mid T(a, c) \leq b\}$, defines the residuum, is a complete residuated lattice. In this article, we will refer to complete residuated lattices determined by the Łukasiewicz t -norm and nilpotent minimum, i.e.,

$$T_L(a, b) = \max(a + b - 1, 0),$$

$$T_{nM}(a, b) = \begin{cases} 0, & \text{if } a + b \leq 1, \\ \min(a, b), & \text{otherwise,} \end{cases}$$

respectively. Their residua are as follows:

$$a \rightarrow_L b = \min(1, 1 - a + b),$$

$$a \rightarrow_{nM} b = \begin{cases} 1, & \text{if } a \leq b, \\ \max(1 - a, b), & \text{otherwise.} \end{cases}$$

In the first case, the complete residuated lattice will be denoted by L_L . Note that L_L is a complete MV-algebra called the *Lukasiewicz algebra* (on $[0, 1]$), where, for example, the distributivity of \otimes over \wedge is satisfied.² The residuated lattice determined by the nilpotent minimum is an example of a residuated lattice in which the above-mentioned distributivity fails.

Example 2. Let $a, b \in [0, \infty]$ be such that $a < b$. One checks easily that $L_{[a,b]} = \langle [a, b], \min, \max, \min, \rightarrow, a, b \rangle$, where

$$c \rightarrow d = \begin{cases} b, & \text{if } c \leq d, \\ d, & \text{otherwise,} \end{cases} \tag{2}$$

is a complete residuated lattice. Note that $L_{[a,b]}$ is a special example of a more general residuated lattice called a *Heyting algebra*.³

In the end of this section, we introduce two families of subsets of L from which important algebras of sets are later generated. Let $u : \mathcal{P}(L) \rightarrow \mathcal{P}(L)$ be defined as

$$u(X) = \{x \in L \mid \exists a \in X, a \leq x\} \tag{3}$$

for any $X \in \mathcal{P}(L)$. Obviously, $X \subseteq u(X)$. A set $X \in \mathcal{P}(L)$, for which $u(X) = X$ holds, is called the *upper set* or *upset*. We use $\mathcal{U}(L)$ to denote the family of all upsets in L , i.e., $\mathcal{U}(L) = \{u(X) \mid X \in \mathcal{P}(L)\}$.⁴ Similarly, let $\ell : \mathcal{P}(L) \rightarrow \mathcal{P}(L)$ be defined as

$$\ell(X) = \{x \in L \mid \exists a \in X, a \geq x\} \tag{4}$$

for any $X \in \mathcal{P}(L)$. A set $X \in \mathcal{P}(L)$ for which $\ell(X) = X$ holds is called the *lower set* or *loset*. The family of all losets in L is denoted $\mathcal{L}(L)$.

2.2 Fuzzy Sets

Let L be a complete residuated lattice, and let X be a non-empty universe of discourse. A function $A : X \rightarrow L$ is called a *fuzzy set* (*L-fuzzy set*) on X . A value $A(x)$ is called a *membership degree of x in the fuzzy set A* . The set of all fuzzy sets on X is denoted by $\mathcal{F}(X)$. A fuzzy set A on X is called *crisp* if $A(x) \in \{0, 1\}$ for any $x \in X$. Obviously, a crisp fuzzy set can be uniquely identified with a subset of X . The symbol \emptyset denotes the empty fuzzy set on X , i.e., $\emptyset(x) = 0$ for any $x \in X$. The set of all crisp fuzzy sets on X (i.e., the power set of X) is denoted by $\mathcal{P}(X)$. A constant fuzzy set A on X (denoted as a_X) satisfies $A(x) = a$ for any $x \in X$, where $a \in L$. The sets $\text{Supp}(A) = \{x \mid x \in X \ \& \ A(x) > 0\}$ and $\text{Core}(A) = \{x \mid x \in X \ \& \ A(x) = 1\}$ are called the *support* and the *core* of a fuzzy set A , respectively. A fuzzy set A is called *normal* if $\text{Core}(A) \neq \emptyset$.

² Here we mean that $\bigwedge_{i \in I} (a \otimes b_i) = a \otimes \bigwedge_{i \in I} b_i$ holds.

³ A Heyting algebra is a residuated lattice with $\otimes = \wedge$.

⁴ In [8], a type of topological spaces derived from upsets in L was proposed.

Let A, B be fuzzy sets on X . The extension of the operations \wedge, \vee, \otimes and \rightarrow on L to the operations on $\mathcal{F}(X)$ is given by

$$\begin{aligned} (A \wedge B)(x) &= A(x) \wedge B(x) & \text{and} & & (A \vee B)(x) &= A(x) \vee B(x) \\ (A \otimes B)(x) &= A(x) \otimes B(x) & \text{and} & & (A \rightarrow B)(x) &= A(x) \rightarrow B(x) \end{aligned} \tag{5}$$

for any $x \in X$. Obviously, $A \wedge B$ and $A \vee B$ are the standard definitions of the intersection and union of fuzzy sets A and B , respectively, but we prefer here the symbols of infimum (\wedge) and supremum (\vee) over the classical \cap and \cup .

Let X, Y be non-empty universes. A fuzzy set $K : X \times Y \rightarrow L$ is called a (binary) fuzzy relation. A fuzzy relation K is said to be *normal*, whenever $\text{Core}(K) \neq \emptyset$, and *normal in the first coordinate*, whenever $\text{Core}(K(x, \cdot)) \neq \emptyset$ for any $x \in X$. Similarly, a fuzzy relation is normal in the second component. A fuzzy relation K is said to be *complete normal* whenever K is normal in the first and the second coordinates. A relaxation of the normality of fuzzy relation is a *semi-normal* fuzzy relation defined as $K \neq \emptyset$, i.e., $K(x, y) > 0$ for certain $(x, y) \in X \times Y$. Similarly one can define *semi-normal in the first (second) coordinate* and *complete semi-normal* fuzzy relation.

2.3 Fuzzy Measure Spaces

Measurable spaces and functions. Let us consider algebras of sets as follows.

Definition 1. *Let X be a non-empty set. A subset \mathcal{F} of $\mathcal{P}(X)$ is an algebra of sets on X provided that.*

- (A1) $X \in \mathcal{F}$,
- (A2) if $A \in \mathcal{F}$, then $X \setminus A \in \mathcal{F}$,
- (A3) if $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$.

Definition 2. *An algebra \mathcal{F} of sets on X is a σ -algebra of sets if*

- (A4) if $A_i \in \mathcal{F}$, $i = 1, 2, \dots$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

It is easy to see that if \mathcal{F} is an algebra (σ -algebra) of sets, then the intersection of finite (countable) number of sets belongs to \mathcal{F} . A pair (X, \mathcal{F}) is called a *measurable space* (on X) if \mathcal{F} is an algebra (σ -algebra) of sets on X . Let (X, \mathcal{F}) be a measurable space and $A \in \mathcal{F}(X)$. We say that A is *\mathcal{F} -measurable* if $A \in \mathcal{F}$. Obviously, the sets $\{\emptyset, X\}$ and $\mathcal{P}(X)$ are σ -algebras of fuzzy sets on X .

A beneficial tool how to introduce an algebra or a σ -algebra of sets on X is an algebra (σ -algebra) generated by a non-empty family of sets.

Definition 3. *Let $\mathcal{G} \subseteq \mathcal{P}(X)$ be a non-empty family of sets. The smallest algebra (σ -algebra) on X containing \mathcal{G} is denoted by $\text{alg}(\mathcal{G})$ ($\sigma(\mathcal{G})$) and is called the generated algebra (σ -algebra) by the family \mathcal{G} .*

Note that the intersection of algebras (σ -algebras) is again an algebra (σ -algebra), hence, the smallest algebra (σ -algebra) on X containing \mathcal{G} always exists and its unique. Moreover, the generated algebra $\text{alg}(\mathcal{G})$, in contrast to

$\sigma(\mathcal{G})$, can be simply constructed from the elements \mathcal{G} as the set which consists of all finite unions applied on the set of all finite intersections over the elements of \mathcal{G} and their complements. Note that the construction of a generated σ -algebra needs a transfinite approach. In this article, we will consider the algebras generated from the families of upsets $\mathcal{U}(L)$ and losets $\mathcal{L}(L)$.

Let (X, \mathcal{F}) and (Y, \mathcal{G}) be measurable spaces, and let $f : X \rightarrow Y$ be a function. We say that f is \mathcal{F} - \mathcal{G} -measurable if $f^{-1}(Z) \in \mathcal{F}$ for any $Z \in \mathcal{G}$. The following theorem shows that the verification of \mathcal{F} - \mathcal{G} -measurability of functions can be simplified if \mathcal{G} is a generated algebra (σ -algebra).

Theorem 2. *Let $\mathcal{G} \subseteq \mathcal{P}(Y)$ be a subset such that $Y \in \mathcal{G}$, and let (X, \mathcal{F}) be a measurable space. A function $f : X \rightarrow Y$ is \mathcal{F} - $\text{alg}(\mathcal{G})$ -measurable if and only if $f^{-1}(Z) \in \mathcal{F}$ for any $Z \in \mathcal{G}$.*

Proof. (\Rightarrow) The implication is a simple consequence of $\mathcal{G} \subseteq \text{alg}(\mathcal{G})$.

(\Leftarrow) Let $\mathcal{Q} = \{Z \mid f^{-1}(Z) \in \mathcal{F}\}$. Note that \mathcal{Q} is called the preimage algebra on Y . From the definition of the generated algebra $\text{alg}(\mathcal{G})$ by the family \mathcal{G} , we find that $\text{alg}(\mathcal{G}) \subseteq \mathcal{Q}$. Hence, we obtain that $f^{-1}(Z) \in \mathcal{F}$ for any $Z \in \text{alg}(\mathcal{G})$, which means that f is \mathcal{F} - $\text{alg}(\mathcal{G})$ -measurable.

Note that the previous theorem remains true if $\text{alg}(\mathcal{G})$ is replaced by $\sigma(\mathcal{G})$. In the following three statements we provide sufficient conditions under which the functions obtained applying the operations to measurable functions remain measurable. For the purpose of this article, we restrict to fuzzy sets and algebras determined by upsets and losets.

Theorem 3. *Let L be linearly ordered, let (X, \mathcal{F}) be an algebra, and let $\mathcal{B} \subseteq \mathcal{F}(X)$ be a set of all \mathcal{F} - $\text{alg}(\mathcal{U}(L))$ -measurable fuzzy sets. Then $f \wedge g, f \vee g \in \mathcal{B}$ for any $f, g \in \mathcal{B}$.*

Theorem 4. *Let (X, \mathcal{F}) be an algebra, and let $\mathcal{B} \subseteq \mathcal{F}(X)$ be a set of all \mathcal{F} - $\text{alg}(\mathcal{U}(L))$ -measurable fuzzy sets. If \mathcal{F} is closed over arbitrary unions, then*

$$f \otimes g, f \wedge g, f \vee g, \quad f, g \in \mathcal{B}.$$

Theorem 5. *Let L be linearly ordered and dense. Let (X, \mathcal{F}) be an algebra, and let $\mathcal{B} \subseteq \mathcal{F}(X)$ be a set of all \mathcal{F} - $\text{alg}(\mathcal{U}(L))$ -measurable fuzzy sets. If \mathcal{F} is closed over arbitrary unions, then*

$$f \rightarrow g \in \mathcal{B}, \quad f, g \in \mathcal{B}.$$

The previous theorems become true if the algebra $\text{alg}(\mathcal{U}(L))$ is replaced by $\text{alg}(\mathcal{L}(L))$ and the \mathcal{F} - $\text{alg}(\mathcal{L}(L))$ -measurability is considered.

Fuzzy Measures. The concept of a fuzzy measure on a measurable space (X, \mathcal{F}) is a slight extension of the standard definition of the normed measure where the unit interval (or the real line) is replaced by a complete residuated lattice L (e.g., [6, 12]).

Definition 4. A map $\mu : \mathcal{F} \rightarrow L$ is called a fuzzy measure on a measurable space (X, \mathcal{F}) if

- (i) $\mu(\emptyset) = 0$ and $\mu(X) = 1$,
- (ii) if $A, B \in \mathcal{F}$ such that $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

A triplet (X, \mathcal{F}, μ) is called a fuzzy measure space whenever (X, \mathcal{F}) is a measurable space and μ is a fuzzy measure on (X, \mathcal{F}) .

Example 3. Let L_T be an algebra from Example 1, where T is a continuous t -norm. Let $X = \{x_1, \dots, x_n\}$ be a finite non-empty set, and let \mathcal{F} be an arbitrary algebra. A relative fuzzy measure μ^r on (X, \mathcal{F}) can be given as

$$\mu^r(A) = \frac{|A|}{|X|}$$

for all $A \in \mathcal{F}$, where $|A|$ and $|X|$ denote the cardinality of A and X , respectively. Let $\varphi : L \rightarrow L$ be a monotonically non-decreasing map with $\varphi(0) = 0$ and $\varphi(1) = 1$. The relative measure μ^r can be generalized as a fuzzy measure μ_φ^r on (X, \mathcal{F}) given by $\mu_\varphi^r(A) = \varphi(\mu^r(A))$ for any $A \in \mathcal{F}$.

3 Residuum Based Fuzzy Integrals

In the following part, we introduce two types of fuzzy (qualitative) integrals based on the operation of residuum. The first type of this fuzzy integral was proposed by Dvořák and Holčapek in [4] for fuzzy quantifiers modelling, the second type was proposed by Dubois, Prade and Rico in [3], known also under the name *desintegral*, for the reasoning with a decreasing evaluation scale. A comparison of both fuzzy integrals can be found in [10].

3.1 \rightarrow_{DH} -Fuzzy Integral

The integrated functions are fuzzy sets on X . We consider a modified version of the original definition of the residuum based fuzzy integral presented in [4].

Definition 5. Let (X, \mathcal{F}, ν) be a complementary fuzzy measure space, and let $f : X \rightarrow L$. The \rightarrow_{DH} -fuzzy integral of f on X is given by

$$\int_{\text{DH}}^{\rightarrow} f \, d\nu = \bigwedge_{A \in \mathcal{F}} \left(\bigwedge_{x \in A} f(x) \right) \rightarrow \nu(A). \tag{6}$$

Note that the original definition in [4] and the previous definition of residuum based integrals coincide on MV-algebras. The following statement presents basic properties of \rightarrow_{DH} -fuzzy integral.

Theorem 6. For any $f, g \in \mathcal{F}(X)$ and $a \in L$, we have

- (i) $\int_{\text{DH}}^{\rightarrow} f \, d\nu \geq \int_{\text{DH}}^{\rightarrow} g \, d\nu$ if $f \leq g$;
- (ii) $\int_{\text{DH}}^{\rightarrow} a_X \, d\nu = \neg a$;

- (iii) $\int_{\text{DH}}^{\rightarrow} a_X \otimes f \, d\nu \leq a \rightarrow \int_{\text{DH}}^{\rightarrow} f \, d\nu$;
- (iv) $\int_{\text{DH}}^{\rightarrow} a_X \rightarrow f \, d\nu \geq a \otimes \int_{\text{DH}}^{\rightarrow} f \, d\nu$.

Note that the inequality (iii) of the previous theorem becomes the equality in a complete MV-algebra. An equivalent, and useful from the practical point of view, definition of \rightarrow_{DH} -fuzzy integrals can be obtained for $\mathcal{F}\text{-alg}(\mathcal{U}(L))$ -measurable functions.

Theorem 7. *If $f : X \rightarrow L$ be $\mathcal{F}\text{-alg}(\mathcal{U}(L))$ -measurable, then*

$$\int_{\text{DH}}^{\rightarrow} f \, d\nu = \bigwedge_{a \in L} (a \rightarrow \nu(\{x \in X \mid f(x) \geq a\})). \tag{7}$$

We say that $f, g \in \mathcal{F}(X)$ are *comonotonic* if and only if there is no pair $x_1, x_2 \in X$ such that $f(x_1) < f(x_2)$ and simultaneously $g(x_1) > g(x_2)$. Note that the Sugeno integral preserves the infimum and supremum for the commonotonic functions, i.e., it is comonotonically minitive and comonotonically maxitive, (see, [7, Theorem 4.44]). A similar result for the residuum based fuzzy integral can be simply derived using the following lemma whose proof can be found in [9].

Lemma 1. *Let L be linearly ordered, and let $f, g \in \mathcal{F}(X)$. Denote $C_f = \{C_f(a) \mid a \in L\}$, where $C_f(a) = \{x \in X \mid f(x) \geq a\}$. Then C_f is a chain with respect to \subseteq , and if f and g are comonotonic, then $C_{f \star g}(a) = C_f(a)$ or $C_{f \star g}(a) = C_g(a)$ for any $a \in L$, where $\star \in \{\wedge, \vee\}$.*

Theorem 8. *Let L be linearly ordered, and let $f, g \in \mathcal{F}(X)$ be comonotonic $\mathcal{F}\text{-alg}(\mathcal{U}(L))$ -measurable functions. Then*

$$\int_{\text{DH}}^{\rightarrow} (f \vee g) \, d\nu = \int_{\text{DH}}^{\rightarrow} f \, d\nu \wedge \int_{\text{DH}}^{\rightarrow} g \, d\nu. \tag{8}$$

Note that a dual formula to (8), where the infimum is replaced by the supremum and vice versa, is not true in general even if we restrict ourselves to linearly ordered Heyting algebra (cf. Theorem 3.4 in [9] for the multiplication based fuzzy integral).

3.2 \rightarrow_{DPR} -Fuzzy Integrals

The integrated functions are again fuzzy sets on X .

Definition 6. *Let (X, \mathcal{F}, μ) be a fuzzy measure space, and let $f : X \rightarrow L$. The \rightarrow_{DPR} -fuzzy integral of f on X is given by*

$$\int_{\text{DPR}}^{\rightarrow} f \, d\mu = \bigwedge_{A \in \mathcal{F}} \left(\mu^c(A) \rightarrow \bigvee_{x \in A} f(x) \right). \tag{9}$$

Note that if $A = \emptyset$, then $\mu^c(X \setminus \emptyset) \rightarrow \bigvee \emptyset = 0 \rightarrow 0 = 1$, hence, the empty set has no influence on the value of the \rightarrow_{DPR} -fuzzy integral. The following statement presents basic properties of \rightarrow_{DPR} -fuzzy integral.

Theorem 9. For any $f, g \in \mathcal{F}(X)$ and $a \in L$, we have

- (i) $\int_{\text{DPR}}^{\rightarrow} f \, d\mu \leq \int_{\text{DPR}}^{\rightarrow} g \, d\mu$ if $f \leq g$;
- (ii) $\int_{\text{DPR}}^{\rightarrow} a_X \, d\mu = a$;
- (iii) $\int_{\text{DPR}}^{\rightarrow} a_X \otimes f \, d\mu \geq a \otimes \int_{\text{DPR}}^{\rightarrow} f \, d\mu$;
- (iv) $\int_{\text{DPR}}^{\rightarrow} a_X \rightarrow f \, d\mu \leq a \rightarrow \int_{\text{DPR}}^{\rightarrow} f \, d\mu$.

Note that the inequality (iv) of the previous theorem becomes the equality in a complete MV-algebra. An equivalent formula to (9) under the assumption of $\mathcal{F}\text{-alg}(\mathcal{L}(L))$ -measurability of functions is as follows.

Theorem 10. Let (X, \mathcal{F}, μ) be a fuzzy measure space, and let $f : X \rightarrow L$ be $\mathcal{F}\text{-alg}(\mathcal{L}(L))$ -measurable. Then

$$\int_{\text{DPR}}^{\rightarrow} f \, d\mu = \bigwedge_{a \in L} (\mu^c(\{x \in X \mid f(x) \leq a\}) \rightarrow a). \tag{10}$$

Lemma 2. Let L be linearly ordered, and let $f, g \in \mathcal{F}(X)$. Denote $B_f = \{B_f(a) \mid a \in L\}$, where $B_f(a) = \{x \in X \mid f(x) \leq a\}$. Then B_f is a chain with respect to \subseteq , and if f and g are comonotonic, then $B_{f \star g}(a) = B_f(a)$ or $B_{f \star g}(a) = B_g(a)$ for any $a \in L$, where $\star \in \{\wedge, \vee\}$.

Theorem 11. Let L be linearly ordered, and let $f, g \in \mathcal{F}(X)$ be comonotonic $\mathcal{F}\text{-alg}(\mathcal{L}(L))$ -measurable functions. Then

$$\int_{\text{DPR}}^{\rightarrow} (f \wedge g) \, d\mu = \int_{\text{DPR}}^{\rightarrow} f \, d\mu \wedge \int_{\text{DPR}}^{\rightarrow} g \, d\mu.$$

4 Integral Transforms for Lattice-Valued Functions

In this section, we propose four types of integral transforms for functions whose function values are evaluated in a complete residuated lattice. For their definitions, we use the residuum based fuzzy integral introduced in Sect. 3. The integral transforms transform fuzzy sets from $\mathcal{F}(X)$ to fuzzy sets from $\mathcal{F}(Y)$.

4.1 \rightarrow_{DH} -Integral Transforms

In this part, we propose two types of integral transform based on \rightarrow_{DH} -fuzzy integral. For their definitions we are inspired by a straightforward generalization of the lower and upper fuzzy transforms in terms of the multiplication based fuzzy integral presented in [9]. We start with the definition of \rightarrow_{DH} -integral transforms merging an integral kernel and a transformed function by the multiplication operation.

Definition 7. Let (X, \mathcal{F}, ν) be a complementary fuzzy measure space, and let $K : X \times Y \rightarrow L$ be a semi-normal in the second component fuzzy relation. A map $F_{(K, \rightarrow_{\text{DH}})}^{\otimes} : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ defined by

$$F_{(K, \rightarrow_{\text{DH}})}^{\otimes}(f)(y) = \int_{\text{DH}}^{\rightarrow} K(x, y) \otimes f(x) \, d\nu \tag{11}$$

is called a $(K, \otimes, \rightarrow_{\text{DH}})$ -integral transform.

It is easy to see that a complementary measure ν and a semi-normal in the second component fuzzy relation K are parameters of $(K, \otimes, \rightarrow_{\text{DH}})$ -integral transform. The fuzzy relation K will be called the *integral kernel*, which corresponds to the standard notation in the theory of integral transforms. Note that the semi-normality in the second component of integral kernels is considered as a natural assumption avoiding the trivial case, namely, if $K(x, y) = 0$ for any $x \in X$ and some $y \in Y$, we trivially obtain $F_{(K, \rightarrow_{\text{DH}})}^{\otimes}(f)(y) = 1$ as a consequence of $\int_{\text{DH}}^{\rightarrow} 0_X \, d\nu = \neg 0 = 1$ (see Theorem 6(b)).

Remark 1. If an integral kernel K is normal in the second component for any $y \in Y$ and, moreover, the family of sets $\{\text{Core}(K(\cdot, y)) \mid y \in Y\}$ forms a partition of X , the family of fuzzy sets $\{K(\cdot, y) \mid y \in Y\}$ is called a *fuzzy partition* of X , which is a crucial concept in the definition of lower and upper fuzzy transforms [20]. In this article, we significantly relax the concept of fuzzy partition because we require only that $K(\cdot, y) > 0$ for any $y \in Y$. Nevertheless, the fulfillment of certain integral transforms properties usually forces to introduce specific conditions for integral kernels (see Theorems 4.4 and 4.7 in [9]).

The following theorem shows basic properties of $(K, \otimes, \rightarrow_{\text{DH}})$ -integral transforms.

Theorem 12. *For any $f, g \in \mathcal{F}(X)$ and $a \in L$, we have*

- (i) $F_{(K, \rightarrow_{\text{DH}})}^{\otimes}(f) \geq F_{(K, \rightarrow_{\text{DH}})}^{\otimes}(g)$ if $f \leq g$;
- (ii) $F_{(K, \rightarrow_{\text{DH}})}^{\otimes}(f \wedge g) \geq F_{(K, \rightarrow_{\text{DH}})}^{\otimes}(f) \vee F_{(K, \rightarrow_{\text{DH}})}^{\otimes}(g)$;
- (iii) $F_{(K, \rightarrow_{\text{DH}})}^{\otimes}(f) \wedge F_{(K, \rightarrow_{\text{DH}})}^{\otimes}(g) \geq F_{(K, \rightarrow_{\text{DH}})}^{\otimes}(f \vee g)$;
- (iv) $F_{(K, \rightarrow_{\text{DH}})}^{\otimes}(a_X \otimes f) \leq a \rightarrow F_{(K, \rightarrow_{\text{DH}})}^{\otimes}(f)$;
- (v) $F_{(K, \rightarrow_{\text{DH}})}^{\otimes}(a_X \rightarrow f) \geq a \otimes F_{(K, \rightarrow_{\text{DH}})}^{\otimes}(f)$.

Moreover, if L is a complete MV-algebra, the equality in (iv) holds.

Proof. The first three statements are trivial consequences of the monotonicity of the operation \otimes (i.e., monotonically non-decreasing) and the \rightarrow_{DH} -fuzzy integral (Theorem 6(i)). Using Theorem 6(iii) and the commutativity of \otimes , for any $y \in Y$, we have

$$\begin{aligned} F_{(K, \rightarrow_{\text{DH}})}^{\otimes}(a_X \otimes f)(y) &= \int_{\text{DH}}^{\rightarrow} K(x, y) \otimes (a_X(x) \otimes f(x)) \, d\nu \\ &\leq a \rightarrow \int_{\text{DH}}^{\rightarrow} K(x, y) \otimes f(x) \, d\nu = a \rightarrow F_{(K, \rightarrow_{\text{DH}})}^{\otimes}(f)(y). \end{aligned}$$

Moreover, if L is a complete MV-algebra, the previous inequality becomes the equality and hence, the equality in (iv) holds. Since $K(x, y) \otimes (a \rightarrow f(x)) \leq a \rightarrow (K(x, y) \otimes f(x))$, using (i) and (iv) of Theorem 6, one can simply prove (v). \square

Let us continue with another type of \rightarrow_{DH} -integral transforms, where the integral kernels are combined with the transformed functions using the residuum operation.

Definition 8. Let (X, \mathcal{F}, ν) be a complementary fuzzy measure space, and let $K : X \times Y \rightarrow L$ be a semi-normal in the second component fuzzy relation. A map $F_{(K, \rightarrow_{\text{DH}})}^{\rightarrow} : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ defined by

$$F_{(K, \rightarrow_{\text{DH}})}^{\rightarrow}(f)(y) = \int_{\text{DH}}^{\rightarrow} K(x, y) \rightarrow f(x) \, d\nu \tag{12}$$

is called a $(K, \rightarrow, \rightarrow_{\text{DH}})$ -integral transform.

Note that if K is not semi-normal in the second component, i.e., $K(x, y) = 0$ for some $y \in Y$ and any $x \in X$, we trivially obtain $F_{(K, \rightarrow_{\text{DH}})}^{\rightarrow}(f)(y) = 0$ as a consequence of $\int_{\text{DH}}^{\rightarrow} 1_X \, d\nu = -1 = 0$ (see Theorem 6(b)). In what follows, we present some basic properties of the $(K, \rightarrow, \rightarrow_{\text{DH}})$ -integral transform.

Theorem 13. For any $f, g \in \mathcal{F}(X)$ and $a \in L$, we have

- (i) $F_{(K, \rightarrow_{\text{DH}})}^{\rightarrow}(f) \geq F_{(K, \rightarrow_{\text{DH}})}^{\rightarrow}(g)$ if $f \leq g$;
- (ii) $F_{(K, \rightarrow_{\text{DH}})}^{\rightarrow}(f \wedge g) \geq F_{(K, \rightarrow_{\text{DH}})}^{\rightarrow}(f) \vee F_{(K, \rightarrow_{\text{DH}})}^{\rightarrow}(g)$;
- (iii) $F_{(K, \rightarrow_{\text{DH}})}^{\rightarrow}(f \vee g) \leq F_{(K, \rightarrow_{\text{DH}})}^{\rightarrow}(f) \wedge F_{(K, \rightarrow_{\text{DH}})}^{\rightarrow}(g)$;
- (iv) $F_{(K, \rightarrow_{\text{DH}})}^{\rightarrow}(a_X \otimes f) \leq a \rightarrow F_{(K, \rightarrow_{\text{DH}})}^{\rightarrow}(f)$;
- (v) $F_{(K, \rightarrow_{\text{DH}})}^{\rightarrow}(a_X \rightarrow f) \geq a \otimes F_{(K, \rightarrow_{\text{DH}})}^{\rightarrow}(f)$.

Proof. The first three statements are trivial consequences of the monotonicity of the operation \rightarrow (i.e., monotonically non-decreasing in the second component) and the \rightarrow_{DH} -fuzzy integral (Theorem 6(i)). Since $K(x, y) \rightarrow (a \otimes f(x)) \geq a \otimes (K(x, y) \rightarrow f(x))$, then using (i) and (iii) of Theorem 6, for any $y \in Y$, we obtain

$$\begin{aligned} F_{(K, \rightarrow_{\text{DH}})}^{\otimes}(a_X \otimes f)(y) &= \int_{\text{DH}}^{\rightarrow} K(x, y) \rightarrow (a \otimes f(x)) \, d\nu \\ &\leq \int_{\text{DH}}^{\rightarrow} a \otimes (K(x, y) \rightarrow f(x)) \, d\nu \leq a \rightarrow F_{(K, \rightarrow_{\text{DH}})}^{\otimes}(f)(y). \end{aligned}$$

Since $K(x, y) \rightarrow (a \rightarrow f(x)) = a \rightarrow (K(x, y) \rightarrow f(x))$, using Theorem 6(iv), one can simply prove (v). □

One could see that, although, the \rightarrow_{DH} -integral transforms are defined by different operations, i.e., \otimes and \rightarrow , their basic properties coincide.

We showed in Theorem 8 that under the assumption of the linearity of complete residuated lattices, the \rightarrow_{DH} -fuzzy integral is a linear operator in the sense that the \rightarrow_{DH} -fuzzy integral of the supremum of comonotonic functions is the infimum of \rightarrow_{DH} -fuzzy integrals of these functions. The linearity property of \rightarrow_{DH} -fuzzy integral can be used to prove the analogous property for \rightarrow_{DH} -integral transforms.

Theorem 14. Let L be a linearly ordered and assume that the algebra \mathcal{F} is closed over arbitrary unions. Let $f, g, K(\cdot, y)$ be \mathcal{F} -alg($\mathcal{U}(L)$)-measurable for any $y \in Y$. If $K(\cdot, y) \star f$ and $K(\cdot, y) \star g$ are comonotonic for $\star \in \{\otimes, \rightarrow\}$, then

$$F_{(K, \rightarrow_{\text{DH}})}^{\star}(f \vee g) = F_{(K, \rightarrow_{\text{DH}})}^{\star}(f) \wedge F_{(K, \rightarrow_{\text{DH}})}^{\star}(g) \tag{13}$$

4.2 \rightarrow_{DPR} -Integral Transform

Similarly to the previous subsection we propose two types of integral transforms based now on the \rightarrow_{DPR} -fuzzy integral. Again we start with the definition of integral transform, where integral kernels and transformed functions are merged by the multiplication operation.

Definition 9. Let (X, \mathcal{F}, μ) be a fuzzy measure space, and let $K : X \times Y \rightarrow L$ be a semi-normal in the second component fuzzy relation. A map $F_{(K, \rightarrow_{\text{DPR}})}^{\otimes} : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ defined by

$$F_{(K, \rightarrow_{\text{DPR}})}^{\otimes}(f)(y) = \int_{\text{DPR}}^{\rightarrow} K(x, y) \otimes f(x) d\mu \tag{14}$$

is called a $(K, \otimes, \rightarrow_{\text{DPR}})$ -integral transform.

The following theorem shows several basic properties of $(K, \otimes, \rightarrow_{\text{DPR}})$ -integral transforms.

Theorem 15. For any $f, g \in \mathcal{F}(X)$ and $a \in L$, we have

- (i) $F_{(K, \rightarrow_{\text{DPR}})}^{\otimes}(f) \leq F_{(K, \rightarrow_{\text{DPR}})}^{\otimes}(g)$ if $f \leq g$;
- (ii) $F_{(K, \rightarrow_{\text{DPR}})}^{\otimes}(f \wedge g) \leq F_{(K, \rightarrow_{\text{DPR}})}^{\otimes}(f) \wedge F_{(K, \rightarrow_{\text{DPR}})}^{\otimes}(g)$;
- (iii) $F_{(K, \rightarrow_{\text{DPR}})}^{\otimes}(f \vee g) \geq F_{(K, \rightarrow_{\text{DPR}})}^{\otimes}(f) \vee F_{(K, \rightarrow_{\text{DPR}})}^{\otimes}(g)$;
- (iv) $F_{(K, \rightarrow_{\text{DPR}})}^{\otimes}(a_X \otimes f) \geq a \otimes F_{(K, \rightarrow_{\text{DPR}})}^{\otimes}(f)$;
- (v) $F_{(K, \rightarrow_{\text{DPR}})}^{\otimes}(a_X \rightarrow f) \leq a \rightarrow F_{(K, \rightarrow_{\text{DPR}})}^{\otimes}(f)$;

Proof. Similarly to the proof of Theorem 12 one can simply prove all the statements using the properties of \rightarrow_{DPR} -fuzzy integral presented in Theorem 9. \square

Definition 10. Let (X, \mathcal{F}, μ) be a fuzzy measure space, and let $K : X \times Y \rightarrow L$ be a semi-normal in the second component fuzzy relation. A map $F_{(K, \rightarrow_{\text{DPR}})}^{\rightarrow} : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ defined by

$$F_{(K, \rightarrow_{\text{DPR}})}^{\rightarrow}(f)(y) = \int_{\text{DPR}}^{\rightarrow} K(x, y) \rightarrow f(x) d\mu \tag{15}$$

is called a $(K, \rightarrow, \rightarrow_{\text{DPR}})$ -integral transform.

Some of basic properties of $(K, \rightarrow, \rightarrow_{\text{DPR}})$ -integral transform are presented in the following theorem.

Theorem 16. For any $f, g \in \mathcal{F}(X)$ and $a \in L$, we have

- (i) $F_{(K, \rightarrow_{\text{DPR}})}^{\rightarrow}(f) \leq F_{(K, \rightarrow_{\text{DPR}})}^{\rightarrow}(g)$ if $f \leq g$;
- (ii) $F_{(K, \rightarrow_{\text{DPR}})}^{\rightarrow}(f \wedge g) \leq F_{(K, \rightarrow_{\text{DPR}})}^{\rightarrow}(f) \wedge F_{(K, \rightarrow_{\text{DPR}})}^{\rightarrow}(g)$;
- (iii) $F_{(K, \rightarrow_{\text{DPR}})}^{\rightarrow}(f \vee g) \geq F_{(K, \rightarrow_{\text{DPR}})}^{\rightarrow}(f) \vee F_{(K, \rightarrow_{\text{DPR}})}^{\rightarrow}(g)$;
- (iv) $F_{(K, \rightarrow_{\text{DPR}})}^{\rightarrow}(a \otimes f) \geq a \otimes F_{(K, \rightarrow_{\text{DPR}})}^{\rightarrow}(f)$;

$$(v) F_{(K, \rightarrow_{\text{DPR}})}^{\rightarrow}(a \rightarrow f) \leq a \rightarrow F_{(K, \rightarrow_{\text{DPR}})}^{\rightarrow}(f).$$

Moreover, if L is a complete MV-algebra, the equality in (v) holds.

Again one could notice that although, the \rightarrow_{DPR} -integral transforms are defined by different operations, their basic properties are identical. The following linear property ensured for comonotonic functions is a straightforward consequence of Theorem 11.

Theorem 17. *Let L be a linearly ordered and assume that the algebra \mathcal{F} is closed over arbitrary unions. Let $f, g, K(\cdot, y)$ be $\mathcal{F}\text{-alg}(\mathcal{L}(L))$ -measurable for any $y \in Y$. If $K(\cdot, y) \star f$ and $K(\cdot, y) \star g$ are comonotonic for $\star \in \{\otimes, \rightarrow\}$, then*

$$F_{(K, \rightarrow_{\text{DPR}})}^{\star}(f \wedge g) = F_{(K, \rightarrow_{\text{DPR}})}^{\star}(f) \wedge F_{(K, \rightarrow_{\text{DPR}})}^{\star}(g) \quad (16)$$

5 Conclusion

In this article, we introduced four types of integral transforms, where we used the residuum based fuzzy (qualitative) integrals, namely, the \rightarrow_{DH} -fuzzy integral proposed by Dvořák and Holčapek in [4] and the \rightarrow_{DPR} -fuzzy integral proposed by Dubois, Prade and Rico in [3]. We presented some of the basic properties of the residuum based fuzzy integrals including a linearity property for the comonotonic functions which holds in the linearly ordered complete residuated lattices. Using these properties we provided an initial analysis of elementary properties of proposed integral transforms. The further development of the theory of integral transforms for residuated lattice-valued functions is a subject of our future research, where, among others, we want to focus on the seeking of inverse integral kernels to be able to approximate the original functions from the transformed functions. Our motivation comes from the relationship between the lower and upper fuzzy transforms and their related inverse fuzzy transforms.

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