Gold Price: Trend-Cycle Analysis Using Fuzzy Techniques

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Abstract. In this paper, we apply special fuzzy techniques to analyze the gold price historical data. The main tools are the higher degree fuzzy transform and specific methods of fuzzy natural logic. First, we show how to apply the former for the estimation of the trend-cycle. Then, we provide methodologies for identifying monotonous periods in the trend-cycle and describe them by sentences in natural language.

Keywords: Fuzzy transform · Fuzzy modeling · Financial time series · Data mining

1 Introduction

The crises of the global financial environment in recent years have put many financial markets into complicated situations, e.g., the drop in oil prices, the unusual fluctuation of gold prices, the unprecedented boom of digital currencies. These make it difficult for people who would like to optimize their profit when trading in such financial markets. Facing this situation, any knowledge of how a market behaves is significantly helpful. In this paper, we devote several techniques to mine information behind a market based on its historical data. Our focus is on the gold market.

First, we apply the fuzzy transform (F-transform) technique for estimation of the trend-cycle of the gold price historical data. The F-transform is a fuzzy approximation technique proposed by I. Perfilieva in [21] and later elaborated by several authors in [5, 10, 13, 22] and elsewhere. It has been successfully applied in many branches of applied sciences [6, 8, 12, 14, 25, 26], and especially, in time series analysis [3, 4, 17, 18, 27] that inspires our investigation in this paper. From those contributions, it is known that the trend-cycle of a time series can be estimated by the F-transform technique, successfully. However, the quality of this estimation strongly depends on practical experience in setting parameters of the latter. Therefore, in addition to the application of the fuzzy transform to the trend-cycle estimation, we introduce a technique for choosing parameters to make it possible to achieve an efficient estimation without requiring much practical experience.

To help investors have a better understanding of the behavior of a market (particularly the gold market), we introduce the concept of bull and bear periods on the trend-cycle to characterize the monotonous stages on it. These concepts are inspired through the notions of bull and bear markets in finance [2, 19]. For practical purposes, an algorithm is provided for identifying bull and bear periods on the estimated trend-cycle.

Finally, we employ one of the important tasks in mining information from time series that is to extract linguistic characterization of the trend-cycle. Let us note that mining linguistic information or time series summarization has been studied for quite a while, recently. There are several approaches that have been proposed to this issue such as [1, 9, 15, 20, 24, 28] and elsewhere. Being motivated by fuzzy natural logic techniques used for describing the behavior of the trend-cycle [15], we develop a methodology to extract linguistic characteristics of bull and bear periods on the trend-cycle of the gold price market and represent them by sentences.

The paper is structured as follows. The next section provides a brief introduction to the (higher degree) F-transform and specific tools in fuzzy natural logic. The main contribution of this paper is described in Sect. 3, where we analyze the trend-cycle of the gold price data. This consists of the estimating of the trend-cycle, identifying of monotonous periods on it, and describing its course in natural language (by sentences). The last section is the conclusion.

2 Preliminaries

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the set of natural numbers, integers and real numbers, respectively.

2.1 The Higher Degree Fuzzy Transform

The central notion in the theory of fuzzy transform is the fuzzy partition. Standardly, a fuzzy partition is a set of fuzzy sets on $\mathbb{R}$ that satisfy the Ruspini's condition.¹ Together with the development of the fuzzy transform, this concept has several modified versions such as: the generalized (uniform) fuzzy partition [7], the adjoint fuzzy partition [23]. For the practical purposes of this paper, we restrict our consideration on a particularly simple one, called the triangular generalized uniform fuzzy partition.

Definition 1. Let $t_0 \in \mathbb{R}$, and two positive constants h and r be such that $h/r \in \mathbb{Z}$. Let $\mathcal{A} = \{A_k \mid k \in \mathbb{Z}\}$ be a set of fuzzy sets on $\mathbb{R}$, defined by

$$A_k(t) = \frac{r}{h} \cdot \max \left\{ 1 - \left| \frac{t - t_0 - kr}{h} \right|, 0 \right\}, \quad k \in \mathbb{Z}.$$

¹ The sum of all membership functions corresponding to fuzzy sets in a fuzzy partition over each point in $\mathbb{R}$ is one.

Then, $\mathcal{A}$ is called a *triangular generalized uniform fuzzy partition of the real line* $\mathbb{R}$ determined by the triplet (t_0, h, r) . Each fuzzy set in this fuzzy partition is called a *basic function*.

Note that basic functions determined in Definition 1 need not to be normal, because there can be more than one basic function covering their peak, in general (see [7]).

In the sequel, since t_0 does not affect the theoretical results concerning the fuzzy transform, for the sake of simplicity, we restrict our investigation to the fuzzy partitions with $t_0 = 0$. Moreover, we fix $r = h/2$. Therefore, we omit the reference both to t_0 as well as r in the triplet (h, r, t_0) and deal with h only. We call h the *bandwidth* of the fuzzy partition.

The higher degree fuzzy transform (F^m -transform, $m \in \mathbb{N}$) is a fuzzy approximation technique that can provide both local and global approximation of functions. The first phase, called *direct F^m -transform*, transforms a given function to a family of polynomials of degree up to m . These polynomials are orthogonal projections of the given function onto polynomial approximation spaces concerning specific inner products defined with respect to basic functions of a fuzzy partition. Below, we briefly introduce how the direct F^m -transform of a function is computed.

Let f be a locally square Lebesgue integrable function on $\mathbb{R}$, and $\mathcal{A}$ be a fuzzy partition of the latter in the sense of Definition 1. Let $K : \mathbb{R} \rightarrow [0, 1]$ be defined by²

$$K(t) = \frac{1}{2} \cdot \max\{1 - |t|, 0\}.$$

The *direct F^m -transform of f* with respect to $\mathcal{A}$ is the family

$$F_{\mathcal{A}}^m[f] = \{F_k^m[f] \mid k \in \mathbb{Z}\},$$

where, for any $k \in \mathbb{Z}$,

$$F_k^m[f](t) = C_{k,0} + C_{k,1} \left(\frac{t - t_k}{h} \right) + \dots + C_{k,m} \left(\frac{t - t_k}{h} \right)^m, \quad t \in [t_k - h, t_k + h],$$

with $t_k = \frac{kh}{2}$,

$$(C_{k,0}, C_{k,1}, \dots, C_{k,m})^T = (\mathcal{Z}_m)^{-1} \cdot \mathcal{Y}_{m,k}, \quad (1)$$

where $\mathcal{Z}_m = (Z_{ij})$ is an $(m+1) \times (m+1)$ invertible matrix defined by

$$Z_{ij} = \int_{-1}^1 t^{i+j-2} K(t) dt, \quad i, j = 1, \dots, m+1,$$

and $\mathcal{Y}_{m,k} = (Y_{k,1}, \dots, Y_{k,m+1})^T$ is defined by

$$Y_{k,\ell} = \int_{-1}^1 f(ht + t_k) \cdot t^{\ell-1} K(t) dt, \quad \ell = 1, \dots, m+1. \quad (2)$$

² This function is known as a generating function of the fuzzy partition $\mathcal{A}$ (see in [7]).

The polynomial $F_k^m[f]$ is called the k -th component of the direct F^m -transform of f . It provides an approximation of the latter on the region covered by the basic function A_k .

The second phase called the *inverse F^m -transform* of f transforms the vector of components to a function defined as the linear-like combination of the direct F -transform components and the corresponding basic functions, i.e.,

$$\hat{f}_A^m(t) = \sum_{k \in \mathbb{Z}} F_k^m[f](t) \cdot A_k(t), \quad t \in \mathbb{R}.$$

Being motivated by fuzzy natural logic techniques used for characterizing behavior of the trend-cycle in time series, we develop a methodology to extract linguistic characteristics of bull and bear periods on the trend-cycle of the gold price market and represent them by sentences. This function provides an approximation to f on its domain $\mathbb{R}$ (global approximation). The quality of this approximation depends on the settings of fuzzy partition $\mathcal{A}$, particularly, the bandwidth h . By contrast, under reasonable setting of the bandwidth, the inverse F^m -transform can suppress high frequencies in the function f . Namely, $\hat{f}_A^m$ is a smoothed function of the latter. In the sequel, when needing to emphasize the influence of the bandwidth to the inverse function, we use the notation $\hat{f}_h^m$.

2.2 Evaluative Linguistic Expressions: A Formal Theory in Fuzzy Natural Logic

The fuzzy natural logic (FNL) was established as the formal logic aiming at modeling of natural human reasoning which proceeds in natural language. It is an extension of *mathematical fuzzy logic*, and its paradigm extends the classical concepts of *natural logic* suggested by Lakoff in [11]. FNL is a class of several formal theories such as: theory of evaluative linguistic expressions, theory of fuzzy IF-THEN rules, etc. For the purposes of this paper, we only focus on the theory of evaluative linguistic expressions.

Evaluative linguistic expressions are special expressions of natural language that are used to specify the course of development of some processes, to evaluate a phenomenon or to make a decision. These expressions may have a complicated structure, but in this paper, we simply consider evaluative linguistic expressions of the following form:

$$\langle \text{linguistic hedge} \rangle \langle \text{atomic evaluative expression} \rangle, \quad (3)$$

where the atomic evaluative expression comprises any of the *canonical adjectives*: small, medium, big, and the *linguistic hedge* is a specific adverb, for example, extremely, significantly, very, rather, roughly, very roughly. The linguistic hedge makes the meaning of the atomic expression more or less precise. An evaluative expression of the form (3) will be denoted by a script letter, for example, $\mathcal{A}$ or $\mathcal{B}$.

When using an evaluative expression to evaluate values of a variable X , the resulting expression is of the form

$$X \text{ is } \mathcal{A}, \quad (4)$$

is called the *evaluative (linguistic) predication*. To make such a predication meaningful, one must specify the *context* (the state of the world) in which variable X is considered. The context is characterized by a triplet $w = \langle v_L, v_M, v_R \rangle$, where $v_L, v_M, v_R \in \mathbb{R}$ and $v_L < v_M < v_R$. These numbers characterize the minimal, typically middle, and maximal values, respectively, of the evaluated characteristics (e.g., “height”, “distance”) in the specified context of use. Only when a context is specified to X , the meaning of the predication in (4) is well-defined and interpreted by a fuzzy set on $\mathbb{R}$. Moreover, if a context w is known then one can determine an evaluative linguistic expression $\mathcal{A}$ characterizing a given value x_0 of X by using a function of local perception

$$LPerc(x_0, w) = \mathcal{A}.$$

For the details of this function as well as tools in FNL, we refer to the book [16].

3 Gold Price: Trend-Cycle Model and Analysis

Our investigation employs the Gold Future Historical Data published in the site [investing.com](https://www.investing.com). This is the daily data of the price, the open, high, and low prices in the US Dollar of one Troy Ounce of gold. Our methodology is focused on the price only. A similar analysis can be applied to the rest of the data. This section aims at developing techniques based on the fuzzy transform and tools developed in the theory of evaluative linguistic expressions for estimation of the trend-cycle of the data, classifying periods in the trend-cycle into specific stages concerning its monotonousness, and finally, for extraction of the linguistic characterization of the course of the trend-cycle. Established methods are applied to the data from October 1, 2018, to December 3, 2019, of the length 312, as presented in Fig. 1.

Fig. 1. The daily gold price from October 1, 2018, to December 3, 2019.

3.1 Time Series Model

Let us consider the gold price values as a time series $\{X(t), t \in \mathbb{T}\}$, where $\mathbb{T} = \{0, 1, \dots, T\}$. Standardly, a time series can be additively (multiplicatively) decomposed into a trend-cycle, a seasonal component and an irregular fluctuation. However, we know that the influence of the seasonality to the gold market is weak, and even, unclear. Therefore, let us assume that the time series X can be decomposed as follows:

$$X(t) = TC(t) + R(t), \quad t \in \mathbb{T}, \quad (5)$$

where TC is the trend-cycle component and $R(t)$ is a realization of a random process characterized by quick decay of its autocorrelation function ρ , i.e.,

$$\int_0^\ell |\rho(\tau)| d\tau = o(\ell), \quad \text{for } \ell \rightarrow \infty. \quad (6)$$

3.2 Trend-Cycle Estimation

This task has been elaborated in several papers as mentioned in the introduction. Most of these investigations are to show that the (higher degree) F-transform is a good technique for estimating the trend-cycle of a time series. More precisely, the inverse F^m -transform of X provides an estimation of the trend-cycle TC ,

$$TC(t) \approx \hat{X}_h^m(t), \quad t \in \mathbb{T}. \quad (7)$$

However, one can see that the quality of this estimation depends on the degree m of the fuzzy transform and the construction of the fuzzy partition (the bandwidth h). In what follows, we describe a methodology for setting these parameters such that the trend-cycle is well estimated by (7) without requiring much practical experience.

- *Choosing of the degree m :* The more the time series changes its course (or has higher volatility), the higher the degree m should be chosen. A rule of thumb says that if the observed trend-cycle is a nearly linear function then we should choose $m = 0$ or $m = 1$; otherwise, choose $m = 2$ or $m = 3$.
- *Choosing of the bandwidth h :* Let $\tilde{R}_h(t) = X(t) - \hat{X}_h^m(t)$ and $\hat{\rho}_h$ be its sample autocorrelation function. The chosen bandwidth h_0 is the value satisfying that

$$\sum_{k=0}^{N_{max}} |\hat{\rho}_{h_0}(k)| = \min \left\{ \sum_{k=0}^{N_{max}} |\hat{\rho}_h(k)| \mid h = 1, 2, \dots, H_{max} \right\},$$

where N_{max} and H_{max} are two positive integers chosen by the users.

Applying these rules to the gold price data displayed in Fig. 1, we find $m = 2$ with the bandwidth $h_0 = 6$. Namely, the trend-cycle of the considered data is estimated by the inverse F^2 -transform with respect to the fuzzy partition determined as in Definition 1 with $h = 6$. The estimated trend-cycle is depicted in Fig. 2.

Fig. 2. The trend-cycle estimated by the fuzzy transform (dark/blue line). (Color figure online)

3.3 Identification of Monotonous Periods on the Trend-Cycle

This section provides a method for identification of monotonous periods on the trend-cycle TC of time series X . These periods are determined based on duration and market move constraints. Inspired by the notion of bull and bear markets in finance [2, 19], we classify monotonous periods into three categories: bull, bear and neutral ones, corresponding to the increasing, decreasing and stagnating of the trend-cycle. To characterize the monotonousness of the trend-cycle, we use the forward difference defined as follows.

Definition 2. Let $f(t)$, $t = 1, 2, \dots, N$ be a discrete function. The forward difference $\Delta[f]$ of f is a discrete function determined by

$$\Delta[f](t) = f(t+1) - f(t), \quad t = 1, 2, \dots, N-1.$$

Below, we define the bull, bear and neutral periods in the trend-cycle.

Definition 3. Let $S = \{s, s+1, \dots, s+p\}$ be a time period in $\mathbb{T}$. Let β_{bull} , β_{bear} and L_{min} be three positive real numbers. The period $S = \{TC(t) \mid t \in S\}$ in TC is said to be a bull (or bear) period with respect to β_{bull} (or β_{bear}) and L_{min} if the following statements hold true:

- (i) $\Delta[TC](t) > 0$ (or $\Delta[TC](t) < 0$), for any $t \in S$,
- (ii) If $S' \supseteq S$ is a time period satisfying that $\Delta[TC](t) > 0$ (or $\Delta[TC](t) < 0$), for any $t \in S'$, then $S' = S$,
- (iii) $\frac{TC(s+p) - TC(s)}{p} \geq \beta_{bull}$ (or $\frac{TC(s+p) - TC(s)}{p} \leq -\beta_{bear}$) and $p+1 \geq L_{min}$.

S is said to be a neutral period if it is neither bull nor bear period.

In Definition 3, L_{min} is the smallest length of the bull and bear periods, while β_{bull} and β_{bear} are thresholds characterize the steepness of the trend-cycle. From (iii), one can see that β_{bull} and β_{bear} are chosen to categorize long

stages with little change of the trend-cycle to the neutral periods. In practice, L_{min} is the shortest time period, chosen by traders based on the time frame of their trading strategy. Knowing bull and bear periods with lengths greater than L_{min} is significantly insightful. For example, if one would like to trade on the gold market weekly based on the given data, he would choose $L_{min} = 5$. Moreover, β_{bull} and β_{bear} are chosen based on the market move and the duration constraints. Let us assume that one is interested to discover bull (or bear) periods in the gold market with at least 20\$ jump in at most a half of the month (10 days). In this specific case, he should choose $\beta_{bull} = 2$ (or $\beta_{bear} = 2$).³

In what follows, we describe an informal algorithm for identifying the bull and bear periods on a trend-cycle.

Algorithm 1. Identification of the bull, bear and neutral period in the trend-cycle TC of X

Input:

trend-cycle $TC(t)$, $t = 0, 1, \dots, T$;
three thresholds β_{bull} , β_{bear} , and L_{min} ;

Processing:

compute $\Delta[TC](t)$, $t = 0, 1, \dots, T - 1$;
classify points $TC(t)$, $t = 0, 1, \dots, T - 1$ into distinct segments based on the sign (positive and negative) of $\Delta[TC](t)$;
evaluate each obtained segment with respect to the thresholds β_{bull} , β_{bear} , and L_{min} to determine bull, bear and neutral periods;

Output:

bull, bear and neutral periods in TC and their position

Apply Algorithm 1 to the considered gold price data where the inputs are the estimated trend-cycle obtained in the previous subsection, $L_{min} = 5$ and $\beta_{bull} = \beta_{bear} = 2$. We obtain the result as in Fig. 3.

3.4 Evaluation of the Trend-Cycle in Natural Language

In this subsection, we describe how the course of the trend-cycle can be evaluated in natural language. Namely, we focus on the bull and bear periods of the trend-cycle TC and introduce a method for generating the linguistic characterization of these stages. Each bull and bear period is characterized by the following sentence,

$$\begin{aligned} \text{Period is } \langle \text{price change evaluation} \rangle \\ \text{in } \langle \text{time period evaluation} \rangle \text{ period,} \end{aligned} \quad (8)$$

³ β_{bull} and β_{bear} can be different. The setting is up to users.

Fig. 3. The bull (dark/blue lines), bear (dotted/red lines) and neutral (grey/green lines) periods on the estimated trend-cycle. (Color figure online)

where

$$\langle \text{price change evaluation} \rangle := \langle \text{evaluative expression 1} \rangle \langle \text{sign} \rangle,$$

$$\langle \text{time period evaluation} \rangle := \langle \text{evaluative expression 2} \rangle,$$

with

$$\langle \text{evaluative expression 1} \rangle := \langle \text{linguistic hedge} \rangle \langle \text{weak} \mid \text{moderate} \mid \text{strong} \rangle,$$

$$\langle \text{sign} \rangle := \langle \text{increasing} \mid \text{decreasing} \rangle,$$

$$\langle \text{evaluative expression 2} \rangle := \langle \text{linguistic hedge} \rangle \langle \text{short} \mid \text{medium} \mid \text{long} \rangle.$$

Neutral stages are evaluated as “stagnating”.

Let $S = \{TC(t) \mid t = s, s+1, \dots, s+p\}$ be a bull (bear) period on the trend-cycle TC . From Definition 3, this stage is characterized by two essential components: the change of price and the length of the stage. Let δ_S and l_S be the change of price and the length of S , respectively. In this case, $\delta_S = TC(s+p) - TC(s)$ and $l_S = p+1$. Then,

$$\langle \text{evaluative expression 1} \rangle = LPerc(|\delta_S|, w_1),$$

$$\langle \text{sign} \rangle = \begin{cases} \text{increasing,} & \delta_S > 0 \\ \text{decreasing,} & \delta_S < 0 \end{cases},$$

$$\langle \text{evaluative expression 2} \rangle = LPerc(l_S, w_2),$$

where w_1 and w_2 are the contexts corresponding to the change of price and the length of periods, respectively. For details how to set the context, we refer the readers to [16].

In what follows, we apply the proposed technique for extracting the linguistic characteristics of the trend-cycle of the considered gold price data depicted

Table 1. Linguistic characterization of the trend-cycle.

Time period	Identified stage	Linguistic characterization
[0, 3]	neutral	stagnating
[3, 12]	bull	ML moderate increasing in VR short period
[12, 24]	neutral	stagnating
[24, 30]	bear	Ro weak decreasing in Ra short period
[30, 38]	neutral	stagnating
[38, 48]	bull	QR weak increasing in ML medium period
[48, 51]	neutral	stagnating
[51, 74]	bull	Ty moderate increasing in Ra long period
[74, 80]	bear	Ro weak decreasing in Ra short period
[80, 89]	bull	Ra moderate increasing in VR short period
[89, 98]	neutral	stagnating
[98, 104]	bull	ML weak increasing in Ra short period
[104, 113]	bear	Ra moderate decreasing in VR short period
[113, 126]	bull	QR weak increasing in Ty medium period
[126, 132]	bear	QR weak decreasing in Ra short period
[132, 137]	neutral	stagnating
[137, 145]	bear	VR weak decreasing in QR short period
[145, 151]	bull	Si weak increasing in Ra short period
[151, 155]	neutral	stagnating
[155, 161]	bull	Ex weak increasing in Ra short period
[161, 168]	bear	Ra weak decreasing in ML short period
[168, 171]	neutral	stagnating
[171, 180]	bull	Ra moderate increasing in VR short period
[180, 185]	neutral	stagnating
[185, 194]	bull	ML strong increasing in VR short period
[194, 202]	neutral	stagnating
[202, 212]	bull	Ro weak increasing in ML medium period
[212, 216]	neutral	stagnating
[216, 229]	bull	Si strong increasing in Ty medium period
[229, 233]	neutral	stagnating
[233, 239]	bull	QR weak increasing in Ra short period
[239, 244]	neutral	stagnating
[244, 251]	bear	Ra moderate decreasing in ML short period
[251, 258]	bull	ML weak increasing in ML short period
[258, 264]	bear	VR weak decreasing in Ra short period
[264, 268]	neutral	stagnating
[268, 275]	bear	Ra weak decreasing in ML short period
[275, 287]	neutral	stagnating
[287, 293]	bear	Ra moderate decreasing in Ra short period
[293, 299]	bull	Ve weak increasing in Ra short period
[299, 305]	bear	Ra weak decreasing in Ra short period
[305, 311]	neutral	stagnating

in Fig. 3. Since the exact trend-cycle is unknown, we mine the information on its estimation. Namely, we form the linguistic characterization of the bull, bear, and neutral periods (cf. Fig. 3). To do this, we set $w_1 = \langle 9, 43, 92 \rangle$ and $w_2 = \langle 5, 14, 26 \rangle$. The obtained results are presented in Table 1, where linguistic hedges are abbreviated as follows: Ex (Extremely), Si (Significantly), Ra (Rather), Ve (Very), ML (More or Less), Ro (Roughly), QR (Quite Roughly), Ty (Typically) and VR (Very Roughly). From this table, one can extract sentences characterizing the behavior of the bull and bear stages on the trend-cycle of the given data, for example, “the bull period on [51, 74] is Ty moderate increasing in Ra long period”, or “the bear period on [137, 145] is VR weak decreasing in QR short period”.

4 Conclusions

We applied special fuzzy techniques, consisting of the F-transform and the FNL techniques, for analyzing the gold price historical data. We focused on three critical tasks in mining information of financial time series, including in estimating the trend-cycle, classifying periods in the trend-cycle into bull, bear and neutral stages, and extracting linguistic characterization of the course of the trend-cycle. In this paper, we restricted our consideration on the gold market. However, since the suggested methodologies are described in general schemes, this makes it possible to apply them to other financial data. This is one of the topics for our future researches.

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