On the Relationship Among Relational Categories of Fuzzy Topological Structures

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Abstract. Relational variants of categories of Čech closure or interior L -valued operators, categories of L -fuzzy pretopological and L -fuzzy coretopological operators, category of L -valued fuzzy relation, categories of upper and lower F -transforms and the category of spaces with fuzzy partitions are introduced. The existence of relationships defined by functors among these categories are investigated and a key role of a relational category of spaces with fuzzy partitions is described.

1 Introduction

In this paper we want to build on our previous paper [8] and to analyse the relationships between categories that represent various structures generally included under the term fuzzy topological structures. These structures include variants of fuzzy topological spaces, fuzzy rough sets, fuzzy approximation spaces, fuzzy closure operators, fuzzy pretopological operators and their dual terms. In contrast to the original paper [8], however, the relationships between these fuzzy structures are not represented by classical mappings between supports of these fuzzy structures, but in a more general way, i.e., as a fuzzy relations or a fuzzy relations with other properties. This more general approach is based on current trends in the field of fuzzy structures which are based on the application of *fuzzy relations* as morphisms in suitable categories. A typical example of that use of fuzzy relations is the category of sets as objects and L -valued fuzzy relations between sets as morphisms which is frequently used in approximation theory.

The main result of this paper are two theorems describing the existence of functors among categories of these generalized lattice-valued fuzzy topological structures, where a given lattice L is either a complete residuated lattice or a complete MV -algebra. From both these theorems it follows the key position of a relational version of the category of spaces with fuzzy partitions which represents a structure from which all other considered fuzzy topological structures can be derived.

This research was partially supported by the ERDF/ESF project CZ.02.1.01/0.0/0.0/17-049/0008414.

2 Preliminaries

In this section we repeat basic terminology from residuated lattices and we also introduce principal categories and some subcategories we use in the paper. To be self-contained, in this section we repeat most of definitions related to these structures.

We refer to [6, 9] for additional details regarding residuated lattices.

Definition 1. A residuated lattice is an algebra $\mathcal{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ such that

1. $(L, \wedge, \vee, 0, 1)$ is a bounded lattice with the least element 0 and the greatest element 1;
2. $(L, \otimes, 1)$ is a commutative monoid, and
3. $\forall a, b, c \in L, a \otimes b \leq c \iff a \leq b \rightarrow c$.

A residuated lattice $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is *complete* if it is complete as a lattice.

The following is the derived unary operations of *negation* $\neg$:

$$\neg a = a \rightarrow 0,$$

A residuated lattice $\mathcal{L}$ is called an *MV-algebra* if it satisfies $(a \rightarrow b) \rightarrow b = a \vee b$. Throughout this paper, a complete residuated lattice $\mathcal{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ will be fixed. For simplicity, instead of $\mathcal{L}$ we use only L if there is no danger of misunderstanding.

Let X be a nonempty set and L^X be a set of all L -fuzzy sets (= L -valued functions) of X . For all $\alpha \in L$, $\underline{\alpha}(x) = \alpha$ is a constant L -fuzzy set on X . For all $u \in L^X$, the *core*(u) is a set of all elements $x \in X$, such that $u(x) = 1$. An L -fuzzy set $u \in L^X$ is called *normal*, if $\text{core}(u) \neq \emptyset$. An L -fuzzy set $\chi_{\{y\}}^X \in L^X$ is a *singleton*, if it has the following form

$$\chi_{\{y\}}^X(x) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{otherwise.} \end{cases}$$

We repeat basic definitions of above mentioned fuzzy topological structures. The original notions of Kuratowski closure and interior operators were introduced in several papers, see [1–5]. In the paper we use a more general form of these operators, called Čech operators or preclosure operators, where the idempotence of operators is not required.

Definition 2. The map $i : L^X \rightarrow L^X$ is called a Čech (L -fuzzy) interior operator, if for every $\underline{\alpha}, u, v \in L^X$, it fulfills

1. $i(\underline{\alpha}) = \underline{\alpha}$,
2. $i(u) \leq u$,
3. $i(u \wedge v) = i(u) \wedge i(v)$.

We say that a Čech interior operator $i : L^X \rightarrow L^X$ is a *strong Čech-Alexandroff interior operator*, if

$$i(\underline{\alpha} \rightarrow u) = \alpha \rightarrow i(u) \quad \text{and} \quad i\left(\bigwedge_{j \in J} u_j\right) = \bigwedge_{j \in J} i(u_j).$$

Definition 3. The map $c : L^X \rightarrow L^X$ is called a *Čech (L -fuzzy) closure operator*, if for every $\underline{\alpha}, u, v \in L^X$, it fulfils

1. $i(\underline{\alpha}) = \underline{\alpha}$,
2. $i(u) \geq u$,
3. $i(u \vee v) = i(u) \vee i(v)$.

We say that a Čech closure operator $c : L^X \rightarrow L^X$ is a *strong Čech-Alexandroff closure operator*, if

$$c(\underline{\alpha} \otimes u) = \alpha \otimes c(u) \quad \text{and} \quad c\left(\bigvee_{j \in J} u_j\right) = \bigvee_{j \in J} c(u_j).$$

We remind the notion of an L -fuzzy pretopological space and L -fuzzy co-pretopological space as it has been introduced in [11].

Definition 4. An L -fuzzy pretopology on X is a set of functions $\tau = \{p_x \in L^{L^X} : x \in X\}$, such that for all $u, v \in L^X, \alpha \in L$ and $x \in X$,

1. $p_x(\underline{\alpha}) = \alpha$,
2. $p_x(u) \leq u(x)$,
3. $p_x(u \wedge v) = p_x(u) \wedge p_x(v)$.

We say that an L -fuzzy pretopological space (X, τ) is a *strong Čech-Alexandroff L -fuzzy pretopological space*, if

$$p_x(\underline{\alpha} \rightarrow u) = \alpha \rightarrow p_x(u) \quad \text{and} \quad p_x\left(\bigwedge_{j \in J} u_j\right) = \bigwedge_{j \in J} p_x(u_j).$$

Definition 5. An L -fuzzy co-pretopology on X is a set of functions $\eta = \{p^x \in L^{L^X} : x \in X\}$, such that for all $u, v \in L^X, \alpha \in L$ and $x \in X$,

1. $p^x(\underline{\alpha}) = \alpha$,
2. $p^x(u) \geq u(x)$,
3. $p^x(u \vee v) = p^x(u) \vee p^x(v)$.

We say that an L -fuzzy co-pretopological space (X, τ) is a *strong Čech-Alexandroff L -fuzzy co-pretopological space*, if

$$p^x(\underline{\alpha} \otimes u) = \alpha \otimes p^x(u) \quad \text{and} \quad p^x\left(\bigvee_{j \in J} u_j\right) = \bigvee_{j \in J} p^x(u_j).$$

We recall the notion of an L -fuzzy partition (see [7] or [10]), which is the basic structure for lattice-valued fuzzy transform, introduced in [10].

Definition 6. A set $\mathcal{A}$ of normal fuzzy sets $\{A_\alpha : \alpha \in \Lambda\}$ in X is an L -fuzzy partition of X , if

1. the corresponding set of ordinary subsets $\{\text{core}(A_\alpha) : \alpha \in \Lambda\}$ is a partition of X , and
2. $\text{core}(A_\alpha) = \text{core}(A_\beta)$ implies $A_\alpha = A_\beta$.

Instead of the index set Λ from $\mathcal{A}$ we use $|\mathcal{A}|$.

We need the following notation. If $R : X \times Y \rightarrow L$ is an L -fuzzy relation, then the upper and lower approximation maps $R^\uparrow : L^X \rightarrow L^Y$ and $R^\downarrow : L^Y \rightarrow L^X$ are defined by

$$\begin{aligned} t \in L^X, y \in Y, \quad R^\uparrow(t)(y) &= \bigvee_{x \in X} t(x) \otimes R(x, y), \\ s \in L^Y, x \in X, \quad R^\downarrow(s)(x) &= \bigwedge_{y \in Y} R(x, y) \rightarrow s(y). \end{aligned}$$

Finally, from a set X and a lattice-valued fuzzy partition $\mathcal{A}$ defined on X we can construct examples of upper and lower approximation maps, defined by special fuzzy relations which is derived from a space with a fuzzy partition $(X, \mathcal{A})$. In fact, we can define a fuzzy relation $R : X \times |\mathcal{A}| \rightarrow L$ by

$$x \in X, \lambda \in |\mathcal{A}|, \quad R(x, \lambda) = A_\lambda(x),$$

where $\mathcal{A} = \{A_\lambda : \lambda \in |\mathcal{A}|\}$. Fuzzy approximation operators $R^\uparrow$ and $R^\downarrow$ derived from R are then called upper and lower F -transforms based on a fuzzy partition $\mathcal{A}$. Namely, we have

Definition 7. An upper F -transform on a set X defined by a fuzzy partition $\mathcal{A}$ is a map $F_{X, \mathcal{A}}^\uparrow : L^X \rightarrow L^{|\mathcal{A}|}$, such that

$$\forall u \in L^X, \lambda \in |\mathcal{A}|, \quad F_{X, \mathcal{A}}^\uparrow(u)(\lambda) = \bigvee_{x \in X} A_\lambda(x) \otimes u(x).$$

Definition 8. A lower F -transform on a set X defined by a fuzzy partition $\mathcal{A}$ is a map $F_{X, \mathcal{A}}^\downarrow : L^X \rightarrow L^{|\mathcal{A}|}$, such that

$$\forall u \in L^X, \lambda \in |\mathcal{A}|, \quad F_{X, \mathcal{A}}^\downarrow(u)(\lambda) = \bigwedge_{x \in X} A_\lambda(x) \rightarrow u(x).$$

In the previous paper [8] we discussed the issue of relations, i.e., functors, between categories that have relationships to categories of fuzzy topological structures and whose common features are that morphisms in these categories are special mappings between sets. In this paper, which is a natural continuation of the previous paper [8], we will look at the relationships between analogous categories, where, unlike the original categories, morphisms are defined as special L -valued relations. This shift in understanding relationships between fuzzy objects corresponds to the current trend, where fuzzy relations between different fuzzy structures are of increasing importance.

The categories we will deal with in the paper have the same objects as the categories introduced in [8], the only difference concerns morphisms. Instead of classical maps between sets we use special fuzzy relations as morphisms. To emphasize the link between the original categories in [8], if the original category was labelled **K**, the new category with relational morphisms will be labelled **RK**.

Definition 9. *In what follows by X, Y we denote sets from the standard category **Set** and by $\cdot$ a composition of morphisms in this category.*

1. The category **RCInt** is defined by
 - (a) objects are pairs (X, i) , where $i : L^X \rightarrow L^X$ is a Čech L -fuzzy interior operator,
 - (b) $R : (X, i) \rightarrow (Y, j)$ is a morphism, if $R : X \times X \rightarrow L$ is an L -fuzzy relation and

$$i.R^\downarrow \geq R^\downarrow.j.$$

2. The category **RCClo** is defined by
 - (a) objects are pairs (X, c) , where $c : L^X \rightarrow L^X$ is a Čech L -fuzzy closure operator,
 - (b) $R : (X, c) \rightarrow (Y, d)$ is a morphism, if $R : X \times Y \rightarrow L$ is an L -fuzzy relation, and

$$R^\uparrow.c \leq d.R^\uparrow.$$

3. The category **RFPreTop** is defined by
 - (a) objects are L -fuzzy pretopological spaces (X, τ) ,
 - (b) $R : (X, \tau) \rightarrow (Y, \sigma)$ is a morphism, where $\tau = \{p_x \in L^{L^X} : x \in X\}$, $\sigma = \{q_y \in L^{L^Y} : y \in Y\}$, if $R : X \times Y \rightarrow L$ is an L -fuzzy relation, and for all $x \in X$,

$$\bigwedge_{z \in Y} (R(x, z) \rightarrow q_z) \leq p_x.R^\downarrow.$$

4. The category **RFcoPreTop** is defined by
 - (a) objects are L -fuzzy co-pretopological spaces (X, τ) ,
 - (b) $R : (X, \tau) \rightarrow (Y, \sigma)$ is a morphism, where $\tau = \{p^x \in L^{L^X} : x \in X\}$, $\sigma = \{q^y \in L^{L^Y} : y \in Y\}$, if $R : X \times Y \rightarrow L$ is an L -fuzzy relation, and for all $x \in X, y \in Y$,

$$q^y.R^\uparrow \geq p^x \otimes R(x, y).$$

5. The category **RFRel** is defined by
 - (a) objects are pairs (X, r) , where r is a reflexive L -fuzzy relation on X ,
 - (b) $R : (X, r) \rightarrow (Y, s)$ is a morphism, if $R : X \times Y \rightarrow L$ is an L -fuzzy relation, and

$$s \circ R \geq R \circ r,$$

where $\circ$ is the composition of L -fuzzy relations.

6. The category **RSFP** is defined by
 - (a) objects are sets with an L -fuzzy partition $(X, \mathcal{A})$,

- (b) $(R, \Sigma) : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ is a morphism if $R : X \times Y \rightarrow L$ and $\Sigma : |\mathcal{A}| \times |\mathcal{B}| \rightarrow L$ are L -fuzzy relations and for each $\alpha \in |\mathcal{A}|, \beta \in |\mathcal{B}|, x \in X, y \in Y$,

$$\begin{aligned} A_\alpha(x) \otimes \Sigma(\alpha, \beta) &\leq B_\beta(y) \otimes R(x, y), \\ A_\alpha(x) \otimes R(x, y) &\leq B_\beta(y) \otimes \Sigma(\alpha, \beta). \end{aligned}$$

7. The category $\mathbf{RFTrans}^\uparrow$ is defined by
- (a) objects are upper F -transforms $F_{X, \mathcal{A}}^\uparrow : L^X \rightarrow L^{|\mathcal{A}|}$, where $(X, \mathcal{A})$ are sets with L -fuzzy partitions,
 - (b) $(R, \Sigma) : F_{X, \mathcal{A}}^\uparrow \rightarrow F_{Y, \mathcal{B}}^\uparrow$ is a morphism if $R : X \times Y \rightarrow L$ and $\Sigma : |\mathcal{A}| \times |\mathcal{B}| \rightarrow L$ are L -fuzzy relations and

$$\Sigma^\uparrow \cdot F_{X, \mathcal{A}}^\uparrow \leq F_{Y, \mathcal{B}}^\uparrow \cdot R^\uparrow.$$

8. The category $\mathbf{RFTrans}^\downarrow$ is defined by
- (a) objects are lower F -transforms $F_{X, \mathcal{A}}^\downarrow : L^X \rightarrow L^{|\mathcal{A}|}$, where $(X, \mathcal{A})$ are sets with L -fuzzy partitions,
 - (b) $(R, \Sigma) : F_{X, \mathcal{A}}^\downarrow \rightarrow F_{Y, \mathcal{B}}^\downarrow$ is a morphism if $R : X \times Y \rightarrow L$ and $\Sigma : |\mathcal{A}| \times |\mathcal{B}| \rightarrow L$ are L -fuzzy relations and

$$\Sigma^\downarrow \cdot F_{Y, \mathcal{B}}^\downarrow \leq F_{X, \mathcal{A}}^\downarrow \cdot R^\downarrow.$$

Analogously as in the paper [8] we consider the following full subcategories of the above categories:

1. The full subcategory **RsACClo** of **RCClo** with strong Čech-Alexandroff L -fuzzy closure operators as objects.
2. The full subcategory **RsACInt** of **RCInt** with strong Čech-Alexandroff L -fuzzy interior operators as objects.
3. The full subcategory **RsAFPreTop** of **RFPPreTop** with strong Čech-Alexandroff L -fuzzy pretopological spaces as objects.
4. The full subcategory **RsAFcoPreTop** of **RFcoPreTop** with strong Čech-Alexandroff L -fuzzy co-pretopological spaces.

3 Relationships Among Relational Categories of L -valued Fuzzy Topological Structures

The main result of the paper are the following two theorems, which use functors to describe the relationships between individual categories from the Definitions 7 and 8. Because each of the category $\mathbf{K}$ listed in [8]; Theorem 1 and Theorem 2, can be embedded into the corresponding category $\mathbf{RK}$ and the embedding is done using a graph of the morphisms from $\mathbf{K}$, these new theorems generalize results from [8]. In fact, the following simple proposition holds.

Proposition 1. *Let $\mathbf{K}$ be any of categories listed in [8]; Theorem 1 and Theorem 2. Then there exists an embedding functor $I_{\mathbf{K}} : \mathbf{K} \hookrightarrow \mathbf{RK}$.*

From these two theorems it follows that relationships among relational versions of categories associated with F-transforms are analogical to relationships among categories with maps as morphisms.

The first theorem describes the relationships between these categories of L -valued fuzzy topological structures where L is a complete residuated lattice.

Theorem 1. *Let L be a complete residuated lattice. Then the following diagram of functors commutes,*

$$\begin{array}{ccccc}
 & & \mathbf{RFcoPreTop} & \xrightarrow{F} & \mathbf{RCClo} \\
 & & \uparrow & \xleftarrow{F^{-1}} & \uparrow \\
 \mathbf{RFTrans}^\uparrow & \xrightarrow{Q^\uparrow} & \mathbf{RsAFcoPreTop} & \xrightarrow{G} & \mathbf{RsACClo} \\
 & \searrow U^\uparrow & \uparrow W & \xleftarrow{G^{-1}} & \uparrow M \\
 & & \mathbf{RSFP} & \xrightarrow{T} & \mathbf{RFRel} \\
 & \swarrow U^\downarrow & \downarrow V & & \downarrow M^{-1} \\
 \mathbf{RFTrans}^\downarrow & \xrightarrow{Q^\downarrow} & \mathbf{RsAFPPreTop} & &
 \end{array}$$

where (F, F^{-1}) , (G, G^{-1}) and (M, M^{-1}) are inverse pairs of functors.

The rather long proof of this theorem will be published elsewhere. We show only how the object functions of the functors are defined. For $(X, \mathcal{A}) \in \mathbf{RSFP}$ we set

$$\begin{aligned}
 T(X, \mathcal{A}) &= (X, r), \quad r(x, x') = A_{w_{\mathcal{A}}(x)}(x'), \\
 W(X, \mathcal{A}) &= (X, \{p^x : x \in X\}), \quad p^x(u) = \bigvee_{t \in X} u(t) \otimes A_{w_{\mathcal{A}}(t)}(x), \\
 V(X, \mathcal{A}) &= (X, \{p_x : x \in X\}), \quad p_x(u) = F_{X, \mathcal{A}}^\downarrow(u)(w_{\mathcal{A}}(x)), \\
 U^\uparrow(X, \mathcal{A}) &= F_{X, \mathcal{A}}^\uparrow, \\
 U_\downarrow(X, \mathcal{A}) &= F_{X, \mathcal{A}}^\downarrow,
 \end{aligned}$$

where $w_{\mathcal{A}}(t) \in |\mathcal{A}|$ is such that $A_{w_{\mathcal{A}}(t)}(t) = 1$. For other functors we set

$$\begin{aligned}
 (X, \{p^x : x \in X\}) &\in \mathbf{RFcoPrTop}, \quad F(X, \{p^x : x \in X\}) = (c, u), \quad c(u)(x) = p^x(u), \\
 (X, r) &\in \mathbf{RFRel}, \quad M(X, r) = (X, c), \quad c(u)(x) = r^\uparrow(u)(x), \\
 (X, c) &\in \mathbf{RsACClo}, \quad M^{-1}(X, c) = (X, r), \quad r(x, x') = c(\chi_{\{x\}}^X)(x').
 \end{aligned}$$

If $\mathcal{L}$ is a complete MV -algebra, we obtain a stronger form of the previous theorem.

Theorem 2. *Let $\mathcal{L}$ be a complete MV -algebra. Then the following diagram of functors from Theorem 1 and new functors commutes, where (H, H^{-1}) and (N, N^{-1}) are inverse pairs of functors.*

In the following examples we show how a fuzzy topological structure of one type can be transformed into a fuzzy topological structure of another type using these theorems.

Example 1. Let $\mathcal{L}$ be a complete residuated lattice. Using Theorem 1 we show how a strong Čech-Alexandroff L -fuzzy closure operator c in a set X can be constructed from an equivalence relation σ on X . In fact, using σ we can define a fuzzy partition $\mathcal{A} = \{A_\alpha : \alpha \in X/\sigma\}$, where X/σ is the set of equivalence classes defined by σ and $A_\alpha(x) = 1$ iff $x \in \alpha$, otherwise the value is 0. Using functors from the Theorem 1, the strong Čech-Alexandroff L -fuzzy closure operator c in X can be defined by $(X, c) = M.T(X, \mathcal{A})$, i.e., for arbitrary $u \in L^X, x \in X$,

$$c(u)(x) = \bigvee_{t \in X} u(t) \otimes A_{w_{\mathcal{A}}(x)}(t) = \bigvee_{t \in X, (t, x) \in \sigma} u(t).$$

Example 2. Let $\mathcal{L}$ be a complete MV -algebra. Using the Theorem 2 we show how from a strong Čech-Alexandroff L -fuzzy pretopological space (X, τ) a strong Čech-Alexandroff L -fuzzy co-pretopological space (X, ρ) can be defined. In fact, we can put $(X, \rho) = G^{-1}.M.N.H(X, \tau)$. If $\tau = \{p_x : x \in X\}$ then $\rho = \{p^x : x \in X\}$ is defined by

$$p^x(u) = \bigvee_{t \in X} u(t) \otimes \neg p_x(\neg \chi_{\{x\}}^X)(t),$$

as it can be verified by a simple calculation. If $R : (X, \tau) \rightarrow (Y, \sigma)$ is a morphisms in **RsAFPreTOP**, then R is also a morphism $G^{-1}.M.N.H(X, \tau) \rightarrow G^{-1}.M.N.H(Y, \sigma)$ in **RsAFcoPreTop**.

4 Conclusions

The article follows our previous work [8], in which we dealt with the issue of relationships between categories motivated by topological structures. We looked at a more general situation where morphisms in these categories are not mappings, but L -fuzzy relations. In detail, we considered categories and some of

their subcategories of Čech closure or interior L -valued operators, categories of L -fuzzy pretopological and L -fuzzy co-pretopological operators, category of L -valued fuzzy relation, categories of upper and lower F -transforms and the category of spaces with fuzzy partitions, where morphisms between objects are based on L -valued relations. As an interesting consequence of these relationships among relational categories, it follows that the category of relational spaces with fuzzy partitions plays a key role, i.e., the objects of this category can be used to create objects of any of the above categories of topological structures.

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