Graded Cube of Opposition with Intermediate Quantifiers in Fuzzy Natural Logic

Petra Murinová^(✉) and Vilém Novák

Institute for Research and Applications of Fuzzy Modeling,
University of Ostrava, 30. dubna 22, 701 03 Ostrava 1, Czech Republic
{petra.murinova,vilem.novak}@osu.cz
<http://irafm.osu.cz/>

Abstract. In our previous papers, we formally analyzed the generalized Aristotle’s square of opposition using tools of higher-order fuzzy logic. Namely, we introduced general definitions of selected intermediate quantifiers, constructed a generalized square of opposition consisting of them and syntactically analyzed the emerged properties. The main objective of this paper is to extend the graded Peterson’s square of opposition into the graded cube of opposition with intermediate quantifiers.

Keywords: Intermediate quantifiers · Fuzzy natural logic · Evaluative linguistic expressions · Generalized peterson square · Graded cube of opposition

1 Introduction

This paper continues the work on intermediate quantifiers. *Fuzzy natural logic* (FNL) is a formal mathematical theory that consists of three theories: (1) a formal theory of evaluative linguistic expressions explained in detail in [1], (2) a formal theory of fuzzy IF-THEN rules and approximate reasoning presented in [2, 3], and (3) a formal theory of intermediate and generalized fuzzy quantifiers presented in [4–7]. This paper is a contribution to the latter.

Intermediate quantifiers are special linguistic expressions, for example, *almost all*, *a few*, *many*, *a large part of*, etc. which were introduced and deeply studied by Thompson in [8] and later by Peterson in his book in [9]. Peterson introduced a square of opposition as a generalization of the Aristotle’s one [10–13]. It consists of five basic intermediate quantifiers.

Formalization of Peterson’s square was introduced by Murinová and Novák in [14, 15]. The main objective of this paper is to extend this approach to a graded *5-cube of opposition* and prove its forming properties.

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Note that the idea to extend the square of opposition to a cube was already studied by Dubois in [16–18]. The authors introduced a graded Aristotle’s square of opposition extended to a cube that associates the traditional square of opposition with a dual one. Ciucci, Dubois, and Prade [19, 20] then introduced an application of the graded cube within the possibility theory.

2 Preliminaries

In this section, we will remind the main concepts and properties of the fuzzy type theory (higher-order fuzzy logic) and the theory of evaluative linguistic expressions. The reader can find details in several papers [1, 14, 21].

2.1 Fuzzy Type Theory

The formal theory of intermediate quantifiers is developed within Łukasiewicz fuzzy type theory (L-FTT). The algebra of truth values is a linearly ordered MV_{Δ} -algebra extended by the delta operation (see [22, 23]). A special case is the standard Łukasiewicz MV_{Δ} -algebra.

$$\mathcal{L} = \langle [0, 1], \vee, \wedge, \otimes, \rightarrow, 0, 1, \Delta \rangle \quad (1)$$

where

$$\begin{aligned} \wedge &= \text{minimum}, & \vee &= \text{maximum}, \\ a \otimes b &= 0 \vee (a + b - 1), & a \rightarrow b &= 1 \wedge (1 - a + b), \\ \neg a &= a \rightarrow 0 = 1 - a, & \Delta(a) &= \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The basic syntactical objects of L-FTT are classical (cf. [24]), namely the concepts of type and formula. The atomic types are ϵ (elements) and o (truth values). General types are denoted by Greek letters $\alpha, \beta, \dots$. We will omit the type whenever it is clear from the context. A set of all types is denoted by *Types*. The (meta-)symbol “:=” used below means “is defined by”.

The *language* consists of variables $x_{\alpha}, \dots$, special constants $c_{\alpha}, \dots$ ($\alpha \in \text{Types}$), symbol λ , and parentheses. The connectives (which are special constants) are *fuzzy equality/equivalence* $\equiv$, *conjunction* $\&$, *implication* $\Rightarrow$, *negation* $\neg$, *strong conjunction* $\&$, *strong disjunction* ∇ , *disjunction* $\vee$, and *delta* Δ .

Formulas are formed of variables, constants (each of specific type), and the symbol λ . Each formula A is assigned a type and we write it as A_{α} .¹ A set of formulas of type α is denoted by $Form_{\alpha}$. The set of all formulas is $Form = \bigcup_{\alpha \in \text{Types}} Form_{\alpha}$.

¹ Each formula has a unique type assigned to it. Hence, if α, β are different types then A_{α} and A_{β} are different formulas. To increase clarity of explanation, however, we will usually denote different formulas by different letters.

A *model* is $\mathcal{M} = \{(M_\alpha, \overset{\circ}{=}_\alpha) \mid \alpha \in \text{Types}\}$ where $\overset{\circ}{=}_\alpha$ is a fuzzy equality on a set M_α . If $\mathcal{M}$ is a model then $\mathcal{M}(A_o) \in M_o$ is a truth value, $\mathcal{M}(A_\epsilon) \in M_\epsilon$ is some element and $\mathcal{M}(A_{\beta\alpha}) : M_\alpha \rightarrow M_\beta$ is a function. For example, $\mathcal{M}(A_{o\alpha}) : M_\alpha \rightarrow M_o$ is a fuzzy set and $\mathcal{M}(A_{(o\alpha)\alpha}) : M_\alpha \times M_\alpha \rightarrow M_o$ a fuzzy relation. A formula A_o is *true* in T , $T \models A_o$, if it is true in the degree 1 in all models of T .

The fuzzy type theory is *complete*, i.e., a theory T is consistent iff it has a (Henkin) model. We sometimes apply its equivalent version: $T \vdash A_o$ iff $T \models A_o$.

The following special formulas play a role in our theory below:

$$\mathcal{T}_{oo} \equiv \lambda z_o \cdot \neg \Delta(\neg z_o), \quad (\text{nonzero truth value})$$

$$\hat{\mathcal{T}}_{oo} \equiv \lambda z_o \cdot \neg \Delta(z_o \vee \neg z_o). \quad (\text{general truth value})$$

Thus, $\mathcal{M}(\mathcal{T}(A_o)) = 1$ iff $\mathcal{M}(A_o) > 0$, and $\mathcal{M}(\hat{\mathcal{T}}(A_o)) = 1$ iff $\mathcal{M}(A_o) \in (0, 1)$ holds in any model $\mathcal{M}$.

2.2 Theory of Evaluative Linguistic Expressions

Evaluative linguistic expressions are expressions of a natural language such as *small*, *medium*, *big*, *very short*, *more or less deep*, *quite roughly strong*, *extremely high*, etc. Their theory is the basic constituent of the fuzzy natural logic.

The semantics of evaluative linguistic expressions is formulated in a special formal theory T^{Ev} of L-FTT that was introduced in [1] and less formally explained in [25] where also formulas for direct computation are provided.

The evaluative expressions are construed by special formulas $Sm \in \text{Form}_{oo(oo)}$ (*small*), $Me \in \text{Form}_{oo(oo)}$ (*medium*), $Bi \in \text{Form}_{oo(oo)}$ (*big*), and $Ze \in \text{Form}_{oo(oo)}$ (*zero*) that can be extended by several selected linguistic hedges. Recall that a *hedge*, i.e., usually (but not necessarily) an adverb such as “very, significantly, about, roughly”, etc. is in general construed by a formula $\nu \in \text{Form}_{oo}$ with specific properties. To classify that a given formula is a hedge, we introduced a formula $Hedge \in \text{Form}_{o(oo)}$. Then $T^{\text{Ev}} \vdash Hedge \nu$ means that ν is a hedge. We refer the reader to [1] for the technical details. We assume that the following is provable: $T^{\text{Ev}} \vdash Hedge \nu$ for all $\nu \in \{Ex, Si, Ve, ML, Ro, QR, VR\}$.

2.3 Theory of Intermediate Quantifiers

The theory of intermediate quantifiers is a special formal theory $T^{\text{IQ}}[\mathcal{S}]$ of L-FTT extending T^{Ev} . A detailed structure of $T^{\text{IQ}}[\mathcal{S}]$ and precise definitions can be found in [5, 6, 14].

As discussed in the Introduction, the semantics of the intermediate quantifiers requires the idea of a “size” of a (fuzzy) set that can be characterized by the concept of a measure.

Definition 1. Let $R \in \text{Form}_{o(o\alpha)(o\alpha)}$ be a formula².

(i) A formula $\mu \in \text{Form}_{o(o\alpha)(o\alpha)}$ defined by

$$\mu_{o(o\alpha)(o\alpha)} := \lambda z_{o\alpha} \lambda x_{o\alpha} (Rz_{o\alpha})x_{o\alpha} \quad (2)$$

represents a measure on fuzzy sets in the universe of type $\alpha \in \text{Types}$ if it has the following properties:

$$(M1) \Delta(x_{o\alpha} \subseteq z_{o\alpha}) \& \Delta(y_{o\alpha} \subseteq z_{o\alpha}) \& \Delta(x_{o\alpha} \subseteq y_{o\alpha}) \Rightarrow ((\mu z_{o\alpha})x_{o\alpha} \Rightarrow (\mu z_{o\alpha})y_{o\alpha}),$$

$$(M2) \Delta(x_{o\alpha} \subseteq z_{o\alpha}) \Rightarrow ((\mu z_{o\alpha})(z_{o\alpha} \setminus x_{o\alpha}) \equiv \neg(\mu z_{o\alpha})x_{o\alpha}),$$

$$(M3) \Delta(x_{o\alpha} \subseteq y_{o\alpha}) \& \Delta(x_{o\alpha} \subseteq z_{o\alpha}) \& \Delta(y_{o\alpha} \subseteq z_{o\alpha}) \Rightarrow ((\mu z_{o\alpha})x_{o\alpha} \Rightarrow (\mu y_{o\alpha})x_{o\alpha}).$$

(ii) The following formula characterizes measurable fuzzy sets of a given type α :

$$\mathbf{M}_{o(o\alpha)} := \lambda z_{o\alpha} \cdot \Delta \neg(z_{o\alpha} \equiv \emptyset_{o\alpha}) \& \Delta(\mu z_{o\alpha})z_{o\alpha} \& (\forall x_{o\alpha})(\forall y_{o\alpha}) \Delta((M1) \& (M3)) \& (\forall x_{o\alpha}) \Delta(M2) \quad (3)$$

where, for the simplicity of expression, we write (M1)–(M3) to stand for the axioms from (i).

Definition 2. Let $\mathcal{S} \subseteq \text{Types}$ be a selected set of types, $P = \{R \in \text{Form}_{o(o\alpha)(o\alpha)} \mid \alpha \in \mathcal{S}\}$ be a set of new constants. Let T be a consistent extension of the theory T^{Ev} in the language $J(T) \supseteq J^{Ev} \cup P$. We say that the theory T contains intermediate quantifiers w.r.t. the set of types $\mathcal{S}$ if for all $\alpha \in \mathcal{S}$ the following is provable:

(i)

$$T \vdash (\exists z_{o\alpha}) \mathbf{M}_{o(o\alpha)} z_{o\alpha}. \quad (4)$$

(ii)

$$T \vdash (\forall z_{o\alpha})(\exists x_{o\alpha})(\mathbf{M}_{o(o\alpha)} z_{o\alpha} \Rightarrow (\Delta(x_{o\alpha} \subseteq z_{o\alpha}) \& \hat{\gamma}((\mu z_{o\alpha})x_{o\alpha}))). \quad (5)$$

Formula (5) assures the existence of fuzzy sets in each measurable fuzzy set that have non-trivial measure. Obviously, formulas (4) and (5) can be also introduced as special axioms of T . In the sequel, we will denote a theory that contains intermediate quantifiers due to Definition 2 by T^{IQ} .

For the definition of intermediate quantifiers, we need to define a special operation called *cut of a fuzzy set*, which will be formally defined as follows: Let $y, z \in \text{Form}_{o\alpha}$. The cut of y by z is the fuzzy set

$$y|z \equiv \lambda x_{\alpha} \cdot zx \& \Delta(\gamma(zx) \Rightarrow (yx \equiv zx)).$$

The following lemma can be proved.

² This formula can be understood as a procedure providing computation of the output (a value in L) on the basis of a given input—two fuzzy sets. Formula (2) says that the measure is a function.

Lemma 1 ([15]). *Let $\mathcal{M}$ be a model and p an assignment such that $B = \mathcal{M}_p(y) \subseteq M_\alpha$, $Z = \mathcal{M}_p(z) \subseteq M_\alpha$. Then for any $m \in M_\alpha$*

$$\mathcal{M}_p(y|z)(m) = (B|Z)(m) = \begin{cases} B(m), & \text{if } B(m) = Z(m), \\ 0 & \text{otherwise.} \end{cases}$$

One can see that the operation $B|Z$ takes only those elements $m \in M_\alpha$ from the fuzzy set B whose membership $B(m)$ is equal to $Z(m)$, otherwise $(B|Z)(m) = 0$. If there is no such an element, then $B|Z = \emptyset$. We can thus use various fuzzy sets Z to “picking up proper elements” from B .

The following lemma will play a significant role in proofs of properties of the graded cube of opposition.

Lemma 2. *Let $\mathcal{M}$ be a model and p an assignment such that $B = \mathcal{M}_p(y) \subseteq M_\alpha$, $Z = \mathcal{M}_p(z) \subseteq M_\alpha$, $Z' = \mathcal{M}_p(z') \subseteq M_\alpha$. Then for any $m \in M_\alpha$*

- (a) $(B|Z)(m) \otimes (\neg B|Z')(m) = 0$,
- (b) $(B|Z)(m) \otimes \neg(B|Z')(m) = 0$.

Proof. (a) Let $p(x) = m$ and $B(m) = 0$. Then the property is trivially fulfilled.

Let $B(m) = Z(m) > 0$ and $\neg B(m) = Z'(m) > 0$. Then from Lemma 1 it follows that $B(m) = (B|Z)(m)$ as well as $\neg B(m) = (\neg B|Z')(m)$. Because $B(m) \otimes \neg B(m) = 1$ holds for any $m \in M_\alpha$, then $(B|Z)(m) \otimes (\neg B|Z')(m) = 0$.

(b) Obviously as (a).

Definition 3. *Let T^{IQ} be a theory containing intermediate quantifiers w.r.t. a set of types $\mathcal{S}$ due to Definition 2. Let $Ev \in \text{Form}_{oo}$ be an intension of some evaluative expression. Finally, let $z \in \text{Form}_{o\alpha}$, $x \in \text{Form}_\alpha$ be variables and $A, B \in \text{Form}_{o\alpha}$ be formulas where $T^{IQ} \vdash \mathbf{M}_{o(o\alpha)} B_{o\alpha}$, $\alpha \in \mathcal{S}$. An intermediate generalized quantifier construes the sentence “ $\langle \text{Quantifier} \rangle B$ ’s are A ” is one of the following formulas:*

$$(Q_{Ev}^\forall x)(B, A) \equiv (\exists z)[(\forall x)((B|z)x \Rightarrow Ax) \wedge Ev((\mu B)(B|z))], \quad (6)$$

$$(Q_{Ev}^\exists x)(B, A) \equiv (\exists z)[(\exists x)((B|z)x \wedge Ax) \wedge Ev((\mu B)(B|z))]. \quad (7)$$

The following special intermediate quantifiers can be introduced:

$$\mathbf{A}: \text{All } B \text{ are } A := (Q_{Bi\Delta}^\forall x)(B, A) \equiv (\forall x)(Bx \Rightarrow Ax),$$

$$\mathbf{E}: \text{No } B \text{ are } A := (Q_{Bi\Delta}^\forall x)(B, \neg A) \equiv (\forall x)(Bx \Rightarrow \neg Ax),$$

$$\mathbf{P}: \text{Almost all } B\text{'s are } A := (Q_{Bi\ E x}^\forall x)(B, A)$$

$$\mathbf{B}: \text{Almost all } B\text{'s are not } A := (Q_{Bi\ E x}^\forall x)(B, \neg A)$$

$$\mathbf{T}: \text{Most } B\text{'s are } A := (Q_{Bi\ V e x}^\forall x)(B, A)$$

$$\mathbf{D}: \text{Most } B\text{'s are not } A := (Q_{Bi\ V e x}^\forall x)(B, \neg A)$$

$$\mathbf{K}: \text{Many } B\text{'s are } A := (Q_{\neg S m}^\forall x)(B, A)$$

G: Many B 's are not $A := (Q_{\neg sm}^\forall x)(B, \neg A)$

I: Some B are $A := (Q_{Bi\Delta}^\exists x)(B, A) \equiv (\exists x)(Bx \wedge Ax)$,

O: Some B are not $A := (Q_{Bi\Delta}^\exists x)(B, \neg A) \equiv (\exists x)(Bx \wedge \neg Ax)$.

3 Graded Cube of Opposition

3.1 From the Graded Square to the Graded Cube of Opposition

The graded Aristotle's square of opposition is formed by two positive and two negative intermediate quantifiers that fulfil the generalized properties of contraries, contradictories, sub-contraries, and sub-alterns. Below, we recall the main definitions from [14].

Definition 4. Let T be a consistent theory of $L\text{-}FTT$, $\mathcal{M} \models T$ be a model, $p \in \text{Asg}(\mathcal{M})$ be an assignment, and $P_1, P_2 \in \text{Form}_o$ be closed formulas of type o .

(i) P_1 and P_2 are contraries in the model $\mathcal{M}$ if

$$\mathcal{M}_p(P_1) \otimes \mathcal{M}_p(P_2) = 0. \quad (8)$$

They are contraries in the theory T if $T \vdash \neg(P_1 \& P_2)$. By completeness, this is equivalent to (8) for every model $\mathcal{M} \models T$.

(ii) P_1 and P_2 are subcontraries in the model $\mathcal{M}$ if

$$\mathcal{M}_p(P_1) \oplus \mathcal{M}_p(P_2) = 1. \quad (9)$$

They are subcontraries in the theory T if $T \vdash (P_1 \nabla P_2)$. By completeness, this is equivalent to (9) for every model $\mathcal{M} \models T$.

(iii) P_1 and P_2 are contradictories in the model $\mathcal{M}$ if both

$$\mathcal{M}_p(\Delta P_1) \otimes \mathcal{M}_p(\Delta P_2) = 0 \text{ as well as } \mathcal{M}_p(\Delta P_1) \oplus \mathcal{M}_p(\Delta P_2) = 1. \quad (10)$$

They are contradictories in the theory T if both $T \vdash \neg(\Delta P_1 \& \Delta P_2)$ as well as $T \vdash \Delta P_1 \nabla \Delta P_2$. By completeness, this means that (10) hold for every model $\mathcal{M} \models T$.

(iv) P_2 is a subaltern of P_1 (P_1 is superaltern of P_2) in the model $\mathcal{M}$ if

$$\mathcal{M}_p(P_1) \leq \mathcal{M}_p(P_2). \quad (11)$$

P_2 is subaltern of P_1 in the theory T (P_1 is superaltern of P_2 in the theory T) if $T \vdash P_1 \Rightarrow P_2$. By completeness, this means that inequality (11) holds true in every model $\mathcal{M} \models T$.

All these definitions were introduced as a generalization of the corresponding classical ones. In our previous paper [14], we syntactically proved all the mentioned properties which form the graded Aristotle's square of opposition.

Theorem 1 ([14]). *The following is true in T^{IQ} :*

- (a) formulas **A** and **O** are contradictories in T^{IQ} ,
- (b) formulas **E** and **I** are contradictories in T^{IQ} .
- (c) formulas **A** and **E** are contraries with the presupposition in T^{IQ} .
- (d) the formula **A** is subaltern of **I**.

Changing B and A into their negation, $\neg B$ and $\neg A$ respectively, leads to another similar square of opposition **aeio**, provided that we also assume that the fuzzy set $\neg B$ is non-empty. It means that we assume *presupposition*³ (existential import) which was in detail discussed in our previous papers. The corresponding quantifier with presupposition is denoted by a star. To extend the graded Aristotle's square to a graded cube of opposition, we have to define new formulas as follows:

$$\textbf{a}: \text{All } \neg B \text{ are not } A \quad (\forall x)(\neg Bx \Rightarrow \neg Ax) \& (\exists x)\neg Bx, \quad (12)$$

$$\textbf{e}: \text{All } \neg B \text{ are } A \quad (\forall x)(\neg Bx \Rightarrow Ax), \quad (13)$$

$$\textbf{i}: \text{Some } \neg B \text{ are not } A \quad (\exists x)(\neg Bx \wedge \neg Ax), \quad (14)$$

$$\textbf{o}: \text{Some } \neg B \text{ are } A \quad (\exists x)(\neg Bx \wedge Ax) \nabla \neg(\exists x)\neg Bx. \quad (15)$$

We can see that the new logical square of opposition (**aeio**) develops from the graded Aristotle's one (**AEIO**) by replacing formulas Bx and Ax by $\neg Bx$ and $\neg Ax$, respectively. It means that the “basic” properties between the intermediate quantifiers inside **aeio** can be proved similarly to **AEIO**.

Theorem 2. *The following is true in T^{IQ} :*

- (a) formulas **a** and **o** are contradictories in T^{IQ} ,
- (b) formulas **e** and **i** are contradictories in T^{IQ} .
- (c) formulas **a** and **e** are contraries with the presupposition in T^{IQ} .
- (d) formula **a** is subaltern of **i**.

Proof. The properties (a)–(d) can be proved similarly as the properties of the graded Aristotle's square **AEIO** in [14].

3.2 Relations Between “AEIO” and “aeio”

Lemma 3. *The following is provable:*

- (a) $T^{IQ} \vdash \neg(\mathbf{A} \& \mathbf{e})^*$,
- (b) $T^{IQ} \vdash \neg(\mathbf{a} \& \mathbf{E})^*$.

³ It is necessary that the universal quantifiers carry a presupposition of existential import for the entailments to their respective particular forms to hold.

Proof. (a) By properties of L-FTT we have

$$T^{IQ} \vdash (Bx \Rightarrow Ax) \Rightarrow (\neg Ax \Rightarrow \neg Bx) \quad (16)$$

as well as

$$T^{IQ} \vdash (\neg Bx \Rightarrow Ax) \Rightarrow (\neg Ax \Rightarrow Bx). \quad (17)$$

By $T^{IQ} \vdash (\neg Ax \&(\neg Ax \Rightarrow Bx)) \Rightarrow Bx$ and $T^{IQ} \vdash (\neg Ax \&(\neg Ax \Rightarrow \neg Bx)) \Rightarrow \neg Bx$ we obtain the following provable formula

$$T^{IQ} \vdash (\neg Ax \&(\neg Ax \Rightarrow Bx)) \&(\neg Ax \&(\neg Ax \Rightarrow \neg Bx)) \Rightarrow \perp$$

which, using the properties of L-FTT, is equivalent with

$$T^{IQ} \vdash ((\neg Ax \Rightarrow Bx) \&(\neg Ax \Rightarrow \neg Bx)) \Rightarrow (\neg Ax)^2 \Rightarrow \perp \quad (18)$$

From (16), (17) and (18) by transitivity and using MP, we conclude that

$$T^{IQ} \vdash ((Bx \Rightarrow Ax) \&(\neg Bx \Rightarrow Ax)) \Rightarrow ((\neg Ax)^2 \Rightarrow \perp). \quad (19)$$

Finally, by the quantifier properties, we have

$$T^{IQ} \vdash ((\forall x)(Bx \Rightarrow Ax) \&(\forall x)(\neg Bx \Rightarrow Ax)) \Rightarrow ((\exists x)(\neg Ax)^2 \Rightarrow \perp) \quad (20)$$

which uses the definition of the negation equivalent with

$$T^{IQ} \vdash \neg((\forall x)(Bx \Rightarrow Ax) \&(\forall x)(\neg Bx \Rightarrow Ax)) \&(\exists x)(\neg Ax)^2$$

(b) It can be proved analogously.

Theorem 3. *The following holds true:*

(a) Formulas **a** and **E** are contraries in T^{IQ} .

(b) Formulas **A** and **e** are contraries in T^{IQ} .

Proof. It follows from Lemma 3.

Lemma 4. *Let the following be true:*

(a) $T^{IQ} \vdash (\mathbf{i} \nabla \mathbf{O})^*$,

(b) $T^{IQ} \vdash (\mathbf{I} \nabla \mathbf{o})^*$.

Proof. (a) From Lemma 3(a), we have the following provable formula

$$T^{IQ} \vdash \neg((\forall x)(Bx \Rightarrow Ax) \&(\forall x)(\neg Bx \Rightarrow Ax)) \&(\exists x)(\neg Ax)^2.$$

Then by properties of the negation, we have

$$T^{IQ} \vdash (\exists x)\neg(Bx \Rightarrow Ax) \nabla (\exists x)\neg(\neg Bx \Rightarrow Ax) \nabla \neg(\exists x)(\neg Ax)^2$$

which is equivalent with

$$T^{IQ} \vdash (\exists x)(Bx \&\neg Ax) \nabla (\exists x)(\neg Bx \&\neg Ax) \nabla \neg(\exists x)(\neg Ax)^2.$$

Finally by properties of $\&$ with $\wedge$, we conclude that

$$T^{IQ} \vdash (\exists x)(Bx \wedge \neg Ax) \nabla (\exists x)(\neg Bx \wedge \neg Ax) \nabla \neg(\exists x)(\neg Ax)^2.$$

(b) Analogously as (a).

Theorem 4. *The following is true:*

- (a) Formulas **i** and **O** are subcontraries in T^{IQ} .
- (b) Formulas **I** and **o** are subcontraries in T^{IQ} .

Proof. It follows from Lemma 4.

Lemma 5. *The following is provable:*

- (a) $T^{IQ} \vdash \mathbf{A}^* \Rightarrow \mathbf{i} \quad T^{IQ} \vdash \mathbf{a} \Rightarrow \mathbf{I}$,
- (b) $T^{IQ} \vdash \mathbf{E} \Rightarrow \mathbf{o}^* \quad T^{IQ} \vdash \mathbf{e} \Rightarrow \mathbf{O}$.

Proof. (a) By the properties of L-FTT, we have

$$T^{IQ} \vdash (\forall x)(Bx \Rightarrow Ax) \Rightarrow (\forall x)(\neg Ax \Rightarrow \neg Bx) \quad (21)$$

as well as

$$T^{IQ} \vdash (\forall x)(\neg Ax \Rightarrow \neg Bx) \Rightarrow ((\exists x)\neg Ax \Rightarrow (\exists x)(\neg Ax \wedge \neg Bx)). \quad (22)$$

Finally, by (21) and (22), we obtain

$$T^{IQ} \vdash (\forall x)(Bx \Rightarrow Ax) \& (\exists x)\neg Ax \Rightarrow (\exists x)(\neg Ax \wedge \neg Bx).$$

(b) is proved analogously.

Theorem 5. *The following is true:*

- (a) Formula **A** is subaltern of **i**, and **I** is superaltern of **a**.
- (b) Formula **E** is subaltern of **o**, and **e** is superaltern of **O**.

Proof. It follows from Lemma 5.

Example 1 (Interpretation of Cube of opposition in T^{IQ}).

(a) Construction of **AEIO** as follows:

Let us consider a model $\mathcal{M} \models T^{IQ}$ such that $T^{IQ} \vdash (\exists x)Bx$. Let $\mathcal{M}(\mathbf{A}) = a > 0$ (e.g., $a = 0.2$). Since $[\mathbf{A}, \mathbf{E}]$ are contraries, we have $\mathcal{M}(\mathbf{E}) = e \leq 1 - a$. Because the formulas $[\mathbf{A}, \mathbf{O}]$ are contradictories, it follows from the definition of contradictories that $\mathcal{M}(\Delta \mathbf{A}) = 0$ and so $\mathcal{M}(\Delta \mathbf{O}) = 1$ because $\mathcal{M}(\Delta \mathbf{A}) \otimes \mathcal{M}(\Delta \mathbf{O}) = 0$ and $\mathcal{M}(\Delta \mathbf{A}) \oplus \mathcal{M}(\Delta \mathbf{O}) = 1$. Consequently, **O** is subaltern of **E**. The **I** is subaltern of **A** and thus $\mathcal{M}(\mathbf{I}) = i \geq 0.2$. However, **I** is contradictory with **E** and so $\mathcal{M}(\mathbf{I}) = i = 1$. Finally, **I** is sub-contrary with **O** because $\mathcal{M}(\mathbf{O} \nabla \mathbf{I}) = 1$ and **I** is subaltern of **A**.

(b) Construction of **aeio** as follows:

Let us consider the same model $\mathcal{M} \models T^{IQ}$ such that $T^{IQ} \vdash (\exists x)\neg Ax$. Let $\mathcal{M}(\mathbf{a}) = a' > 0$ (e.g., $a' = 0.8$), $\mathcal{M}(\mathbf{e}) = e'$, $\mathcal{M}(\mathbf{i}) = i'$ and $\mathcal{M}(\mathbf{o}) = o'$. Since $[\mathbf{A}, \mathbf{e}]$, as well as $[\mathbf{E}, \mathbf{i}]$ are contraries, we have $\mathcal{M}(\mathbf{e}) = e' \leq 0.2$ and $\mathcal{M}(\mathbf{a}) = a' \leq 0.8$. Similarly, formulas $[\mathbf{i}, \mathbf{O}]$ and $[\mathbf{I}, \mathbf{o}]$ are sub-contraries in T^{IQ} then $\mathcal{M}(i) = i' = 1$ as well as $\mathcal{M}(i) = i' = 1$. Both of these results correspond with contradictories of the pairs $[\mathbf{a}, \mathbf{o}]$ and $[\mathbf{e}, \mathbf{i}]$. Finally, **A** is subaltern of **i** as well as **E** is subaltern of **o**.

These results are summarized in the following scheme. Recall that the straight lines mark contradictories, the dashed lines contraries, and the dotted lines sub-contraries. The arrows indicate the relation superaltern–subaltern (Fig. 1).

Fig. 1. The example of graded Aristotle's cube of opposition

4 Graded Cube with Intermediate Quantifiers

We continue with an extension of graded the **5**-square of opposition **AEPBT-DKGIO**, which was introduced as a generalization of Peterson's square, to the graded **5**-cube of opposition **aepbtdkgio** with intermediate quantifiers. Below we introduce new forms of intermediate quantifiers as follows:

$$(Q_{Ev}^{\forall} x)(\neg B, \neg A) \equiv (\exists z)[(\forall x)((\neg B|z)x \Rightarrow \neg Ax) \wedge Ev((\mu(\neg B))(\neg B|z))], \quad (23)$$

$$(Q_{Ev}^{\exists} x)(\neg B, \neg A) \equiv (\exists z)[(\exists x)((\neg B|z)x \wedge \neg Ax) \wedge Ev((\mu(\neg B))(\neg B|z))]. \quad (24)$$

Either of the quantifiers (23) or (24) construes the sentence

“ $\langle \text{Quantifier} \rangle$ not B 's are not A ”.

4.1 Contraries

Lemma 6. *Let $B \in \text{Form}_{o\alpha}$ be a formula and $z, z' \in \text{Form}_{o\alpha}$ be variables. Then the following is provable:*

$$T^{IQ} \vdash \neg[(\exists z)(\exists z')[(\forall x)((B|z)x \Rightarrow Ax) \wedge Ev((\mu B)(B|z))] \& \\ (\forall x)((\neg B|z')x \Rightarrow Ax) \wedge Ev((\mu(\neg B))(\neg B|z')) \& (\exists x)(\neg(Ax))^2]. \quad (25)$$

Proof. The proof of (25) is based on the following provable formulas:

$$T^{IQ} \vdash ((B|z)x \Rightarrow Ax) \Rightarrow (\neg Ax \Rightarrow \neg(B|z)x)$$

and

$$T^{IQ} \vdash ((\neg B|z')x \Rightarrow Ax) \Rightarrow (\neg Ax \Rightarrow \neg(\neg B|z')x).$$

Using quantifier properties and by Lemma 2 we obtain from these formulas that

$$T^{IQ} \vdash (\forall x)((B|z)x \Rightarrow Ax) \& (\forall x)((\neg B|z')x \Rightarrow Ax) \Rightarrow ((\exists x)(\neg Ax)^2 \Rightarrow \perp). \quad (26)$$

By the adjunction, the properties of $\wedge$, the quantifier properties and the definition of negation, we obtain (25) using the rules of L-FTT.

Theorem 6. *The pairs of quantifiers*

- (i) $[(Q_{Bi\ Ex}^\forall x)(B, A), (Q_{Bi\ Ex}^\forall x)(\neg B, A)], (i.e., [\mathbf{P}, \mathbf{b}]),$
- (ii) $[(Q_{Bi\ Ve}^\forall x)(B, A), (Q_{Bi\ Ve}^\forall x)(\neg B, A)], (i.e., [\mathbf{T}, \mathbf{d}]),$
- (iii) $[(Q_{\neg(Sm\ \bar{\nu})}^\forall x)(B, A), (Q_{\neg(Sm\ \bar{\nu})}^\forall x)(B, \neg A)], (i.e., [\mathbf{K}, \mathbf{g}]),$

are contraries in T^{IQ} .

Proof. It follows from Lemma 6 when replacing Ev by concrete evaluative linguistic expressions.

Lemma 7. *Let $B \in Form_{o\alpha}$ be a formula and $z, z' \in Form_{o\alpha}$ be variables. Then the following is provable:*

$$T^{IQ} \vdash \neg[(\exists z)(\exists z')[(\forall x)((\neg B|z)x \Rightarrow \neg Ax) \wedge Ev((\mu(\neg B))((\neg B|z))) \& (\forall x)((B|z')x \Rightarrow \neg Ax) \wedge Ev((\mu B)(B|z')) \& (\exists x)(\neg(Ax))^2]. \quad (27)$$

Proof. Similarly to Lemma 6, the proof of (27) is based on the following provable formula:

$$T^{IQ} \vdash (\forall x)((\neg B|z)x \Rightarrow \neg Ax) \& (\forall x)((B|z')x \Rightarrow \neg Ax) \Rightarrow ((\exists x)(Ax)^2 \Rightarrow \perp). \quad (28)$$

By the adjunction, the properties of $\wedge$, the quantifier properties and the definition of negation, we obtain (28) using the rules of L-FTT.

Theorem 7. *The pairs of quantifiers*

- (i) $[(Q_{Bi\ Ex}^\forall x)(\neg B, \neg A), (Q_{Bi\ Ex}^\forall x)(B, \neg A)], (i.e., [\mathbf{p}, \mathbf{B}]),$
- (ii) $[(Q_{Bi\ Ve}^\forall x)(\neg B, \neg A), (Q_{Bi\ Ve}^\forall x)(B, \neg A)], (i.e., [\mathbf{t}, \mathbf{D}]),$
- (iii) $[(Q_{\neg(Sm\ \bar{\nu})}^\forall x)(\neg B, \neg A), (Q_{\neg(Sm\ \bar{\nu})}^\forall x)(B, \neg A)], (i.e., [\mathbf{k}, \mathbf{G}]),$

are contraries in T^{IQ} .

Proof. It follows from Lemma 7 when replacing Ev by concrete evaluative linguistic expressions.

4.2 Sub-alterns

Lemma 8. *The following is provable in $L\text{-}FTT$:*

- (a) $T^{IQ} \vdash a \Rightarrow p$ $T^{IQ} \vdash e \Rightarrow b$,
- (b) $T^{IQ} \vdash p \Rightarrow t$ $T^{IQ} \vdash b \Rightarrow d$,
- (c) $T^{IQ} \vdash t \Rightarrow k$ $T^{IQ} \vdash d \Rightarrow g$,
- (d) $T^{IQ} \vdash k \Rightarrow i$ $T^{IQ} \vdash g \Rightarrow o$.

Proof. This proceeds similarly as in [14] using monotonicity of the corresponding evaluative linguistic expressions.

Below, we introduce a **5**-graded cube of opposition with five basic intermediate quantifiers as a generalization of the graded Peterson's square (Fig. 2).

Fig. 2. Graded cube of opposition with generalized intermediate quantifiers

5 Future Applications

As we mentioned above, the idea of this paper was to introduce new forms of generalized intermediate quantifiers forming graded cube of opposition. An idea for future is to apply the theory of syllogistic reasoning introduced in our previous papers [6, 26]. Using inferred new forms of valid syllogisms to derive new information which is not included in real data. For example, below we introduce examples of sentences which can be used:

- *Most people who live in an area affected by heavy industry suffer from asthma.*
- *Almost all shares grow with growing economy.*

New idea is to work with examples of natural language linguistic expressions which form graded cube of opposition as follows:

- *Most people who do not smoke have higher lung capacity.*
- *Most children who do not live in an area affected by heavy industry do not suffer from inflammation of the respiratory tract.*

6 Conclusion

In this paper, we extended the theory of the graded classical Aristotle square of opposition to the graded Aristotle cube of opposition. Furthermore, we suggested a generalization of the Peterson's square of opposition to a graded generalized cube, i.e., the cube whose vertices contain intermediate quantifiers.

The future work will focus on a more detailed analysis of the properties of the graded generalized cube of opposition and to extend by new forms of intermediate quantifiers.

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