Construction of Nullnorms Based on Closure and Interior Operators on Bounded Lattices

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Abstract. In this paper, we introduce two rather effective methods for constructing new families of nullnorms with a zero element on the basis of the closure operators and interior operators on a bounded lattice under some additional conditions. Our constructions can be seen as a generalization of the ones in [28]. As a by-product, two types of idempotent nullnorms on bounded lattices are obtained. Several interesting examples are included to get a better understanding of the structure of new families of nullnorms.

Keywords: Bounded lattice · Construction method · Closure operator · Interior operator · Nullnorm

1 Introduction

The definitions of t -operators and nullnorms on the unit interval were introduced by Mas et al. [20] and Calvo et al. [4], respectively. In [21], it was shown that nullnorms coincide with t -operators on the unit interval $[0, 1]$ since both of them have identical block structures on $[0, 1]^2$. As a generalization of t -norms and t -conorms on the unit interval, nullnorms have a zero element s derived from the whole domain, regardless of whether t -norms and t -conorms have zero elements 0 and 1, respectively. In particular, a nullnorm is a t -norm if $s = 0$ while it is a t -conorm if $s = 1$. These operators are effective in various fields of applications, such as expert systems, fuzzy decision making and fuzzy system modeling. They are interesting also from theoretical point of view. For more studies about t -norms, t -conorms, nullnorms and related operators on the unit interval, it can be referred to [5, 10, 14, 17, 19, 22–24, 26, 27, 29].

In recent years, the study of nullnorms on bounded lattices was initiated by Karaçal et al. [18]. They demonstrated the presence of nullnorms with a zero element on the basis of a t -norm and a t -conorm on a bounded lattice. Notice that the families of nullnorms obtained in [18] are not idempotent, in general. For this reason, Çaylı and Karaçal [6] introduced a method showing the presence idempotent nullnorms on bounded lattices such that there is only

one element incomparable with the zero element. Moreover, they proposed that there does not need to exist an idempotent nullnorm on a bounded lattice. After then, Wang et al. [28] and Çaylı [7] presented some methods for constructing idempotent nullnorms on a bounded lattice with some additional conditions on their zero element. Their methods can be viewed as a generalization of the proposed method in [6]. On the contrary to the approaches in [6, 7, 28] based on the only infimum t-norm on $[s, 1]^2$ and supremum t-conorm on $[0, s]^2$, in [8, 9] by using an arbitrary t-norm on $[s, 1]^2$ and an arbitrary t-conorm on $[0, s]^2$, it was described some construction methods for nullnorms on a bounded lattice L having a zero element s with some constraints.

In general topology, by considering a nonempty set A and the set $\wp(A)$ of all subsets of A , the closure operator (resp. interior operator) on $\wp(A)$ is defined as an expansive, isotone and idempotent map $cl : \wp(A) \rightarrow \wp(A)$ (resp. a contractive, isotone and idempotent map $int : \wp(A) \rightarrow \wp(A)$). Both of these operators can be used for constructing topologies on A in general topology [15]. More precisely, a one-to-one correspondence from the set of all topologies on A to the set of all closure (interior) operators on $\wp(A)$. That is, any topology on a nonempty set can induce the closure (interior) operator on its underlying powerset. It should be pointed out that closure and interior operators can be defined on a lattice $(\wp(A), \subseteq)$ of all subsets of a set A with set union as the join and set intersection as the meet. Hence, Everett [16] extended the closure operator (resp. interior operator) on $\wp(A)$ to a general lattice L where the condition $cl(\emptyset) = \emptyset$ (resp. $int(A) = A$) is omitted.

The main aim of this paper is to present some methods for yielding new families of nullnorms with a zero element by means of closure operators and interior operators on a bounded lattice. The remainder of this paper is organized as follows. In Sect. 2, we recall some preliminary details about bounded lattices and nullnorms, interior and closure operators on them. In Sect. 3, considering a bounded lattice L , we propose two new methods for generating nullnorms with a zero element based on the presence of closure operators $cl : L \rightarrow L$ and interior operators $int : L \rightarrow L$. We note that our constructions are a generalization of the constructions in [28]. We also provide some corresponding examples showing that our constructions actually create new types of nullnorms on bounded lattices different from those in [28]. It should be pointed out that our methods need some sufficient and necessary conditions to generate a nullnorm on a bounded lattice. As a by product, two classes of idempotent nullnorms on bounded lattices are obtained when taking the closure operator $cl : L \rightarrow L$ as $cl(x) = x$ for all $x \in L$ and the interior operator $int : L \rightarrow L$ as $int(x) = x$ for all $x \in L$. Furthermore, we exemplify that we cannot force new nullnorms to coincide with another predefined t-conorm $[0, s]^2$ and t-norm $[s, 1]^2$ apart from the t-conorm $S_{\vee} : [0, s]^2 \rightarrow [0, s]$ defined by $S_{\vee}(x, y) = x \vee y$ for all $x, y \in [0, s]$ and the t-norm $T_{\wedge} : [s, 1]^2 \rightarrow [s, 1]$ defined by $T_{\wedge}(x, y) = x \wedge y$. Finally, some concluding remarks are added.

2 Preliminaries

In this part, some basic results about bounded lattices and nullnorms, closure and interior operators on them are recalled.

A lattice L is a nonempty set with the partial order $\leq$ where any two elements $x, y \in L$ have a smallest upper bound (called join or supremum), denoted by $x \vee y$ and a greatest lower bound (called meet or infimum), denoted by $x \wedge y$. For $a, b \in L$, we use the notation $a < b$ where $a \leq b$ and $a \neq b$. Moreover, we use the notation $a \parallel b$ to denote that a and b are incomparable. For $s \in L \setminus \{0, 1\}$, we denote $D_s = [0, s] \times [s, 1] \cup [s, 1] \times [0, s]$ and $I_s = \{x \in L \mid x \parallel s\}$.

A bounded lattice $(L, \leq, \wedge, \vee)$ is a lattice having the top and bottom elements, which are written as 1 and 0, respectively, that is, there are two elements $1, 0 \in L$ such that $0 \leq x \leq 1$ for all $x \in L$.

Given $a, b \in L$ with $a \leq b$, the subinterval $[a, b]$ is a sublattice of L defined by $[a, b] = \{x \in L \mid a \leq x \leq b\}$.

The subintervals $]a, b[$, $[a, b[$ and $]a, b[$ are defined by $]a, b[= \{x \in L \mid a < x \leq b\}$, $[a, b[= \{x \in L \mid a \leq x < b\}$ and $]a, b[= \{x \in L \mid a < x < b\}$ (see [1]).

Definition 1 ([6–9, 18]). *Let L be a bounded lattice. A binary operation $F : L \times L \rightarrow L$ is called a nullnorm on L if, for any $x, y, z \in L$, it satisfies the following properties:*

- (1) $F(x, z) \leq F(y, z)$ for $x \leq y$.
- (2) $F(x, F(y, z)) = F(F(x, y), z)$.
- (3) $F(x, y) = F(y, x)$.
- (4) *There exists an element $s \in L$ called the zero element, such we obtain $F(x, 0) = x$ for all $x \leq s$ and $F(x, 1) = x$ for all $x \geq s$.*

F is called an idempotent nullnorm on L whenever $F(x, x) = x$ for all $x \in L$. We note that a triangular norm T (t-norm for short) on L is a special case of nullnorm with $s = 0$ whereas a triangular conorm S (t-conorm for short) on L is a special case of nullnorm with $s = 1$ (see [2, 3]).

Proposition 1 ([13, 18]). *Let L be a bounded lattice and F be a nullnorm on L with the zero element $s \in L \setminus \{0, 1\}$. Then the following statements hold:*

- i) $F[0, s]^2 : [0, s]^2 \rightarrow [0, s]$ is a t-conorm on $[0, s]^2$.
- ii) $F[s, 1]^2 : [s, 1]^2 \rightarrow [s, 1]$ is a t-norm on $[s, 1]^2$.

Proposition 2 ([9]). *Let L be a bounded lattice and F be an idempotent nullnorm on L with the zero element $s \in L \setminus \{0, 1\}$. Then the following statements hold:*

- i) $F(x, y) = x \vee y$ for all $(x, y) \in [0, s]^2$.
- ii) $F(x, y) = x \wedge y$ for all $(x, y) \in [s, 1]^2$.
- iii) $F(x, y) = x \vee (y \wedge s)$ for all $(x, y) \in [0, s] \times I_s$.
- iv) $F(x, y) = y \vee (x \wedge s)$ for all $(x, y) \in I_s \times [0, s]$.
- v) $F(x, y) = x \wedge (y \vee s)$ for all $(x, y) \in [s, 1] \times I_s$.
- vi) $F(x, y) = y \wedge (x \vee s)$ for all $(x, y) \in I_s \times [s, 1]$.

Definition 2 ([11,12,16]). Let L be a lattice. A mapping $cl : L \rightarrow L$ is called a closure operator if, for any $x, y \in L$, it satisfies the following properties:

- (1) *Expansion*: $x \leq cl(x)$.
- (2) *Preservation of join*: $cl(x \vee y) = cl(x) \vee cl(y)$.
- (3) *Idempotence*: $cl(cl(x)) = cl(x)$.

For a closure operator $cl : L \rightarrow L$ and for any $x, y \in L$, we obtain that $cl(cl(x)) \leq cl(x)$ and $cl(x) \leq cl(y)$ whenever $x \leq y$.

Definition 3 ([11,12,25]). Let L be a lattice. A mapping $int : L \rightarrow L$ is called an interior operator if, for any $x, y \in L$, it satisfies the following properties:

- (1) *Contraction*: $int(x) \leq x$.
- (2) *Preservation of meet*: $int(x \wedge y) = int(x) \wedge int(y)$.
- (3) *Idempotence*: $int(int(x)) = int(x)$.

For an interior operator $int : L \rightarrow L$ and for any $x, y \in L$, we obtain that $int(x) \leq int(int(x))$ and $int(x) \leq int(y)$ whenever $x \leq y$.

3 Construction Methods for Nullnorms

In this section, considering a bounded lattice L , we introduce two methods to construct the classes of nullnorms $F_{cl} : L \times L \rightarrow L$ and $F_{int} : L \times L \rightarrow L$ with the zero element on the basis of the closure operator $cl : L \rightarrow L$ and interior operator $int : L \rightarrow L$, respectively. We note that our constructions require some sufficient and necessary conditions on the bounded lattice and the closure (interior) operator. These conditions play an effective role in our constructions, and they yield a nullnorm on a bounded lattice in only particular cases. We also present some illustrative examples to have a better understanding of the structures of new constructions.

Theorem 1. Let L be a bounded lattice and $s \in L \setminus \{0, 1\}$ such that $a \wedge s = b \wedge s$ and $a \vee s = b \vee s$ for all $a, b \in I_s$. Given a closure operator $cl : L \rightarrow L$ such that $cl(p) \vee cl(q) \in I_s$ for all $p, q \in I_s$, the following function $F_{cl} : L \times L \rightarrow L$ defined by

$$F_{cl}(x, y) = \begin{cases} x \vee y & \text{if } (x, y) \in [0, s]^2, \\ x \wedge y & \text{if } (x, y) \in [s, 1]^2, \\ s & \text{if } (x, y) \in D_s, \\ y \wedge (x \vee s) & \text{if } (x, y) \in I_s \times [s, 1], \\ x \wedge (y \vee s) & \text{if } (x, y) \in [s, 1] \times I_s, \\ y \vee (x \wedge s) & \text{if } (x, y) \in I_s \times [0, s], \\ x \vee (y \wedge s) & \text{if } (x, y) \in [0, s] \times I_s, \\ cl(x) \vee cl(y) & \text{if } (x, y) \in I_s^2 \end{cases} \quad (1)$$

is a nullnorm on L with the zero element s .

Theorem 2. Let L be a bounded lattice, $s \in L \setminus \{0, 1\}$ such that $a \wedge s = b \wedge s$ for all $a, b \in I_s$ and $cl : L \rightarrow L$ be a closure operator. If the function F_{cl} defined by the formula (1) is a nullnorm on L with the zero element s , then there holds $p \vee s = q \vee s$ and $cl(p) \vee cl(q) \in I_s$ for all $p, q \in I_s$.

Proof. Let the function F_{cl} defined by the formula (1) be a nullnorm on L with the zero element s .

Given $p, q \in I_s$, from the monotonicity of F_{cl} , we have $q \leq cl(q) \leq cl(p) \vee cl(q) = F_{cl}(p, q) \leq F_{cl}(p, 1) = p \vee s$ and $p \leq cl(p) \leq cl(p) \vee cl(q) = F_{cl}(p, q) \leq F_{cl}(1, q) = q \vee s$. Then $q \vee s \leq p \vee s$ and $p \vee s \leq q \vee s$. Hence, it holds $p \vee s = q \vee s$ for any $p, q \in I_s$.

Assume that $cl(p) \vee cl(q) \in [0, s]$. Then we have $p \vee q \leq cl(p) \vee cl(q) \leq s$, i.e., $p \leq s$. This is a contradiction. So, $cl(p) \vee cl(q) \in [0, s]$ does not hold. Supposing that $cl(p) \vee cl(q) \in]s, 1]$, we have $F_{cl}(0, F_{cl}(p, q)) = s$ and $F_{cl}(F_{cl}(0, p), q) = F_{cl}(p \wedge s, q) = (p \wedge s) \vee (q \wedge s) = p \wedge s$. From the associativity of F_{cl} , it holds $s = p \wedge s$, i.e., $s \leq p$. This is a contradiction. So, $cl(p) \vee cl(q) \in]s, 1]$ does not hold. Therefore, we have $cl(p) \vee cl(q) \in I_s$ for any $p, q \in I_s$.

Consider a bounded lattice L , $s \in L \setminus \{0, 1\}$ such that $a \wedge s = b \wedge s$ for all $a, b \in I_s$ and a closure operator $cl : L \rightarrow L$. We observe that the conditions $p \vee s = q \vee s$ and $cl(p) \vee cl(q) \in I_s$ for all $p, q \in I_s$ are both sufficient and necessary to yield a nullnorm on L with the zero element s of the function F_{cl} defined by the formula (1). In this case, one can ask whether the condition $a \wedge s = b \wedge s$ for all $a, b \in I_s$ is necessary to be a nullnorm on L with the zero element s of F_{cl} . We firstly give an example to show that in Theorem 1, the condition $a \wedge s = b \wedge s$ for all $a, b \in I_s$ cannot be omitted, in general.

Example 1. Consider the bounded lattice $L_1 = \{0, v, s, r, u, t, 1\}$ depicted by Hasse diagram in Fig. 1. Define the closure operator $cl : L_1 \rightarrow L_1$ by $cl(0) = cl(v) = v$, $cl(r) = cl(u) = cl(t) = t$ and $cl(s) = cl(1) = 1$. Notice that $p \vee s = q \vee s$ and $cl(p) \vee cl(q) \in I_s$ for all $p, q \in I_s$ but $t \wedge s = v \neq 0 = r \wedge s$ for $r, t \in I_s$. In this case, by using the approach in Theorem 1, we have $F_{cl}(0, F_{cl}(u, r)) = F_{cl}(0, cl(u) \vee cl(r)) = F_{cl}(0, t) = t \wedge s = v$ and $F_{cl}(F_{cl}(0, u), r) = F_{cl}(u \wedge s, r) = F_{cl}(0, r) = r \wedge s = 0$. Then, F_{cl} is not associative for the indicated closure operator on L_1 . Therefore, F_{cl} is not a nullnorm on L_1 which does not satisfy the condition $a \wedge s = b \wedge s$ for all $a, b \in I_s$.

From Example 1, we observe that the condition $a \wedge s = b \wedge s$ for all $a, b \in I_s$ is sufficient in Theorem 1. Taking into account the above mentioned question, we state that this condition is not necessary in Theorem 1. In order to show this fact, we provide an example of a bounded lattice violating this condition on which the function F_{cl} defined by the formula (1) is a nullnorm with the zero element s .

Example 2. Consider the lattice L_1 depicted in Example 1 and the closure operator $cl : L_1 \rightarrow L_1$ defined by $cl(x) = x$ for all $x \in L_1$. If we apply the construction in Theorem 1, then we obtain the function $F_{cl} : L_1 \times L_1 \rightarrow L_1$ given as in Table 1.

Fig. 1. Lattice L_1

It is easy to check that F_{cl} is a nullnorm with the zero element s for the chosen closure operator on L_1 .

Table 1. Nullnorm F_{cl} on L_1

F_{cl}	0	v	s	r	u	t	1
0	0	v	s	0	0	v	s
v	v	v	s	v	v	v	s
s	s	s	s	s	s	s	s
r	0	v	s	r	u	t	1
u	0	v	s	u	u	t	1
t	v	v	s	t	t	t	1
1	s	s	s	1	1	1	1

In view of Theorems 1 and 2, when taking the closure operator $cl : L \rightarrow L$ as $cl(x) = x$ for all $x \in L$, we have the following Corollary 1 which shows the presence of idempotent nullnorms on L with the zero element $s \in L \setminus \{0, 1\}$.

Corollary 1. *Let L be a bounded lattice and $s \in L \setminus \{0, 1\}$ such that $a \wedge s = b \wedge s$ for all $a, b \in I_s$. Then, the following function $F_1 : L \times L \rightarrow L$ defined by*

$$F_1(x, y) = \begin{cases} x \vee y & \text{if } (x, y) \in [0, s]^2, \\ x \wedge y & \text{if } (x, y) \in [s, 1]^2, \\ s & \text{if } (x, y) \in D_s, \\ y \wedge (x \vee s) & \text{if } (x, y) \in I_s \times [s, 1], \\ x \wedge (y \vee s) & \text{if } (x, y) \in [s, 1] \times I_s, \\ y \vee (x \wedge s) & \text{if } (x, y) \in I_s \times [0, s], \\ x \vee (y \wedge s) & \text{if } (x, y) \in [0, s] \times I_s, \\ x \vee y & \text{if } (x, y) \in I_s^2 \end{cases}$$

is an idempotent nullnorm on L with the zero element s if and only if $p \vee s = q \vee s$ and $p \vee q \in I_s$ for all $p, q \in I_s$.

Remark 1. Let L be a bounded lattice, $s \in L \setminus \{0, 1\}$, $a \wedge s = b \wedge s$ and $a \vee s = b \vee s$ for all $a, b \in I_s$. Consider a closure operator $cl : L \rightarrow L$ such that $cl(p) \vee cl(q) \in I_s$ for all $p, q \in I_s$. It should be pointed out that the classes of the nullnorms $V_\vee$ introduced in [28, Theorem 2] and F_{cl} defined by the formula (1) in Theorem 1 are different from each other. F_{cl} differs from $V_\vee$ on the domain $I_s \times I_s$. The value of F_{cl} is $cl(x) \vee cl(y)$ whereas $V_\vee$ has the value $x \vee y$ when $(x, y) \in I_s \times I_s$. Both of them have the same value on all remainder domains. From Corollary 1, we can easily observe that the nullnorm F_{cl} coincides with the nullnorm $V_\vee$ when defining the closure operator $cl : L \rightarrow L$ by $cl(x) = x$ for all $x \in L$. More precisely, our construction in Theorem 1 encompass as a special case the construction of $V_\vee$ in [28, Theorem 2]. Furthermore, the nullnorms F_{cl} and $V_\vee$ do not have to coincide with each other. In the following, we present an example to illustrative the correctness of this argument.

Example 3. Given the lattice $L_2 = \{0, u, m, n, s, v, p, q, t, r, 1\}$ characterized by Hasse diagram in Fig. 2, it is clear that $a \wedge s = b \wedge s$ and $a \vee s = b \vee s$ for all $a, b \in I_s$. Take the closure operator $cl : L_2 \rightarrow L_2$ as $cl(0) = cl(v) = v$, $cl(u) = cl(n) = cl(s) = cl(m) = u$, $cl(t) = cl(q) = q$, $cl(p) = cl(r) = r$ and $cl(1) = 1$. Then, by use of the approaches in Theorem 1 and [28, Theorem 2], respectively, the nullnorms $F_{cl}, V_\vee : L_2 \times L_2 \rightarrow L_2$ are defined in Tables 2 and 3, respectively. These nullnorms are different from each other since $F_{cl}(p, t) = r \neq p = V_\vee(p, t)$ for $p, t \in L_2$.

Fig. 2. Lattice L_2

Theorem 3. Let L be a bounded lattice and $s \in L \setminus \{0, 1\}$ such that $a \wedge s = b \wedge s$ and $a \vee s = b \vee s$ for all $a, b \in I_s$. Given an interior operator $int : L \rightarrow L$ such that $int(p) \wedge int(q) \in I_s$ for all $p, q \in I_s$, the following function $F_{int} : L \times L \rightarrow L$ defined by

Table 2. Nullnorm F_{cl} on L_2

F_{cl}	0	v	n	s	t	p	q	r	m	u	1
0	0	v	n	s	v	v	v	v	s	s	s
v	v	v	n	s	v	v	v	v	s	s	s
n	n	n	n	s	n	n	n	n	s	s	s
s	s	s	s	s	s	s	s	s	s	s	s
t	v	v	n	s	q	r	q	r	m	u	u
p	v	v	n	s	r	r	r	r	m	u	u
q	v	v	n	s	q	r	q	r	m	u	u
r	v	v	n	s	r	r	r	r	m	u	u
m	s	s	s	s	m	m	m	m	m	m	m
u	s	s	s	s	u	u	u	u	m	u	u
1	s	s	s	s	u	u	u	u	m	u	1

Table 3. Nullnorm $V_\vee$ on L_2

$V_\vee$	0	v	n	s	t	p	q	r	m	u	1
0	0	v	n	s	v	v	v	v	s	s	s
v	v	v	n	s	v	v	v	v	s	s	s
n	n	n	n	s	n	n	n	n	s	s	s
s	s	s	s	s	s	s	s	s	s	s	s
t	v	v	n	s	t	p	q	r	m	u	u
p	v	v	n	s	p	p	r	r	m	u	u
q	v	v	n	s	q	r	q	r	m	u	u
r	v	v	n	s	r	r	r	r	m	u	u
m	s	s	s	s	m	m	m	m	m	m	m
u	s	s	s	s	u	u	u	u	m	u	u
1	s	s	s	s	u	u	u	u	m	u	1

$$F_{int}(x, y) = \begin{cases} x \vee y & \text{if } (x, y) \in [0, s]^2, \\ x \wedge y & \text{if } (x, y) \in [s, 1]^2, \\ s & \text{if } (x, y) \in D_s, \\ y \wedge (x \vee s) & \text{if } (x, y) \in I_s \times [s, 1], \\ x \wedge (y \vee s) & \text{if } (x, y) \in [s, 1] \times I_s, \\ y \vee (x \wedge s) & \text{if } (x, y) \in I_s \times [0, s], \\ x \vee (y \wedge s) & \text{if } (x, y) \in [0, s] \times I_s, \\ int(x) \wedge int(y) & \text{if } (x, y) \in I_s^2 \end{cases} \quad (2)$$

is a nullnorm on L with the zero element s .

Theorem 4. Let L be a bounded lattice, $s \in L \setminus \{0, 1\}$ such that $a \vee s = b \vee s$ for all $a, b \in I_s$, $int : L \rightarrow L$ be an interior operator. If the function F_{int} defined by the formula (2) is a nullnorm on L with the zero element s , then there holds $p \wedge s = q \wedge s$ and $int(p) \wedge int(q) \in I_s$ for all $p, q \in I_s$.

Proof. Let the function F_{int} defined by the formula (2) be a nullnorm on L with the zero element s .

Given $p, q \in I_s$, from the monotonicity of F_{int} , we have $q \geq int(q) \geq int(p) \wedge int(q) = F_{int}(p, q) \geq F_{int}(p, 0) = p \wedge s$ and $p \geq int(p) \geq int(p) \wedge int(q) = F_{int}(p, q) \geq F_{int}(0, q) = q \wedge s$. In this case, we obtain $q \wedge s \leq p \wedge s$ and $p \wedge s \leq q \wedge s$. So, it holds $p \wedge s = q \wedge s$ for any $p, q \in I_s$.

Assume that $int(p) \wedge int(q) \in [s, 1]$. Then we have $s \leq int(p) \wedge int(q) \leq p \wedge q$. That is, $s \leq p$ which is a contradiction. Hence, $int(p) \wedge int(q) \in [s, 1]$ cannot hold. Suppose that $int(p) \wedge int(q) \in [0, s[$. Then, it is obtained that $F_{int}(1, F_{int}(p, q)) = s$ and $F_{int}(F_{int}(1, p), q) = F_{int}(p \vee s, q) = (p \vee s) \wedge (q \vee s) = p \vee s$. Since F_{int} is associative, we get $s = p \vee s$, i.e., $p \leq s$ which is a contradiction. Hence, $int(p) \wedge int(q) \in [0, s[$ cannot hold. Therefore, it holds $int(p) \wedge int(q) \in I_s$ for any $p, q \in I_s$.

Consider a bounded lattice L , $s \in L \setminus \{0, 1\}$ such that $a \vee s = b \vee s$ for all $a, b \in I_s$ and an interior operator $\text{int} : L \rightarrow L$. It should be pointed out that the conditions $p \wedge s = q \wedge s$ and $\text{int}(p) \wedge \text{int}(q) \in I_s$ for all $p, q \in I_s$ are both sufficient and necessary to generate a nullnorm on L with the zero element s of the function F_{int} defined by the formula (2). Then a natural question arises: is it necessary the condition $a \vee s = b \vee s$ for all $a, b \in I_s$ to be a nullnorm on L with the zero element s of F_{cl} . At first, by the following example, we demonstrate that in Theorem 3, the condition $a \vee s = b \vee s$ for all $a, b \in I_s$ cannot be omitted, in general.

Example 4. Consider the bounded lattice $L_3 = \{0, s, k, n, t, m, 1\}$ with the lattice diagram shown in Fig. 3. Define the interior operator $\text{int} : L_3 \rightarrow L_3$ by $\text{int}(x) = x$ for all $x \in L_3$. It holds $p \wedge s = q \wedge s$ and $\text{int}(p) \wedge \text{int}(q) \in I_s$ for all $p, q \in I_s$, however, $t \vee s = m \neq 1 = n \vee s$ for $n, t \in I_s$. Then, by applying the method in Theorem 3, we obtain $F_{\text{cl}}(1, F_{\text{cl}}(k, n)) = F_{\text{cl}}(1, \text{int}(k) \wedge \text{int}(n)) = F_{\text{cl}}(1, t) = t \vee s = m$ and $F_{\text{cl}}(F_{\text{cl}}(1, k), n) = F_{\text{cl}}(k \vee s, n) = F_{\text{cl}}(1, n) = n \vee s = 1$. In that case, F_{int} is not associative for the indicated interior operator on L_3 . Hence, F_{int} is not a nullnorm on L_3 violating the condition $a \vee s = b \vee s$ for all $a, b \in I_s$.

By Example 4, we observe that the condition $a \vee s = b \vee s$ for all $a, b \in I_s$ is sufficient in Theorem 3. Moreover, we answer the above question so that this is not a necessary condition in Theorem 3. In order to illustrative this observation, we provide an example of a bounded lattice violating this condition on which the function F_{int} defined by the formula (2) is a nullnorm on L with the zero element s .

Example 5. Consider the bounded lattice $L_4 = \{0, s, m, k, t, 1\}$ with the lattice diagram shown in Fig. 4. Notice that $t \vee s = m \neq 1 = k \vee s$ for $k, t \in I_s$. Define the interior operator $\text{int} : L_4 \rightarrow L_4$ by $\text{int}(x) = x$ for all $x \in L_4$. Then, by applying the construction in Theorem 3, we obtain the function $F_{\text{int}} : L_4 \times L_4 \rightarrow L_4$ given as in Table 4. It can be easily seen that F_{int} is a nullnorm on L_4 with the zero element s .

Fig. 3. Lattice L_3 Fig. 4. Lattice L_4

Table 4. Nullnorm F_{int} on L_4

F_{int}	0	s	t	k	m	1
0	0	s	0	0	s	s
s	s	s	s	s	s	s
t	0	s	t	t	m	m
k	0	s	t	k	m	1
m	s	s	m	m	m	m
1	s	s	m	1	m	1

Taking into consideration Theorems 3 and 4, if we choose the interior operator $int : L \rightarrow L$ as $int(x) = x$ for all $x \in L$, then we get the following Corollary 2 which shows the presence of idempotent nullnorms on L with the zero element $s \in L \setminus \{0, 1\}$.

Corollary 2. *Let L be a bounded lattice and $s \in L \setminus \{0, 1\}$ such that $a \vee s = b \vee s$ for all $a, b \in I_s$. Then, the following function $F_2 : L \times L \rightarrow L$ defined by*

$$F_2(x, y) = \begin{cases} x \vee y & \text{if } (x, y) \in [0, s]^2, \\ x \wedge y & \text{if } (x, y) \in [s, 1]^2, \\ s & \text{if } (x, y) \in D_s, \\ y \wedge (x \vee s) & \text{if } (x, y) \in I_s \times [s, 1], \\ x \wedge (y \vee s) & \text{if } (x, y) \in [s, 1] \times I_s, \\ y \vee (x \wedge s) & \text{if } (x, y) \in I_s \times [0, s], \\ x \vee (y \wedge s) & \text{if } (x, y) \in [0, s] \times I_s, \\ x \wedge y & \text{if } (x, y) \in I_s^2 \end{cases} \quad (3)$$

is an idempotent nullnorm on L with the zero element s if and only if $p \wedge s = q \wedge s$ and $p \wedge q \in I_s$ for all $p, q \in I_s$.

Remark 2. Let L be a bounded lattice, $s \in L \setminus \{0, 1\}$, $a \wedge s = b \wedge s$ and $a \vee s = b \vee s$ for all $a, b \in I_s$. Consider an interior operator $int : L \rightarrow L$ such that $int(p) \wedge int(q) \in I_s$ for all $p, q \in I_s$. We note that F_{int} defined by the formula (2) in Theorem 3 can create new type of nullnorm different from $V_\wedge$ described in [28, Theorem 2]. In particular, F_{int} differs from $V_\wedge$ on the domain $I_s \times I_s$. While F_{int} has the value of $int(x) \wedge int(y)$ on $I_s \times I_s$, the value of $V_\wedge$ is $x \wedge y$. Both of them have same value on all remainder domains. By Corollary 2, when considering the interior operator $int : L \rightarrow L$ as $int(x) = x$ for all $x \in L$, we observe that the nullnorm F_{int} coincides with the nullnorm $V_\wedge$. To be more precise, the class of the nullnorm F_{int} is a generalization of the class of the nullnorm $V_\wedge$. Moreover, the nullnorms F_{int} and $V_\wedge$ do not need to coincide with each other. We provide the following example to demonstrate this observation.

Example 6. Consider the lattice L_2 with the given order in Fig. 2 and the interior operator $int : L_2 \rightarrow L_2$ defined by $int(0) = 0$, $int(v) = v$, $int(p) = int(q)$

Table 5. Nullnorm F_{int} on L_2

F_{int}	0	v	n	s	t	p	q	r	m	u	1
0	0	v	n	s	v	v	v	v	s	s	s
v	v	v	n	s	v	v	v	v	s	s	s
n	n	n	n	s	n	n	n	n	s	s	s
s	s	s	s	s	s	s	s	s	s	s	s
t	v	v	n	s	t	t	t	t	m	u	u
p	v	v	n	s	t	t	t	t	m	u	u
q	v	v	n	s	t	t	t	t	m	u	u
r	v	v	n	s	t	t	t	r	m	u	u
m	s	s	s	s	m	m	m	m	m	m	m
u	s	s	s	s	u	u	u	u	m	u	u
1	s	s	s	s	u	u	u	u	m	u	1

Table 6. Nullnorm $V_\wedge$ on L_2

$V_\wedge$	0	v	n	s	t	p	q	r	m	u	1
0	0	v	n	s	v	v	v	v	s	s	s
v	v	v	n	s	v	v	v	v	s	s	s
n	n	n	n	s	n	n	n	n	s	s	s
s	s	s	s	s	s	s	s	s	s	s	s
t	v	v	n	s	t	t	t	t	m	u	u
p	v	v	n	s	t	p	t	p	m	u	u
q	v	v	n	s	t	t	q	q	m	u	u
r	v	v	n	s	t	p	q	r	m	u	u
m	s	s	s	s	m	m	m	m	m	m	m
u	s	s	s	s	u	u	u	u	m	u	u
1	s	s	s	s	u	u	u	u	m	u	1

$= \text{int}(t) = t$, $\text{int}(m) = m$, $\text{int}(r) = r$, $\text{int}(n) = \text{int}(s) = n$ and $\text{int}(u) = \text{int}(1) = u$. Then, by use of the approaches in Theorem 3 and [28, Theorem 2], respectively, the nullnorms F_{int} , $V_\wedge : L_2 \times L_2 \rightarrow L_2$ are defined in Tables 5 and 6, respectively. These nullnorms are different from each other since $F_{int}(q, r) = t \neq q = V_\wedge(q, r)$ for $q, r \in L_2$.

Remark 3. It should be noted that the restriction of the nullnorms F_{cl} and F_{int} on $[0, s]^2$ is the t-conorm $S_\vee : [0, s]^2 \rightarrow [0, s]$ defined by $S_\vee(x, y) = x \vee y$ for all $x, y \in [0, s]$. However, F_{cl} and F_{int} do not need to coincide with another predefined t-conorm except for the t-conorm $S_\vee$ on $[0, s]^2$. To illustrate this argument, considering the lattice $L_5 = \{0, k, t, m, s, n, 1\}$ according to the lattice diagram shown in Fig. 5, we assume that the restriction of the nullnorms F_{cl} and F_{int} on $[0, s]^2$ is the t-conorm $S : [0, s]^2 \rightarrow [0, s]$ given as in Table 7. Then, by applying the construction approaches in Theorems 1 and 3, we have $F_{cl}(F_{cl}(t, k), n) = m$ ($F_{int}(F_{int}(t, k), n) = m$) and $F_{cl}(t, F_{cl}(k, n)) = s$ ($F_{int}(t, F_{int}(k, n)) = s$). Since F_{cl} and F_{int} do not satisfy associativity property, we cannot force F_{cl} and F_{int} to coincide with another prescribed t-conorm except for the t-conorm $S_\vee$ on $[0, s]^2$.

Similarly, we note that the restriction of the nullnorms F_{cl} and F_{int} on $[s, 1]^2$ is the t-norm $T_\wedge : [s, 1]^2 \rightarrow [s, 1]$ defined by $T_\wedge(x, y) = x \wedge y$ for all $x, y \in [s, 1]$. However, F_{cl} and F_{int} do not need to coincide with another predefined t-norm except for the t-norm $T_\wedge$ on $[s, 1]^2$. To illustrate this observation, we consider the lattice $L_6 = \{0, k, s, t, r, m, n, 1\}$ according to the lattice diagram shown in Fig. 6 and assume that the restriction of the nullnorms F_{cl} and F_{int} on $[s, 1]^2$ is the t-norm $T : [s, 1]^2 \rightarrow [s, 1]$ given as in Table 8. Then, by means of the construction approaches in Theorems 1 and 3, we have $F_{cl}(F_{cl}(m, n), k) = r$ ($F_{int}(F_{int}(m, n), k) = r$) and $F_{cl}(m, F_{cl}(n, k)) = t$ ($F_{int}(m, F_{int}(n, k)) = t$).

Fig. 5. Lattice L_5

Fig. 6. Lattice L_6

Table 7. T-conorm S on $[0, s]^2$

S	0	k	t	m	s
0	0	k	t	m	s
k	k	k	m	m	s
t	t	m	s	s	s
m	m	m	s	s	s
s	s	s	s	s	s

Table 8. T-norm T on $[s, 1]^2$

T	s	t	r	m	n	1
s	s	s	s	s	s	s
t	s	t	t	t	t	t
r	s	t	t	t	t	r
m	s	t	t	r	r	m
n	s	t	t	r	r	n
1	s	t	r	m	n	1

Since F_{cl} and F_{int} do not satisfy associativity property, we cannot force F_{cl} and F_{int} to coincide with another prescribed t-norm except for the t-norm $T_{\wedge}$ on $[s, 1]^2$.

4 Concluding Remarks

Following the characterization of nullnorms on the real unit interval $[0, 1]$, the structure of nullnorms concerning algebraic structures on bounded lattices has attracted researchers' attention. The definition of nullnorms was extended to bounded lattices by Karaçal et al. [18]. They also demonstrated the presence of nullnorms based on a t-norm and a t-conorm on bounded lattices. Some further methods for constructing nullnorms (in particular, idempotent nullnorms) on a bounded lattice were introduced in the papers [2, 6–9, 28]. In this paper, we continued to investigate the methods for obtaining new classes of nullnorms on bounded lattices with the zero element different from the bottom and top elements. More particularly, by using the existence of closure operators and interior operators on a bounded lattice L , we proposed two different construction

methods for nullnorms on L with the zero element $s \in L \setminus \{0, 1\}$ under some additional conditions. We also pointed out that our constructions encompass as a special case the ones in [28]. The proposed constructions for nullnorms both in this paper and in [28] coincide with the supremum t-conorm $S_\vee$ on $[0, s]^2$ and the infimum t-norm $T_\wedge$ on $[s, 1]^2$. We demonstrated that these classes of nullnorms do not need to coincide with another predefined t-conorm except for $S_\vee$ on $[0, s]^2$ and t-norm except for $T_\wedge$ on $[s, 1]^2$. Moreover, some specific examples were presented to illustrate more clearly new methods of nullnorms on bounded lattices.

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