An Interval-Valued Divergence for Interval-Valued Fuzzy Sets

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Abstract. Characterizing the degree of similarity or difference between two sets is a very important topic, since it has many applications in different areas, including image processing or decision making. Several studies have been done about the comparison of fuzzy sets and its extensions, in particular for interval-valued fuzzy sets. However, in most of the cases, the results of the comparison is just a number. In order to avoid this reduction of the information, we have introduced a measure for comparing two interval-valued fuzzy sets such that it is an interval itself, which can be reduced to a number if it is necessary. Thus, a richer class of measures is now considered.

Keywords: Interval-valued fuzzy set · Order between intervals · Dissimilarity · Divergence measure

1 Introduction

In many real-life situations, we have to compare two objects, opinions, etc having an incomplete or not precise information about them. One of the most extended approach to model these situations is to consider fuzzy sets and compare them when necessary. Comparison represents an important topic in fuzzy sets such that several measures for comparing fuzzy sets have been proposed in literature. A general study was done by Bouchon-Meunier et al. in 1996 [5]. Since then, a lot of measures have been introduced, some of them as constructive definitions, that is, specific formulas (see, among many others, [1,30,31,34]) and some of them by means of axiomatic definitions (see [12,20,23] for instance).

The necessity of dealing with imprecision in real world problems has been a long-term research challenge that has originated different extensions of the fuzzy sets. Interval-valued fuzzy sets are one of the most challenging extensions.

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They were introduced independently by Zadeh [33], Grattan-Guinness [17], Jahn [19] and Sambuc [24] in the seventies. Interval-valued fuzzy sets can be useful to deal with situations where the classical fuzzy tools are not so efficient as, for instance, when there is not an objective procedure to select the crisp membership degrees. This extension has attracted very quickly the attention of many researchers, since they could see the high potential of them for different applications. Thus, for instance, Sambuc [24] used them in medical diagnosis in thyroidian pathology, Bustince [6] and Gozalczany [16] in approximate reasoning and Cornelis et al. [11] and Turksen [29] in logic.

Based on their utility, several concepts, tools and trends related to this extension have to be studied. In particular, we are specially interested on the measures of similarity, or their dual measures of the difference, between interval-valued fuzzy sets. There are a lot of papers related to this topic in the literature (see e.g. [2, 13, 32, 35]).

Another very related concept is the notion of intuitionistic fuzzy set, introduced by Atanassov [3] about ten years later than interval-valued fuzzy sets. Despite the semantic differences, it was proven by many authors that intuitionistic fuzzy sets and interval-valued fuzzy sets are equipollent ([4, 14]), that is, there is a bijection function mapping one onto the other. Thus, the measures of comparison between intuitionistic fuzzy sets (see, for instance, [21, 22]) could propose us a first idea about the way to compare two interval-valued fuzzy sets, although they cannot be directly used, as it was shown in [10, 25, 26].

Many of the previously introduced measures represent the result of the comparison as a value in the real line. However, if we are dealing with interval-valued fuzzy sets, even the total similarity of incomplete descriptions does not guarantee the total similarity of the described objects. In order to solve this problem, a similarity described by means of a range of values could be more appropriate. However, this is not the usual case and, as far as we know, it has been only considered in a few number of works (see [28, 36]).

Although interval-valued fuzzy sets could be compared by means of distances, the most usual case in the literature is the use of dissimilarity measures. However, it was noticed in [22] that these measures could not be appropriate in some cases, for some counterintuitive examples, when we are comparing Atanassov intuitionistic fuzzy sets. To avoid this problem, it is necessary to introduce a measure of comparison with stronger properties than dissimilarities. Thus, the main aim of this work is to obtain a first idea on dissimilarity and divergence measures which are able to compare two interval-valued fuzzy sets assuming values in the set of closed real intervals and to relate both concepts.

This paper is organized as follows. In Sect. 2, some basic concepts are introduced and the notation is fixed for the remaining parts of the paper. Section 3 is devoted to the new definition of divergence measures between interval-valued fuzzy sets and dissimilarities. Finally, some conclusions and open problems are drawn in Sect. 4.

2 Basic Concepts

Let X denote the universe of discourse. An interval-valued fuzzy subset of X is a mapping $A : X \rightarrow L([0, 1])$ such that $A(x) = [\underline{A}(x), \overline{A}(x)]$, where $L([0, 1])$ denotes the family of closed intervals included in the unit interval $[0, 1]$. Thus, an interval-valued fuzzy set A is totally characterized by two mapping, $\underline{A}$ and $\overline{A}$, from X into $[0, 1]$ such that $\underline{A}(x) \leq \overline{A}(x), \forall x \in X$. These maps represent the lower and upper bound of the corresponding intervals. Let us notice that if $\underline{A}(x) = \overline{A}(x), \forall x \in X$, then A is a classical fuzzy sets. The collection of all the interval-valued fuzzy sets in X is denoted by $IVFS(X)$ and the subset formed by all the fuzzy sets in X is denoted by $FS(X)$.

Several operations have been considered to this concept in the literature. We will consider now the most usual ones, since they are the most usual particular case of the general operations defined by means of t-norms and t-conorms defined on $L([0, 1])$ (see e.g. [15]). Thus, for any $A, B \in IVFS(X)$, we have that:

- The intersection of A and B is the interval-valued fuzzy set defined by $\underline{A \cap B}(x) = \min\{\underline{A}(x), \underline{B}(x)\}$ and $\overline{A \cap B}(x) = \min\{\overline{A}(x), \overline{B}(x)\}$ for any $x \in X$.
- The union of A and B is the interval-valued fuzzy set defined by $\underline{A \cup B}(x) = \max\{\underline{A}(x), \underline{B}(x)\}$ and $\overline{A \cup B}(x) = \max\{\overline{A}(x), \overline{B}(x)\}$ for any $x \in X$.
- The complement of A is the interval-valued fuzzy set defined by $\underline{A^c}(x) = 1 - \overline{A}(x)$ and $\overline{A^c}(x) = 1 - \underline{A}(x)$ for any $x \in X$.
- A is a subset of B if, and only if, $\underline{A}(x) \leq \underline{B}(x)$ and $\overline{A}(x) \leq \overline{B}(x)$ for any $x \in X$. Thus, in fact, we are saying that A is included in B if the membership degree of any element in X is an interval lower than or equal to the interval representing the membership degree of B , when we use the usual lattice-ordering between closed intervals of the real line. We are going to consider here this order but, of course, any other order between intervals (see, for instance, [7]) could be considered to define the inclusion.

Another important partial order on $IVFS(X)$ could be considered:

$$A \sqsubseteq B \text{ iff } A(x) \subseteq B(x), \forall x \in X$$

for the usual inclusion between intervals.

A first intuitive ways to compare two interval-valued fuzzy sets, based on the Hausdorff distance (see [18]), could be:

$$d_H(A, B) = \sum_{x \in X} \max\{|\underline{A}(x) - \underline{B}(x)|, |\overline{A}(x) - \overline{B}(x)|\}$$

where X is a finite universe.

Note that in all the above mentioned families of measures, the degree of difference between two sets is just a number. In the following some axiomatic definitions about how to compare two sets while keeping the original information as much as possible are introduced.

3 Interval-Valued Measures for the Difference for Interval-Valued Fuzzy Sets

The measure of the difference between two interval-valued fuzzy sets is defined axiomatically on the basis of some natural properties:

- It is non negative and symmetric.
- It becomes zero when the two sets are fuzzy and equal, since the equality of two interval-valued fuzzy sets does not imply they are equal.

Every definition of divergence or dissimilarity satisfies the two previous properties. However, divergences and dissimilarities differ in a third axiom, the one representing the idea that the closer sets, the less differences.

On the other hand, when intervals are considered their width should be also considered as an evidence of uncertainty. Thus,

- For a wider interval, the uncertainty is greater and therefore the width of the difference with another interval should be bigger.

The usual way to compare two sets is by means of dissimilarities. If we adapt the previous ideas to our purposes, we could define a dissimilarity between interval-valued fuzzy sets as follows.

Definition 1. [28] *A map $D : IVFS(X) \times IVFS(X) \rightarrow L([0, 1])$ is a dissimilarity on $IVFS(X)$ if for any $A, B, C \in IVFS(X)$ the following conditions are fulfilled:*

- (Dis1) $D(A, B) = [0, 0]$ iff $A, B \in FS(X)$ and $A = B$.
- (Dis2) $D(A, B) = D(B, A)$.
- (Dis3) if $A \subseteq B \subseteq C$, then $D(A, B) \leq D(A, C)$ and $D(B, C) \leq D(A, C)$.
- (Dis4) if $B \sqsubseteq C$, then $D(A, B) \subseteq D(A, C)$.

Example 1. From [28] we can obtain several examples of dissimilarity measures:

- The trivial dissimilarity:

$$D_0(A, B) = \begin{cases} [0, 0] & \text{if } A, B \in FS(X), A = B, \\ [0, 1] & \text{otherwise.} \end{cases}$$

- The dissimilarity induced by a numerical distance:

$$D_1(A, B) = \frac{1}{|X|} \sum_{x \in X} \left[\inf_{a \in A(x), b \in B(x)} |a - b|, \sup_{a \in A(x), b \in B(x)} |a - b| \right]$$

- The dissimilarity induced by the numerical trivial dissimilarity:

$$D_2(A, B) = \begin{cases} [0, 0] & \text{if } A, B \in FS(X), A = B, \\ [0, 1] & \text{if } A \neq B \text{ and } A(x) \cap B(x) \neq \emptyset, \forall x \in X, \\ [1, 1] & \text{if } \exists x \in X \text{ such that } A(x) \cap B(x) = \emptyset. \end{cases}$$

Although dissimilarities are the usual way to compare two sets, they are based on a partial order on $IVFS(X)$, so one of the main properties is only required for a few number of elements in $IVFS(X)$, what represents an important drawback. In fact, we could find some counterintuitive dissimilarities based on the numerical measure given by Chen [9]. Thus, to avoid this circumstance, the third axiom could be replaced as follows.

Definition 2. A map $D : IVFS(X) \times IVFS(X) \rightarrow L([0, 1])$ is a divergence on $IVFS(X)$ if for any $A, B, C \in IVFS(X)$ the following conditions are fulfilled:

- (Dis1) $D(A, B) = [0, 0]$ iff $A, B \in FS(X)$ and $A = B$.
- (Dis2) $D(A, B) = D(B, A)$
- (Div3) $D(A \cap C, B \cap C) \leq D(A, B)$ and $D(A \cup C, B \cup C) \leq D(A, B)$
- (Dis4) if $B \subseteq C$, then $D(A, B) \subseteq D(A, C)$.

A fuzzy set can be seen as a particular case of interval-valued fuzzy set such that the width of the membership interval is always zero. Thus, another logical requirement would be that when we restrict a divergence measure to the set formed by all the fuzzy sets in X , we obtain a divergence measure in the sense introduced in [23], although now the value could be an interval instead of just a number. This requirement was not added, since it is a consequence of the other four axioms, as we can see at the following proposition.

Proposition 1. Let D be a divergence measure in $IVFS(X)$. Then the map $D|_{FS(X)}$ defined by

$$D|_{FS(X)}(A, B) = D(A', B')$$

with $A'(x) = [A(x), A(x)]$ and $B'(x) = [B(x), B(x)]$ for any $x \in X$ and for any $A, B \in FS(X)$, is a divergence measure in $FS(X)$.

The same proposition could be proven for dissimilarities. It is clear that both concepts are very related. In fact, the family of divergence measures is really a subfamily of the dissimilarities, as we can see from the following proposition.

Proposition 2. If a map $D : IVFS(X) \times IVFS(X) \rightarrow L([0, 1])$ is a divergence on $IVFS(X)$, then it is a dissimilarity on $IVFS(X)$.

Thus, any example of divergence is an example of dissimilarity. However, the converse is not true, as we can see at the following example.

Example 2. Let us consider the map $D_3 : IVFS(X) \times IVFS(X) \rightarrow L([0, 1])$ defined by

$$D_3(A, B) = \begin{cases} [0, 0] & \text{if } A, B \in FS(X), A = B, \\ [0, 0.5] & \text{if } A \neq B \text{ and } A \neq X, B \neq X, \\ [0, 1] & \text{otherwise.} \end{cases}$$

First we check that D_3 is a dissimilarity. It is trivial that Axioms (Dis1) and (Dis2) are fulfilled. For Axiom (Dis3) we are going to consider three cases:

- If $D(A, C) = [0, 1]$, then it is trivial.
- If $D(A, C) = [0, 0.5]$, then $A \neq C$, $A \neq X$ and $C \neq X$. Since $B \subseteq C$, then $B \neq X$. Thus, $D(A, B) = [0, 0]$ or $D(A, B) = [0, 0.5]$ and the same happens for $D(B, C)$.
- If $D(A, C) = [0, 0]$, then $A = C$ and $A, C \in FS(X)$. Then, $B = A$ and therefore $D(A, B) = D(B, C) = [0, 0]$.

Finally, for Axiom (Dis4), we have again three cases:

- If $D(A, C) = [0, 1]$, then it is trivial.
- If $D(A, C) = [0, 0.5]$, then $A \neq C$, $A \neq X$ and $C \neq X$. Since $B \sqsubseteq C$, then $B \neq X$. Thus, $D(A, B) = [0, 0]$ or $D(A, B) = [0, 0.5]$.
- If $D(A, C) = [0, 0]$, then $A = C$ and $A, C \in FS(X)$. Since $B \sqsubseteq C$, then $B = C$ and therefore $D(A, B) = D(A, C)$.

Thus, D_3 is a dissimilarity, but it is not a divergence since for $X = \{x, y\}$, if we consider the sets A, B defined by $A(x) = [1, 1]$, $A(y) = [0, 1]$, $B(x) = [0, 1]$ and $B(y) = [1, 1]$, we have that $A \cup B = X$. Then, $D_3(A, B) = [0, 0.5]$ but $D_3(A \cup B, B \cup B) = D_3(X, B) = [0, 1]$ which is not lower than or equal to $D_3(A, B)$. Then Axiom (Div3) is not fulfilled by D_3 .

Example 3. Trivial dissimilarity D_0 in Example 1 is neither a divergence as $D_0(A, B) = [0, 0]$ when $A, B \in FS(X)$, $A = B$, but $D_0(A \cup C, B \cup C)$ is not necessary $[0, 0]$. For example, if $X = \{x\}$, $A(x) = B(x) = \{0.5\}$ and $C(x) = [0.2, 0.6]$, then $A \cup C(x) = [0.2, 0.5]$ and $B \cup C(x) = [0.2, 0.5]$. $A \cup C(x), B \cup C(x) \notin FS(X)$, then $D_0(A \cup C, B \cup C) = [0, 1]$ and thus axiom (Div3) is not satisfied.

As the previous result shows, divergences are particular cases of dissimilarities, with specific properties. Next proposition show several interesting properties satisfied by divergences defined by means of a range of values.

Proposition 3. *Let $D : IVFS(X) \times IVFS(X) \rightarrow L([0, 1])$ be a divergence on $IVFS(X)$. Then, $\forall A, B \in IVFS(X)$*

1. $[0, 0] \leq D(A, B)$.
2. $D(A \cap B, B) \leq D(A, A \cup B)$.
3. $D(A \cap B, B) \leq D(A, B)$.
4. $D(A \cap B, B) \leq D(A \cap B, A \cup B)$.
5. $D(B, A \cup B) \leq D(A \cap B, A \cup B)$.

From these measures we could obtain the ones defined just by a number if, for instance, we aggregate the lower and upper bound of the divergence measure in order to obtain just a number. This could be done by any average function as the arithmetic mean or the median.

4 Concluding Remarks

In this work we have introduced a way to compare two interval-valued fuzzy sets such that the value is itself an interval instead of the usual numbers. Dissimilarities and the particular case of divergences are defined and some properties are studied. Of course, this is just a first approach to this topic and a lot of work is still pending. In particular, we would like to study in detail the importance of the width of the interval, in a similar way as it was done in [8, 27], since distances, dissimilarities and divergences are concepts clearly related. Apart from that, we would like to consider a more general definition of the concepts of intersection, union and complement based on general t-norms and study the behaviour of the proposed measures in that general case. We would also like to develop methods for building divergence measures.

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