Acyclic Matching in Some Subclasses of Graphs

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Abstract. A subset $M \subseteq E$ of edges of a graph $G = (V, E)$ is called a *matching* if no two edges of M share a common vertex. A matching M in a graph G is called an *acyclic matching* if $G[V(M)]$, the subgraph of G induced by the M -saturated vertices of G is acyclic. The ACYCLIC MATCHING PROBLEM is the problem of finding an acyclic matching of maximum size. The decision version of the ACYCLIC MATCHING PROBLEM is known to be NP-complete for general graphs as well as for bipartite graphs. In this paper, we strengthen this result by showing that the decision version of the ACYCLIC MATCHING PROBLEM remains NP-complete for comb-convex bipartite graphs and dually-chordal graphs. On the positive side, we present linear time algorithms to compute an acyclic matching of maximum size in split graphs and proper interval graphs. Finally, we show that the ACYCLIC MATCHING PROBLEM is hard to approximate within a factor of $n^{1-\epsilon}$ for any $\epsilon > 0$, unless $P = NP$ and the ACYCLIC MATCHING PROBLEM is APX-complete for $2k + 1$ -regular graphs for $k \geq 3$, where k is a constant.

Keywords: Matching · Bipartite graphs · Chordal graphs · Graph algorithm · NP-completeness · Approximation algorithm

1 Introduction

A subset $M \subseteq E$ of edges of a graph $G = (V, E)$ is called a *matching* if no two edges of M share a common vertex. Vertices that are incident on the edges of a matching M are called M -saturated vertices and are denoted by $V(M)$. In this paper, we study an important variant of matching called *acyclic matching* (see [3, 5, 9]). A matching M in G is called an *acyclic matching* if $G[V(M)]$, the subgraph of G induced by the M -saturated vertices of G is acyclic. The ACYCLIC MATCHING PROBLEM asks to find an acyclic matching of maximum

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size in a given graph G . The *acyclic matching number* of G , denoted by $\mu_{ac}(G)$ is the maximum size of an acyclic matching in G .

More formally, the ACYCLIC MATCHING PROBLEM and its decision version are defined as follows:

ACYCLIC MATCHING PROBLEM

Instance: A graph $G = (V, E)$.

Solution: An acyclic matching M in G .

Measure: Cardinality of the set M .

ACYCLIC MATCHING DECIDE PROBLEM

Instance: A graph $G = (V, E)$ and a positive integer k .

Question: Does there exist an acyclic matching M in G of size at least k ?

Goddard et al. [5] introduced the concept of *acyclic matching* along with some other variants of the matching and proved that the ACYCLIC MATCHING DECIDE PROBLEM is NP-complete for general graphs. Later, Panda and Pradhan [9] strengthened this result by showing that the ACYCLIC MATCHING DECIDE PROBLEM remains NP-complete for bipartite graphs and even for perfect-elimination bipartite graphs, which is a subclass of bipartite graphs. They also gave a dynamic programming based algorithm to find an acyclic matching of maximum size in bipartite permutation graphs. Baste et al. [2] showed that finding a maximum size 1-degenerate matching in a graph G is equivalent to finding a maximum acyclic matching in G . They further proved that a maximum 1-degenerate matching could be found in polynomial time in chordal graphs, but the time complexity is very high. Recently, Fürst and Rautenbach showed that it is hard to decide whether a given bipartite graph of maximum degree at most four has a maximum matching that is acyclic [4]. They further characterized the graphs for which every maximum matching is acyclic and give linear time algorithms to compute a maximum acyclic matching in graph classes like P_4 -free graphs and $2P_3$ -free graphs [4]. There are no approximation results known for the ACYCLIC MATCHING PROBLEM till now.

In this paper, we study the complexity status of the ACYCLIC MATCHING PROBLEM and the ACYCLIC MATCHING DECIDE PROBLEM in some subclasses of graphs. The main contributions of this paper are summarized below.

1. We prove that the ACYCLIC MATCHING DECIDE PROBLEM is NP-complete for tree-convex bipartite graphs by showing that it is NP-complete for comb-convex bipartite graphs which is a subclass of tree-convex bipartite graphs.
2. We prove that the ACYCLIC MATCHING DECIDE PROBLEM is NP-complete for dually chordal graphs.
3. We prove that a maximum size acyclic matching can be computed in linear time in split graphs and proper interval graphs.
4. We prove that it is hard to approximate the ACYCLIC MATCHING PROBLEM within a factor of $n^{1-\epsilon}$ for any $\epsilon > 0$, unless $P = NP$.
5. We prove that the ACYCLIC MATCHING PROBLEM is APX-complete for $2k + 1$ -regular graphs for $k \geq 3$, where k is a constant.

2 Preliminaries

We consider only simple and connected graphs. For a graph $G = (V, E)$, let n denotes the number of vertices and m denotes the number of edges in G . The open and closed neighborhood of a vertex $u \in V$ are denoted by $N(u)$ and $N[u]$ respectively, where $N(u) = \{w \mid wu \in E\}$ and $N[u] = N(u) \cup \{u\}$. The degree of a vertex u is $|N(u)|$ and is denoted by $d(u)$. For a graph $G = (V, E)$, the subgraph of G induced by $U \subseteq V$ is denoted by $G[U]$, where $G[U] = (U, E_U)$ and $E_U = \{xy \in E \mid x, y \in U\}$.

A graph $G = (V, E)$ is called a k -regular graph if $d(v) = k$ for every vertex v of G . A graph $G = (V, E)$ is called a *bipartite graph* if its vertex set V can be partitioned into two independent sets X and Y , such that every edge of G joins a vertex in X to a vertex in Y . A *comb* is a graph obtained by attaching a pendant vertex (tooth) to every vertex of a path (backbone). A bipartite graph $G = (X, Y, E)$ is said to be a *tree-convex bipartite graph*, if a tree $T = (X, E_X)$ can be defined on X such that for every vertex $y \in Y$, the vertices in $N_G(y)$ induces a subtree of T . It can be noted that tree-convex bipartite graphs are recognizable in linear time and the associated tree T can also be constructed in linear time [11]. If the tree T in a tree-convex bipartite graph is a comb, then G is called a *comb-convex bipartite graph*.

A graph $G = (V, E)$ is called a *chordal graph* if every cycle in G of length at least four has a *chord*, that is, an edge joining two non-consecutive vertices of the cycle. A graph $G = (V, E)$ is called a *split graph* if its vertex set V can be partitioned into two sets I and C such that I is an independent set and C is a clique. Let $\mathcal{F}$ be a family of sets. The *intersection graph* of $\mathcal{F}$ is obtained by taking each set in $\mathcal{F}$ as a vertex and joining two sets in $\mathcal{F}$ if and only if they have a nonempty intersection. A graph G is called a *proper interval graph* if it is the intersection graph of a family $\mathcal{F}$ of intervals on the real line such that no intervals in $\mathcal{F}$ contains another. A vertex $u \in N_G[v]$ in a graph G is called a maximum neighbor of v if for all $w \in N_G[v]$, $N_G[w] \subseteq N_G[u]$. An ordering $\alpha = (v_1, v_2, \dots, v_n)$ of $V(G)$ is called a maximum neighborhood ordering, if v_i has a maximum neighbor in $G_i = G[\{v_i, \dots, v_n\}]$ for all $i, 1 \leq i \leq n$. A graph G is called a *dually chordal graph* if it has a maximum neighborhood ordering.

3 NP-Completeness Results

3.1 Comb-Convex Bipartite Graphs

It has been shown in [9] that the ACYCLIC MATCHING DECIDE PROBLEM is NP-complete for bipartite graphs. In this subsection, we strengthen this result by showing that the ACYCLIC MATCHING DECIDE PROBLEM remains NP-complete for tree-convex bipartite graphs, which is a subclass of bipartite graphs by showing that it is NP-complete for comb-convex bipartite graphs.

Theorem 1. *The ACYCLIC MATCHING DECIDE PROBLEM is NP-complete for comb-convex bipartite graphs.*

Proof. Clearly, the ACYCLIC MATCHING DECIDE PROBLEM belongs to the class NP for comb-convex bipartite graphs. To show the NP-completeness, we give a polynomial reduction from the ACYCLIC MATCHING DECIDE PROBLEM for bipartite graphs, which is already known to be NP-complete [9].

Given a bipartite graph $G = (X, Y, E)$, we construct a comb-convex bipartite graph $H = (X_H, Y_H, E_H)$ as follows:

Let $X_H = X \cup X'$, where $X' = \{x'_i \mid x_i \in X\}$, $Y_H = Y$, and $E_H = E \cup E'$, where $E' = \{x'_i y \mid x'_i \in X' \text{ and } y \in Y\}$. The constructed graph H is a comb-convex bipartite graph if X' is taken as the backbone and X is taken as the teeth of a comb C . Further, note that given a bipartite graph G , the graph H can be constructed in polynomial time.

Now, the following claim is sufficient to complete the proof of the theorem.

Claim. G has an acyclic matching of size at least k if and only if H has an acyclic matching of size at least k .

Proof. Necessity: Let M be an acyclic matching in G of size at least k . Since G is a vertex induced subgraph of graph H , M is acyclic in H . Hence, H has an acyclic matching of size at least k .

Sufficiency: Let M' be an acyclic matching in H of size at least k . If M' does not have an edge from the edge set E' then M' is a required acyclic matching in G . Otherwise, note that M' can include at most one edge from the edge set E' . To the contrary, if $\{x'_i y_i, x'_j y_j\} \subseteq M'$ for some $x'_i, x'_j \in X'$ then $G[\{x'_i, y_i, x'_j, y_j\}]$ forms a cycle, which is a contradiction. Thus, M' can include at most one edge from E' .

Next, let $x'_i y_i \in M'$ for some $x'_i \in X'$. Since G is connected, y_i will have a neighbor (say x_k) in X . Note that x_k must be unsaturated by M' because otherwise if $x_k y_k \in M'$ for some $y_k \in Y$, then $G[\{x_k, y_k, x'_i, y_i\}]$ will form a cycle, which is a contradiction to the fact that M' is acyclic. Let $M = (M' \setminus \{x'_i y_i\}) \cup \{x_k y_i\}$. If $G[V(M)]$ is acyclic, then M is a required acyclic matching in G . Otherwise, let us assume that $G[V(M)]$ contains a cycle C' . If C' does not contain the vertex x_k , then C' is also a cycle in $G[V(M')]$. This contradicts the fact that M' is an acyclic matching. So, C' contains the vertex x_k . Let $x_k y_a, x_k y_b \in E(C')$. Since vertices of set X' are adjacent to every $y \in Y$, $x'_i y_a, x'_i y_b \in E_H$. Now, $C = (C' \setminus \{x_k y_a, x_k y_b\}) \cup \{x'_i y_a, x'_i y_b\}$ is also a cycle in $G[V(M')]$, which is a contradiction. Hence, M is acyclic and it is a required acyclic matching in G of size at least k . $\diamond$

Hence, the ACYCLIC MATCHING DECIDE PROBLEM is NP-complete for comb-convex bipartite graphs. $\square$

Corollary 1. *The ACYCLIC MATCHING DECIDE PROBLEM is NP-complete for tree-convex bipartite graphs.*

3.2 Dually Chordal Graphs

The ACYCLIC MATCHING PROBLEM is polynomial time solvable for chordal graphs [2] and hence for strongly chordal graphs. In this subsection, we show

that the ACYCLIC MATCHING DECIDE PROBLEM is NP-complete for dually chordal graphs which is a superclass of strongly chordal graphs.

Theorem 2. *The ACYCLIC MATCHING DECIDE PROBLEM is NP-complete for dually chordal graphs.*

Proof. Clearly, the ACYCLIC MATCHING DECIDE PROBLEM belongs to the class NP for dually chordal graphs. To show the NP-completeness, we give a polynomial reduction from the ACYCLIC MATCHING DECIDE PROBLEM for general graphs, which is already known to be NP-complete [5].

Given a graph $G = (V, E)$, we construct a dually chordal graph $H = (V_H, E_H)$ as follows: $V_H = V \cup \{v_0\}$, $E_H = E \cup \{v_0v \mid v \in V\}$.

Consider the ordering $\alpha = (v_1, v_2, \dots, v_n, v_0)$. Since $N[v_j] \subseteq N[v_0]$ for all $v_j \in V$, v_0 is a maximum neighbor for all v_j in $G_j = G[\{v_j, \dots, v_n\}]$. Therefore, it is easy to see that the constructed graph $H = (V_H, E_H)$ is a dually chordal graph. Also, note that given a graph G , the graph H can be constructed in polynomial time.

Now, the following claim is sufficient to complete the proof of the theorem.

Claim. G has an acyclic matching of size at least k if and only if H has an acyclic matching of size at least k , where $k > 1$.

Proof. Necessity: Let M be an acyclic matching in G of size at least k . Since G is a vertex induced subgraph of H , so M is an acyclic matching in graph H of size at least k .

Sufficiency: Let M' be an acyclic matching in graph H of size at least k , $k > 1$. Observe that if the vertex v_0 is saturated by M' , that is $v_0v_i \in M'$ for some $v_i \in V$, then $|M'| = 1$. To the contrary, if there exists another edge $v_jv_k \in M'$, then the graph $H[\{v_0, v_j, v_k\}]$ forms a cycle, which is a contradiction.

As $|M'| \geq k > 1$, vertex v_0 is not saturated by M' , that is, M' does not have any edge of the form v_0v_i for any $v_i \in V$. Thus, M' is a required acyclic matching in graph G of size at least k . $\diamond$

Hence, the ACYCLIC MATCHING DECIDE PROBLEM is NP-complete for dually chordal graphs. $\square$

4 Polynomial Time Algorithms

4.1 Split Graphs

In this subsection, we show that an acyclic matching of maximum size can be computed in linear time for split graphs which is a subclass of chordal graphs, where the complexity of computing a maximum size acyclic matching is $O(n^7)$.

Let $G = (V, E)$ be a split graph. Throughout this section, $I \cup C$ represents a given partition of the vertex set V , where I is an independent set and C is a clique in G . Now, the following lemma shows that the cardinality of an acyclic matching in a split graph $G = (V, E)$ can be either 1 or 2 only.

Lemma 1. Let $G = (V, E)$ be a split graph. If M is an acyclic matching in G , then $1 \leq |M| \leq 2$.

Proof. Let M be an acyclic matching in G and let $|M| \geq 3$. Let $\{a_1b_1, a_2b_2, a_3b_3\} \subseteq M$ for some $a_i, b_i \in V$, $1 \leq i \leq 3$. Since I is an independent set, we can assume without loss of generality that $b_1, b_2, b_3 \in C$. This leads to a contradiction as $G[\{b_1, b_2, b_3\}]$ forms a cycle. Thus, $|M| \leq 2$. $\diamond$

Next, we will characterize the split graphs depending on the size of an acyclic matching in G . For this purpose, let us recall the definition of *threshold graphs*, which is a proper subclass of split graphs.

A split graph $G = (V, E)$ is called a *threshold graph* if the vertices in I can be linearly ordered, say $(v_1, v_2, \dots, v_{|I|})$, such that $N(v_1) \subseteq N(v_2) \subseteq \dots \subseteq N(v_{|I|})$. This linear ordering of a threshold graph can be computed in linear time [6].

Lemma 2. Let $G = (V, E)$ be a split graph and let M be a maximum acyclic matching in G . Then, $|M| = 2$ if and only if there exist a pair of vertices $v_i, v_j \in I$ such that $N(v_i) \setminus N(v_j) \neq \emptyset$ and $N(v_j) \setminus N(v_i) \neq \emptyset$.

Proof. Necessity: Let M be an acyclic matching in G and let $M = \{a_ib_i, a_jb_j\}$. Since C is a clique and I is an independent set, exactly two vertices from the set $\{a_i, a_j, b_i, b_j\}$ belong to C and the other two belongs to I . Without loss of generality, let us assume that $a_i, a_j \in C$ and $b_i, b_j \in I$. Since $G[\{a_i, a_j, b_i, b_j\}]$ is acyclic and $a_ib_i, a_jb_j, a_ib_j \in E$, so $a_i \notin N(b_j)$ and $a_j \notin N(b_i)$. Hence, $b_i, b_j \in I$ is the required pair of vertices.

Sufficiency: Let us assume that there exist two vertices $v_1, v_2 \in I$ such that $N(v_1) \setminus N(v_2) \neq \emptyset$ and $N(v_2) \setminus N(v_1) \neq \emptyset$. Let $c_1 \in N(v_1) \setminus N(v_2)$ and $c_2 \in N(v_2) \setminus N(v_1)$. Now, define a matching $M = \{v_1c_1, v_2c_2\}$. It is easy to see that $G[\{v_1, c_1, v_2, c_2\}]$ contains no cycle, and hence M is acyclic. $\diamond$

Theorem 3. Let $G = (V, E)$ be a split graph and let M be a maximum acyclic matching in G . Then, $|M| = 1$ if and only if G is a threshold graph.

Proof. Necessity: Let $G = (V, E)$ be a split graph and let M be a maximum acyclic matching in G such that $|M| = 1$. For the sake of contradiction, let us suppose that G is not a threshold graph. Then, there will exist a pair of vertices $v_i, v_j \in I$ such that $N(v_i) \not\subseteq N(v_j)$ and $N(v_j) \not\subseteq N(v_i)$. Define a matching $M' = \{c_iv_i, c_jv_j\}$, where $c_i \in N(v_i) \setminus N(v_j)$ and $c_j \in N(v_j) \setminus N(v_i)$. It is easy to see that M' is acyclic as $G[\{v_i, v_j, c_i, c_j\}]$ is a path graph. Since $|M'| = 2$, this leads to a contradiction to the fact that M is a maximum acyclic matching in G . Hence, G is a threshold graph.

Sufficiency: Let $G = (V, E)$ be a threshold graph and let $(v_1, v_2, \dots, v_{|I|})$ be an ordering of I , such that $N(v_1) \subseteq N(v_2) \subseteq \dots \subseteq N(v_{|I|})$. Clearly, there does not exist any pair of vertices $v_i, v_j \in I$ such that $N(v_i) \setminus N(v_j) \neq \emptyset$ and $N(v_j) \setminus N(v_i) \neq \emptyset$. Hence, by Lemma 1 and Lemma 2, it is easy to see that $|M| = 1$. $\square$

Based on the above discussions we have the following theorem.

Theorem 4. A maximum size acyclic matching in a split graph $G = (V, E)$ can be computed in $O(|V| + |E|)$ time.

Proof. Due to space restriction, the proof has been deferred to the longer version of the paper. $\square$

4.2 Proper Interval Graphs

In this subsection, we show that an acyclic matching of maximum size can be computed in linear time for proper interval graphs which is a subclass of chordal graphs, where the complexity of computing a maximum size acyclic matching is $O(n^7)$.

Let $G = (V, E)$ be a given graph. A vertex $v \in V$ is called a simplicial vertex, if $N[v]$ induces a clique in G . An ordering $\alpha = (v_1, v_2, \dots, v_n)$ of vertices is called a perfect elimination ordering (PEO) of G if v_i is a simplicial vertex in $G_i = G[\{v_i, v_{i+1}, \dots, v_n\}]$ for all $1 \leq i \leq n$. A PEO $\alpha = (v_1, v_2, \dots, v_n)$ of a graph G is called a bi-compatible elimination ordering (BCO) if $\alpha^{-1} = (v_n, v_{n-1}, \dots, v_1)$ i.e., the reverse of α , is also a PEO of G . It has been characterized in [7] that a graph is proper interval if and only if it has a BCO.

Observation 5. [8] *Let $\sigma = (v_1, v_2, \dots, v_n)$ be a BCO of a proper interval graph G . If $v_i v_j \in E$, then $v_k v_j \in E$ for all $k, i \leq k \leq j - 1$.*

Observation 6. *Let $\sigma = (v_1, v_2, \dots, v_n)$ be a BCO of a proper interval graph G and let $L[v_i]$ denotes the last neighbor of a vertex v_i in σ . If $v_i < v_j$ in σ , then $L[v_i] \leq L[v_j]$.*

Proof. Let us suppose that there exists v_i and v_j such that $v_i < v_j$ in σ and $L[v_i] > L[v_j]$. Then by Observation 5, $v_j L[v_i] \in E$ but since $L[v_j] < L[v_i]$, we reach at a contradiction. $\diamond$

Observation 7. *Let $\sigma = (v_1, v_2, \dots, v_n)$ be a BCO of a proper interval graph G and let $L[v_i]$ denotes the last neighbor of a vertex v_i in σ . If M is an acyclic matching in G , then at most two vertices from the set $\{v_i, v_{i+1}, \dots, L[v_i]\}$ can be saturated by M .*

Proof. The result easily follows from Observation 5. $\diamond$

Lemma 3. *Let G be a proper interval graph with a BCO $\sigma = (v_1, v_2, \dots, v_n)$ and let M be an acyclic matching in G . If the edges $u_1 w_1, u_2 w_2 \in M$ such that $u_1 < w_1$ and $u_2 < w_2$ in σ , then either $w_1 < u_2$ or $w_2 < u_1$.*

Proof. Let us assume without loss of generality that there exist two edges $e_1 = u_1 w_1$ and $e_2 = u_2 w_2$ such that $u_1 < w_2 < w_1$ in σ . Now, the $G[\{u_1, w_2, w_1\}]$ forms a cycle, which is a contradiction. Thus, either $w_1 < u_2$ or $w_2 < u_1$. $\diamond$

Lemma 4. *Let G be a proper interval graph with a BCO $\sigma = (v_1, v_2, \dots, v_n)$ and let M be a maximum acyclic matching in G . Then, there exists an acyclic matching M' in G such that $v_1 v_2 \in M'$ and $|M'| = |M|$.*

Proof. Let G be a proper interval graph with a BCO $\sigma = (v_1, v_2, \dots, v_n)$ and let M be a maximum acyclic matching in G . Let $v_a v_b$ be the first edge with respect to σ that belongs to M . Let us assume without loss of generality that

$v_a < v_b$. It is easy to see that $v_a \leq L[v_2]$ in σ . To the contrary, if $v_a > L[v_2]$ in σ , then $G[\{v_1, v_2, v_a, v_b\}]$ is acyclic and hence the edge v_1v_2 can be added to M . This leads to a contradiction to the fact that M is a maximum acyclic matching in G . Hence, $v_a \leq L[v_2]$ in σ . Now, if $v_a = v_2$, then replace v_2v_b by v_2v_1 in M . If $v_a \neq v_2$, then replace v_av_b by v_av_2 in M . If $v_a = v_1$, then we are done. Otherwise, again replace the edge v_av_2 by v_1v_2 in M .

By Observation 6 and Lemma 3, it is easy to see that we can replace the edge v_av_b with the desired edges in the cases mentioned above. $\diamond$

Let $\sigma = (v_1, v_2, \dots, v_n)$ be a BCO of a proper interval graph G and let $\sigma' = (v_a, v_b, \dots, v_k)$ be an ordering obtained from σ by removing some vertices from σ . Then, σ' is also a BCO of some proper interval graph G' , where G' is a subgraph of G . Hence, we have the following corollary to Lemma 4.

Corollary 2. *If $\sigma' = (v_a, v_b, \dots, v_k)$ is a BCO of a subgraph G' of a proper interval graph G , then the edge v_av_b is contained in some maximum acyclic matching of G' .*

Based on the above lemmas, we now present a linear time algorithm AM-PIG(G), which computes an acyclic matching of maximum size in a given proper interval graph G . The pseudocode of the algorithm is given below:

Algorithm 1. AM-PIG(G)

Input: A proper interval graph G_1 with BCO $\sigma(G_1) = (v_1, v_2, \dots, v_n)$;

Output: A Maximum Acyclic Matching M ;

$M = \emptyset$, $i = 1$, $v + 1 = \text{vertex next to vertex } v \text{ in } \sigma(G_i)$ for $i \geq 1$;

$F[G_i] = \text{first vertex in the BCO } \sigma(G_i) \text{ of graph } G_i \text{ for } i \geq 1$;

$L[v] = \text{last neighbor of vertex } v \text{ in } \sigma(G_i) \text{ for } i \geq 1$;

while ($|G_i| \geq 2$) **do**

$M = M \cup \{F[G_i](F[G_i] + 1)\}$;

if ($(L[F[G_i]] = L[F[G_i] + 1]) \text{ or } (L[F[G_i]] + 1 = L[F[G_i] + 1))$ **then**

$i = i + 1$;

$G_i = G_{i-1} \setminus \{F[G_{i-1}], \dots, L[F[G_{i-1}]]\}$;

else if ($(L[F[G_i]] + 1 = v_k) < L[F[G_i] + 1] < L[v_k]$) **then**

$i = i + 1$;

$G_i = G_{i-1} \setminus (\{F[G_{i-1}], \dots, L[F[G_{i-1}]]\} \cup \{v_{k+1}, \dots, L[F[G_{i-1}] + 1\})$;

else if ($(L[F[G_i]] + 1 = v_k) < L[F[G_i] + 1]$) **and** $L[v_k] = L[F[G_i] + 1]$ **then**

$i = i + 1$;

$G_i = G_{i-1} \setminus \{F[G_{i-1}], \dots, L[F[G_{i-1}]]\}$;

while ($L[v_k] = L[F[G_i] + 1]$) **do**

$temp = v_k$;

$v_k = v_k + 1$;

$G_i = G_i \setminus temp$;

$G_i = G_i \setminus \{v_{k+1}, \dots, L[F[G_i] + 1]\}$;

return M ;

Theorem 8. *Given a proper interval graph G_1 with BCO $\sigma(G_1)$, AM-PIG(G_1) correctly computes a maximum size acyclic matching in G_1 .*

Proof. Due to space restriction, the proof has been deferred to the longer version of the paper. $\square$

5 Inapproximation Results

Let $G = (V, E)$ be a graph with n vertices. It is easy to note that the maximum size of an acyclic matching in G can be at most $\frac{n}{2}$. So, the ACYCLIC MATCHING PROBLEM can be approximated within a factor of n in polynomial time. In this section, we show that for any $\epsilon > 0$, it is hard to approximate the ACYCLIC MATCHING PROBLEM within a factor of $n^{1-\epsilon}$, unless $P = NP$.

To prove the result, we will need the following theorem for the MAXIMUM INDEPENDENT SET PROBLEM.

Theorem 9. [12] *The Maximum Independent Set Problem for a graph G cannot be approximated within a factor of $n^{1-\epsilon}$ for any $\epsilon > 0$, unless $P = NP$.*

Now, consider the following construction:

Construction 1. Let $G = (V, E)$, where $V = \{v_1, v_2, \dots, v_n\}$, be an instance of the MAXIMUM INDEPENDENT SET PROBLEM. We construct a graph $H = (V_H, E_H)$, an instance of the ACYCLIC MATCHING PROBLEM, in the following way:

- $V_H = V \cup V'$, where $V' = \{v'_i \mid v_i \in V\}$.
- $E_H = E \cup \{v_i v'_i \mid 1 \leq i \leq n\} \cup \{v_i v'_j \mid v_i v_j \in E\} \cup \{v'_i v'_j \mid v_i v_j \in E\}$.

Clearly, H can be constructed in polynomial time as $|V_H| = 2|V|$ and $|E_H| = 4|E| + |V|$.

Also, note that the edges in H can be one of the following four types:

1. *Type-I* = $\{v_i v'_i \mid v_i \in V \text{ and } v'_i \in V'\}$.
2. *Type-II* = $\{v_i v_j \mid v_i, v_j \in V\}$.
3. *Type-III* = $\{v'_i v'_j \mid v'_i \in V' \text{ and } v'_j \in V'\}$.
4. *Type-IV* = $\{v'_i v'_j \mid v'_i, v'_j \in V'\}$.

Now, we will discuss some lemmas that will be used in the proof of the main theorem of this section. Let us recall that $V_H(M)$ denotes the set of M -saturated vertices of graph H .

Lemma 5. *Let H be the graph obtained from a given graph G by Construction 1. If M is an acyclic matching in H , then there exists an acyclic matching M' in H such that $|M'| = |M|$ and M' contains edges of Type-I and Type-II only.*

Proof. First, let us suppose that M is an acyclic matching in H and let M contains an edge (say $v'_i v'_j$) of Type-IV. Since M is acyclic and $v'_i v'_j \in M$, both v_i and v_j are unsaturated by M . Let $M' = (M \setminus \{v'_i v'_j\}) \cup \{v'_i v_j\}$. If $G[V_H(M')]$ is acyclic, then we are done. So, assume that $G[V_H(M')]$ contains a cycle C . If C does not contain the vertex v_j , then C is also a cycle in $G[V_H(M)]$. This

contradicts the fact that M is an acyclic matching. So, C contains the vertex v_j . Let $v_j u_a, v_j u_b \in E(C)$. Since $N[v_j] = N[v'_j]$, $v'_j u_a, v'_j u_b \in E_H$. Now, $C' = (C \setminus \{v_j u_a, v_j u_b\}) \cup \{v'_j u_a, v'_j u_b\}$ is a cycle in $G[V_H(M)]$, which is a contradiction. Hence, M' is acyclic. In this way, an acyclic matching of same size can be obtained by replacing an edge of *Type-IV* with a corresponding *Type-III* edge.

Using the similar arguments, we can show that an acyclic matching of same size can be obtained by replacing an edge of *Type-III* with a corresponding *Type-II* edge. $\diamond$

Lemma 6. *Let H be the graph obtained from a given graph G by Construction 1. If M' is an acyclic matching in H containing edges of *Type-I* and *Type-II* only, then there exists an acyclic matching M'' in H such that $|M''| = |M'|$ and M'' contains edges of *Type-I* only.*

Proof. Due to space restriction, the proof has been deferred to the longer version of the paper. $\diamond$

The following lemma shows that the described reduction is exactly what we need.

Lemma 7. *Let H be the graph obtained from a given graph G by Construction 1. Then, G has an independent set of size at least k if and only if H has an acyclic matching of size at least k .*

Proof. Necessity: Let $I = \{v_1, v_2, \dots, v_l\}$ be an independent set in G of size at least k . Define a matching $M = \{v_1 v'_1, v_2 v'_2, \dots, v_l v'_l\}$ in H . It is easy to see that M is an acyclic matching as $G[V(M)]$ is a disjoint union of K'_2 s.

Sufficiency: Let $M = \{e_1, e_2, \dots, e_l\}$ be an acyclic matching in H of size at least k . By Lemma 6, there exists an acyclic matching M' in H such that $|M'| = |M|$ and M' contains edges of *Type-I* only. Define a set $I = \{v_i \mid v_i v'_i \in M'\}$. It is easy to see that I is an independent set of graph G . $\diamond$

Corollary 3. *G has a maximum independent set of size k if and only if H has a maximum acyclic matching of size k .*

Theorem 10. *The ACYCLIC MATCHING PROBLEM for a graph G cannot be approximated within a factor of $n^{1-\epsilon}$ for any $\epsilon > 0$, unless $P = NP$.*

Proof. Let $G = (V, E)$ be a graph with n vertices. Construct a graph $H = (V_H, E_H)$ with $|V_H| = \bar{n}$ from G using Construction 1. Let I^* denotes a maximum independent set in G and M^* denotes a maximum acyclic matching in H .

Now, let us suppose that the ACYCLIC MATCHING PROBLEM can be approximated within a ratio $\alpha \geq 1$ by using an algorithm ALG , where $\alpha = \bar{n}^{1-\epsilon'}$ for some fixed $\epsilon' > 0$.

If $M_{ALG}(H)$ is an acyclic matching in H obtained by applying algorithm ALG , then $|M^*(H)| \leq \alpha |M_{ALG}(H)|$.

By Corollary 3, $|I^*(G)| = |M^*(H)|$. By Lemma 7, we can construct an independent set I_{ALG} of G corresponding to M_{ALG} of H such that $|M_{ALG}(H)| = |I_{ALG}(G)|$.

Hence, we obtain, $|I^*(G)| \leq \alpha |I_{ALG}(G)| = \bar{n}^{1-\epsilon'} |I_{ALG}(G)| = (2n)^{1-\epsilon'} |I_{ALG}(G)| = (2)^{1-\epsilon'} (n)^{1-\epsilon'} |I_{ALG}(G)|$.

If we choose ϵ , such that $2^{1-\epsilon'} < n^{\epsilon'-\epsilon}$, then $|I^*(G)| < (n)^{\epsilon'-\epsilon} (n)^{1-\epsilon'} |I_{ALG}(G)| = (n)^{1-\epsilon} |I_{ALG}(G)|$.

Hence, $|I^*(G)| < (n)^{1-\epsilon} |I_{ALG}(G)|$, which leads to a contradiction to Theorem 9. Therefore, the ACYCLIC MATCHING PROBLEM cannot be approximated within a factor of $n^{1-\epsilon}$ for any $\epsilon > 0$, unless $P = NP$. $\square$

6 APX-Completeness

In this section, we show that the ACYCLIC MATCHING PROBLEM is APX-complete for $2k + 1$ -regular graphs for $k \geq 3$, where k is a constant.

To prove the result, we first show that the ACYCLIC MATCHING PROBLEM is approximable within a constant factor when restricted to k -regular graphs for $k \geq 3$, where k is a constant. For the purpose, consider the following algorithm:

Algorithm 2. APPROX-AM(G)

Input: A graph $G = (V, E)$;

Output: An acyclic matching M_{ac} in G ;

$M_{ac} = \emptyset$;

while ($E \neq \emptyset$) **do**

 Choose an edge $e = uv$ from E ;

$M_{ac} = M_{ac} \cup \{uv\}$;

$V = V \setminus (N_G(u) \cup N_G(v))$;

return M_{ac} .

Lemma 8. *The algorithm APPROX-AM(G) produces an acyclic matching of G in polynomial time.*

Proof. For any pair of edges in M_{ac} , say $e_i = a_i b_i$ and $e_j = a_j b_j$, $G[\{a_i, b_i, a_j, b_j\}]$ is a disjoint union of K'_2 s. $\diamond$

Lemma 9. *The ACYCLIC MATCHING PROBLEM for a k -regular graph G can be approximated with an approximation ratio of $\frac{2k(k-1)+1}{k}$, where k is a constant.*

Proof. Given a k -regular graph G , construct an acyclic matching M_{ac} of G by using algorithm APPROX-AM(G). In each step, after adding an edge in the matching M_{ac} , we are removing at most k^2 edges, hence $\frac{kn}{2[2k(k-1)+1]} \leq |M_{ac}|$. Moreover, it is easy to see that the size of any matching can be at most $\frac{n}{2}$.

Hence, the ACYCLIC MATCHING PROBLEM is approximable within a factor of $\frac{2k(k-1)+1}{k}$ in k -regular graphs, where k is a constant. $\diamond$

To prove the result, we will need the following theorem for the MAXIMUM INDEPENDENT SET PROBLEM.

Theorem 11. [1, 10] *The MAXIMUM INDEPENDENT SET PROBLEM is APX-complete for k -regular graphs for $k \geq 3$.*

Observation 12. *If G is a k -regular graph in Construction 1, then the constructed graph H is a $2k + 1$ -regular graph for $k \geq 3$.*

Now, we are ready to prove the APX-completeness of the ACYCLIC MATCHING PROBLEM for $2k + 1$ -regular graphs for $k \geq 3$, where k is a constant. For this purpose, we recall the concept of L -reduction. Given two NP optimization problems π_1 and π_2 and a polynomial time transformation f from instances of π_1 to instances of π_2 , we say that f is an L -reduction if there are positive constants α and β such that for every instance x of π_1 :

1. $opt_{\pi_2}(f(x)) \leq \alpha \cdot opt_{\pi_1}(x)$;
2. for every feasible solution y of $f(x)$ with objective value $m_{\pi_2}(f(x), y) = c_2$, we can find a solution y' of x in polynomial time with $m_{\pi_1}(x, y') = c_1$ such that $|opt_{\pi_1}(x) - c_1| \leq \beta \cdot |opt_{\pi_2}(f(x)) - c_2|$.

Theorem 13. *The ACYCLIC MATCHING PROBLEM is APX-complete for $2k + 1$ -regular graphs for $k \geq 3$, where k is a constant.*

Proof. By Lemma 9, it is clear that the ACYCLIC MATCHING PROBLEM for $2k + 1$ -regular graphs for $k \geq 3$ belongs to the class APX. By Theorem 11, it is enough to construct an L -reduction from the instances of the MAXIMUM INDEPENDENT SET PROBLEM for k -regular graphs to the instances of the ACYCLIC MATCHING PROBLEM for $2k + 1$ -regular graphs. Given a k -regular graph $G = (V, E)$, where $V = \{v_1, v_2, \dots, v_n\}$. We construct a graph $H = (V_H, E_H)$, an instance of the ACYCLIC MATCHING PROBLEM by Construction 1. It is easy to see by Lemma 7 and Corollary 3 that the reduction described in Construction 1 is an L -reduction with $\alpha = 1$ and $\beta = 1$.

Therefore, the ACYCLIC MATCHING PROBLEM is APX-complete for $2k + 1$ -regular graphs for $k \geq 3$, where k is a constant. $\square$

7 Conclusion

In this paper, we have shown that the ACYCLIC MATCHING DECIDE PROBLEM is NP-complete for comb-convex bipartite graphs and dually chordal graphs. On the positive side, we have shown that the ACYCLIC MATCHING PROBLEM can be solved in linear time in split graphs and proper interval graphs. Apart from these, we have shown that the ACYCLIC MATCHING PROBLEM cannot be approximated within a factor of $n^{1-\epsilon}$ for any $\epsilon > 0$, unless $P = NP$. We have also shown that the ACYCLIC MATCHING PROBLEM is APX-complete for $2k + 1$ -regular graphs for $k \geq 3$, where k is a constant. Further, it will be interesting to study better approximation algorithms for this problem for bipartite graphs and other important graph classes.

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