Equitable d -degenerate Choosability of Graphs

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Abstract. Let $\mathcal{D}_d$ be the class of d -degenerate graphs and let L be a list assignment for a graph G . A colouring of G such that every vertex receives a colour from its list and the subgraph induced by vertices coloured with one color is a d -degenerate graph is called the $(L, \mathcal{D}_d)$ -colouring of G . For a k -uniform list assignment L and $d \in \mathbb{N}_0$, a graph G is equitably $(L, \mathcal{D}_d)$ -colorable if there is an $(L, \mathcal{D}_d)$ -colouring of G such that the size of any colour class does not exceed $\lceil |V(G)|/k \rceil$. An equitable $(L, \mathcal{D}_d)$ -colouring is a generalization of an equitable list coloring, introduced by Kostochka et al., and an equitable list arboricity presented by Zhang. Such a model can be useful in the network decomposition where some structural properties on subnets are imposed. In this paper we give a polynomial-time algorithm that for a given (k, d) -partition of G with a t -uniform list assignment L and $t \geq k$, returns its equitable $(L, \mathcal{D}_{d-1})$ -colouring. In addition, we show that 3-dimensional grids are equitably $(L, \mathcal{D}_1)$ -colorable for any t -uniform list assignment L where $t \geq 3$.

Keywords: Equitable choosability · d -degenerate graph

1 Motivation and Preliminaries

In last decades, a social network graphs, describing relationship in real life, started to be very popular and present everywhere. Understanding key structural properties of large-scale data networks started to be crucial for analyzing and optimizing their performance, as well as improving their security. This topic has been attracting attention of many researches, recently (see [1, 6, 7, 11]). We consider one of problems connected with the decomposition of networks into smaller pieces fulfilling some structural properties. For example, we may desire that, for some security reason, the pieces are acyclic or even independent. This is because of in such a piece we can easily and effectively identify a node failure since the local structure around such a node in this piece is so clear that it

can be easily tested using some classic algorithmic tools [11]. Sometimes, it is also desirable that the sizes of pieces are balanced. It helps us to maintain the whole communication network effectively. Such a problem can be modeled by minimization problems in graph theory, called an *equitable vertex arboricity* or an *equitable vertex colourability* of graphs. Sometimes we have some additional requirements on vertices/nodes that can be modeled by a list of available colours. So, we are interested in the list version, introduced by Kostochka, Pelsmajer and West [5] (an independent case), and by Zhang [10] (an acyclic case).

In colourability and arboricity models the properties of a network can be described in the language of the upper bound on the minimum degree, i.e. each colour class induces a graph whose each induced subgraph has the minimum degree bounded from above by zero or one, respectively. In the paper we consider the generalization of these models in which each colour class induces a graph whose each induced subgraph has the minimum degree bounded from above by some natural constant. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $d \in \mathbb{N}_0$, the graph G is *d-degenerate* if $\delta(H) \leq d$ for any subgraph H of G , where $\delta(H)$ denotes the minimum degree of H . The class of all *d-degenerate* graphs is denoted by $\mathcal{D}_d$. In particular, $\mathcal{D}_0$ is the class of all edgeless graphs and $\mathcal{D}_1$ is the class of all forests. A $\mathcal{D}_d$ -coloring is a mapping $c : V(G) \rightarrow \mathbb{N}$ such that for each $i \in \mathbb{N}$ the set of vertices coloured with i induces a *d-degenerate* graph. A *list assignment* L , for a graph G , is a mapping that assigns a nonempty subset of $\mathbb{N}$ to each vertex $v \in V(G)$. Given $k \in \mathbb{N}$, a list assignment L is *k-uniform* if $|L(v)| = k$ for every $v \in V(G)$. A colouring $c : V(G) \rightarrow \mathbb{N}$ such that $c(v) \in L(v)$ for each $v \in V(G)$ is called an *L-colouring*. Given $d \in \mathbb{N}_0$, a graph G is $(L, \mathcal{D}_d)$ -colourable if there exists such an *L-colouring* $c : V(G) \rightarrow \mathbb{N}$ that is also $\mathcal{D}_d$ -coloring. Such a mapping c is called an $(L, \mathcal{D}_d)$ -colouring of G . If f is any function defined on the set X , then its restriction to Y , $Y \subseteq X$, is denoted by $f|_Y$. For a partially coloured graph G , let $N_G^{col}(d, v) = \{w \in N_G(v) : w \text{ has } d \text{ neighbors coloured with } c(v)\}$, where $N_G(v)$ denotes the set of vertices of G adjacent to v . We refer the reader to [2] for terminology not defined in this paper.

Given $k \in \mathbb{N}$ and $d \in \mathbb{N}_0$, a graph G is *equitably (k, D_d)-choosable* if for any *k-uniform* list assignment L there is an $(L, \mathcal{D}_d)$ -colouring of G such that the size of any colour class does not exceed $\lceil |V(G)|/k \rceil$. The notion of equitable $(k, \mathcal{D}_0)$ -choosability was introduced by Kostochka et al. [5] whereas the notation of equitable $(k, \mathcal{D}_1)$ -choosability was introduced by Zhang [10].

Let $k, d \in \mathbb{N}$. A partition $S_1 \cup \dots \cup S_{\eta+1}$ of $V(G)$ is called a (k, d) -partition of G if $|S_1| \leq k$, and $|S_j| = k$ for $j \in \{2, \dots, \eta+1\}$, and for each $j \in \{2, \dots, \eta+1\}$, there is such an ordering $\{x_1^j, \dots, x_k^j\}$ of vertices of S_j that

$$|N_G(x_i^j) \cap (S_1 \cup \dots \cup S_{j-1})| \leq di - 1, \quad (1)$$

for every $i \in \{1, \dots, k\}$. Observe that if $S_1 \cup \dots \cup S_{\eta+1}$ is a (k, d) -partition of G , then $\eta + 1 = \lceil |V(G)|/k \rceil$. Moreover, immediately by the definition, each (k, d) -partition of G is also its $(k, d + 1)$ -partition. Surprisingly, the monotonicity of the (k, d) -partition with respect to the parameter k is not so easy to analyze. We illustrate this fact by Example 1. Note that for integers k, d the complexity

of deciding whether G has a (k, d) -partition is unknown. The main result of this paper is as follows.

Theorem 1. *Let $k, d, t \in \mathbb{N}$ and $t \geq k$. If a graph G has a (k, d) -partition, then it is equitably $(t, \mathcal{D}_{d-1})$ -choosable. Moreover, there is a polynomial-time algorithm that for any graph with a given (k, d) -partition and for any t -uniform list assignment L returns an equitable $(L, \mathcal{D}_{d-1})$ -colouring of G .*

The first statement of Theorem 1 generalizes the result obtained in [3] for $d \in \{1, 2\}$. In this paper we present an algorithm that confirms both, the first and second statements of Theorem 1 for all possible d . The algorithm, given in Sect. 2, for a given (k, d) -partition of G with t -uniform list assignment L returns its equitable $(L, \mathcal{D}_{d-1})$ -colouring. Moreover, in Sect. 3 we give a polynomial-time algorithm that for a given 3-dimensional grid finds its $(3, 2)$ -partition, what, in consequence, implies $(t, \mathcal{D}_1)$ -choosability of 3-dimensional grids for every $t \geq 3$.

2 The Proof of Theorem 1

2.1 Background

For $S \subseteq V(G)$ by $G - S$ we denote a subgraph of G induced by $V(G) \setminus S$. We start with a generalization of some results given in [5, 9, 10] for classes $\mathcal{D}_0$ and $\mathcal{D}_1$.

Proposition 1. *Let $k, d \in \mathbb{N}$ and let S be a set of distinct vertices $x_1, \dots, x_k$ of a graph G . If $G - S$ is equitably $(k, \mathcal{D}_{d-1})$ -choosable and*

$$|N_G(x_i) \setminus S| \leq di - 1$$

holds for every $i \in \{1, \dots, k\}$, then G is equitably $(k, \mathcal{D}_{d-1})$ -choosable.

Proof. Let L be a k -uniform list assignment for G and let c be an equitable $(L|_{V(G) \setminus S}, \mathcal{D}_{d-1})$ -colouring of $G - S$. Thus each colour class in c has the cardinality at most $\lceil (|V(G)| - k)/k \rceil$ and induces in $G - S$, and consequently in G , a graph from $\mathcal{D}_{d-1}$. We extend c to $(V(G) \setminus S) \cup \{x_k\}$ by assigning to x_k a colour from $L(x_k)$ that is used on vertices in $N_G(x_k) \setminus S$ at most $d - 1$ times. Such a colour always exists because $|N_G(x_k) \setminus S| \leq dk - 1$ and $|L(x_k)| = k$. Next, we colour vertices $x_{k-1}, \dots, x_1$, sequentially, assigning to x_i a colour from its list that is different from colours of all vertices $x_{i+1}, \dots, x_k$ and that is used at most $d - 1$ times in $N_G(x_i) \setminus S$. Observe that there are at least i colours in $L(x_i)$ that are different from $c(x_{i+1}), \dots, c(x_k)$, and, since $|N_G(x_i) \setminus S| \leq di - 1$ for $1 \leq i \leq k - 1$, then such a choice of $c(x_i)$ is always possible. Next, the colouring procedure forces that the cardinality of every colour class in the extended colouring c is at most $\lceil |V(G)|/k \rceil$. Let $G_i = G[(V(G) \setminus S) \cup \{x_i, \dots, x_k\}]$. It is easy to see that for each i each colour class in $c|_{V(G_i)}$ induces a graph belonging to $\mathcal{D}_{d-1}$, $1 \leq i \leq k$. In particular this condition is satisfied for G_1 , i.e. for G . Hence c is an equitable $(L, \mathcal{D}_{d-1})$ -colouring of G and G is equitably $(k, \mathcal{D}_{d-1})$ -choosable. $\square$

Note that if a graph G has a (k, d) -partition, then one can prove that G is equitably $(k, \mathcal{D}_{d-1})$ -choosable by applying Proposition 1 several times. In general, the equitable $(k, \mathcal{D}_{d-1})$ -choosability of G does not imply the equitable $(t, \mathcal{D}_{d-1})$ -choosability of G for $t \geq k$. Unfortunately, if G has a (k, d) -partition, then G may have neither a $(k+1, d)$ -partition nor a $(k-1, d)$ -partition. The infinite family of graphs defined in Example 1 confirms the last fact.

Example 1. Let $q \in \mathbb{N}$ and let $G_1, \dots, G_{2q+1}$ be vertex-disjoint copies of K_6 such that $V(G_i) = \{v_1^i, \dots, v_6^i\}$ for $i \in \{1, \dots, 2q+1\}$. Let $G(q)$ (cf. Fig. 1) be the graph resulted by adding to $G_1, \dots, G_{2q+1}$ edges that join vertices of G_i with vertices of G_{i-1} , $i \in \{2, \dots, 2q+1\}$, in the following way:

for i even:

$$N_{G_{i-1}}(v_1^i) = \emptyset$$

$$N_{G_{i-1}}(v_2^i) = \{v_1^{i-1}\}$$

$$N_{G_{i-1}}(v_3^i) = \{v_2^{i-1}, v_3^{i-1}\}$$

$$N_{G_{i-1}}(v_4^i) = \{v_1^{i-1}, v_2^{i-1}, v_3^{i-1}\}$$

$$N_{G_{i-1}}(v_5^i) = \{v_1^{i-1}, v_4^{i-1}, v_5^{i-1}, v_6^{i-1}\}$$

$$N_{G_{i-1}}(v_6^i) = \{v_2^{i-1}, v_3^{i-1}, v_4^{i-1}, v_5^{i-1}, v_6^{i-1}\}$$

for i odd:

$$N_{G_{i-1}}(v_1^i) = \{v_2^{i-1}, v_3^{i-1}, v_4^{i-1}, v_5^{i-1}, v_6^{i-1}\}$$

$$N_{G_{i-1}}(v_2^i) = \{v_1^{i-1}, v_4^{i-1}, v_5^{i-1}, v_6^{i-1}\}$$

$$N_{G_{i-1}}(v_3^i) = \{v_1^{i-1}, v_2^{i-1}, v_3^{i-1}\}$$

$$N_{G_{i-1}}(v_4^i) = \{v_2^{i-1}, v_3^{i-1}\}$$

$$N_{G_{i-1}}(v_5^i) = \{v_1^{i-1}\}$$

$$N_{G_{i-1}}(v_6^i) = \emptyset$$

Fig. 1. A draft of the graph $G(2)$ from Example 1.

The construction of $G(q)$ immediately implies that for every $q \in \mathbb{N}$ the graph $G(q)$ has a $(6, 1)$ -partition. Also, observe that $\deg_{G(q)}(v) \geq 5$ for each vertex v of $G(q)$. Suppose that $G(q)$ has a $(5, 1)$ -partition $S_1 \cup \dots \cup S_{\eta+1}$ with $S_{\eta+1} = \{x_1^{\eta+1}, \dots, x_5^{\eta+1}\}$ such that $|N_{G(q)}(x_i^{\eta+1}) \cap (S_1 \cup \dots \cup S_{\eta})| \leq i-1$. Thus $|N_{G(q)}(x_1^{\eta+1}) \cap (S_1 \cup \dots \cup S_{\eta})| = 0$ and consequently $\deg_{G(q)}(x_1^{\eta+1}) \leq 4$, contradicting our previous observation. Hence, $G(q)$ has no $(5, 1)$ -partition. In [4] we show that $G(q)$ has no $(7, 1)$ -partition for $q \geq 2$.

2.2 Algorithm

Now we are ready to present the algorithm that confirms both statements of Theorem 1. Note that Proposition 1 and the induction procedure could be used to prove the first statement of Theorem 1 for $t = k$ but, this approach seems to

be useless for $t > k$, as we have observed in Example 1. Mudrock et al. [8] proved the lack of monotonicity for the equitable $(k, \mathcal{D}_0)$ -choosability with respect to the parameter k . It motivates our approach for solving the problem. To simplify understanding we give the main idea of the algorithm presented in the further part. It can be expressed in a few steps (see also Fig. 2):

- on the base of the given (k, d) -partition $S_1 \cup \dots \cup S_{\eta+1}$ of G , we create the list S , consisting of the elements from $V(G)$, whose order corresponds to the order in which the colouring is expanded to successive vertices in each S_j (cf. the proof of Proposition 1);
- let $|V(G)| = \beta t + r_2$, $1 \leq r_2 \leq t$; we colour r_2 vertices from the beginning of S taking into account the lists of available colours; we delete the colour assigned to v from the lists of available colours for vertices from $N_G^{col}(d, v)$;
- let $|V(G)| = \beta(\gamma k + r) + r_2 = \beta\gamma k + (\rho k + x) + r_2$; observe that $r \equiv t \pmod{k}$ and $\rho k + x \equiv 0 \pmod{r}$; we colour $\rho k + x$ vertices taking into account the lists of available colours in such a way that every sublist of length k is formed by vertices coloured differently (consequently, every sublist of length r is coloured differently); we divide the vertices colored here into β sets each one of cardinality r ; we delete $c(v)$ from the lists of vertices from $N_G^{col}(d, v)$;
- we extend the list colouring into the uncoloured $\beta\gamma k$ vertices by colouring β groups of γk vertices; first, we associate each group of γk vertices with a set of r vertices coloured in the previous step (for different groups these sets are disjoint); next, we color the vertices of each of the group using γk different colors that are also different from the colors of r vertices of the set associated with this group;
- our final equitable list colouring is the consequence of a partition of $V(G)$ into $\beta + 1$ coloured sets, each one of size at most t and each one formed by vertices coloured differently.

Fig. 2. An exemplary illustration of the input of $\text{EQUITABLE}(L, \mathcal{D}_{d-1})$ -COLOURING.

Now we illustrate $\text{EQUITABLE}(L, \mathcal{D}_{d-1})$ -COLOURING using a graph from Example 2.

Algorithm 1: EQUITABLE $(L, \mathcal{D}_{d-1})$ -COLOURING(G)

Input : Graph G on n vertices; L - t -uniform list assignment; a (k, d) -partition $S_1 \cup \dots \cup S_{\eta+1}$ of G , given by lists $S_1 = (x_1^1, \dots, x_{t-1}^1)$ and $S_j = (x_1^j, \dots, x_k^j)$ for $j \in \{2, \dots, \eta + 1\}$.

Output : Equitable $(L, \mathcal{D}_{d-1})$ -colouring of G .

```

1 initialization;
2  $S := \text{empty}$ ;  $L_R := \text{empty}$ ;  $L_X := \text{empty}$ ;
3 for  $j := 1$  to  $\eta + 1$  do
4   | add REVERSE( $S_j$ ) to  $S$ ;      //REVERSE is the procedure for reversing lists
5 end
6  $\beta := \lceil n/t \rceil - 1$ ;
7 if  $n \equiv 0 \pmod{t}$  then
8   |  $r_2 := t$ 
9 else
10  |  $r_2 := n \pmod{t}$ ;
11 end
12  $\gamma := t \div k$ ;  $r := t \pmod{k}$ ;  $\rho := \beta r \div k$ ;  $x := \beta r \pmod{k}$ ;
13 take and delete  $r_2$  elements from the beginning of  $S$ , and add them, vertex
14   by vertex, to list  $L_R$ ;
15 COLOUR_LIST( $L_R, r_2$ );
16 take and delete  $x$  elements from the beginning of  $S$ , and add them, vertex
17   by vertex, to list  $L_X$ ;
18 COLOUR_LIST( $L_X, x$ );
19  $S_{col} := L_X$ ;
20 for  $j = 1$  to  $\rho$  do
21   |  $S' := \text{empty}$ ; take and delete  $k$  elements from the beginning of  $S$ , and add
22     them, vertex by vertex, to list  $S'$ ;
23   | COLOUR_LIST( $S', k$ );
24   |  $S_{col} := S_{col} + S'$ ;
25 end
26 REORDER( $S_{col}$ );
27  $\bar{S} := S$ ;                                     //an auxiliary list
28 COLOUR_LIST( $S, \gamma k$ );

```

Fig. 3. An exemplary graph G depicted in Example 2 with an exemplary colouring returned by EQUITABLE $(L, \mathcal{D}_{d-1})$ -COLOURING(G).

Procedure 2: COLOUR_LIST(S', p)

Input : List S' of vertices; integer p . //the length of S' is multiple of p **Output** : L -colouring of the vertices from S' .//The procedure also modifies a global variable of the list assignment L .

```

1 initialization;
2 while  $S' \neq \text{empty}$  do
3   let  $S''$  be the list of the  $p$  first elements of  $S'$ ;
4    $C := \emptyset$ ; //set  $C$  is reserved for the colours being assigned to the vertices of
       $S''$ 
5   while  $S'' \neq \text{empty}$  do
6     let  $v$  be the first element of  $S''$ ;
7      $L(v) := L(v) \setminus C$ ;
8      $c(v) := \text{COLOUR\_VERTEX}(v)$ ;
9     delete the vertex  $v$  from  $S'$  and  $S''$ ;  $C := C \cup \{c(v)\}$ ;
10  end
11 end

```

Procedure 3: COLOUR_VERTEX(v)

Input : Vertex v of the graph G .**Output** : L -colouring of the vertex v .//The procedure modifies also a global variable of the list assignment L .

```

1 initialization;
2  $c(v) :=$  any colour from  $L(v)$ ;
3 delete  $c(v)$  from  $L(w)$  for all  $w \in N_G^{col}(d, v)$ ; //  $d$  is a global variable
4 return  $c(v)$ ;

```

Procedure 4: REORDER(S')

Input : List S' of coloured vertices of G .**Output** : List S' - reordered in such a way that every its sublist of length k is formed by vertices being coloured with different colours.

```

1 initialization;
2  $S_{aux} := \text{empty}$ ; //an auxiliary list
3 take and delete  $r_1$  elements from  $S'$ , and add them, vertex by vertex, to  $S_{aux}$ ;
4 for  $j = 1$  to  $\eta - \beta\gamma$  do
5    $P := \emptyset$ ;
6   take and delete first  $k$  elements from  $S'$ , and add them to set  $P$ ;
7   for  $i = 1$  to  $k$  do
8     let  $v$  be a vertex from  $P$  such that  $c(v)$  is different from the colours of
       the last  $k - 1$  vertices of  $S_{aux}$ ; add  $v$  to the end of  $S_{aux}$ ;
9   end
10 end
11  $S' := S_{aux}$ ;

```

Procedure 5: MODIFY_COLOURLISTS(L_1, L_2)

Input : List L_1 of βr coloured vertices and list L_2 of $\beta\gamma k$ uncoloured vertices.

Output : Modified colour list assignment L for vertices of L_2 .

// L is a global variable

```

1 initialization;
2  $C := \emptyset$ ; //  $C$  is a set of colours of vertices from the depicted part of  $L_1$ 
3 for  $i = 1$  to  $\beta$  do
4   take and delete first  $r$  vertices from  $L_1$ ;
5   let  $C$  be the set of colours assigned to them;
6   for  $j = 1$  to  $\gamma k$  do
7     let  $v$  be the first vertex from  $L_2$ ;
8      $L(v) := L(v) \setminus C$ ; delete  $v$  from  $L_2$ ;
9   end
10 end
```

Example 2. Let G_1, G_2 be two vertex-disjoint copies of K_5 and $V(G_i) = \{v_1^i, \dots, v_5^i\}$ for $i \in \{1, 2\}$. We join every vertex v_j^2 to $v_j^1, v_{j+1}^1, \dots, v_5^1$ for $j \in \{1, 2, 3, 4, 5\}$. Next, we add a vertex w_j^i and join it with v_j^i for $i \in \{1, 2\}$ $j \in \{1, 2, 3, 4, 5\}$. In addition, we join w_j^i to arbitrary two vertices in $\{v_p^q : (q < i) \vee (q = i \wedge p < j)\} \cup \{w_p^q : (q < i) \vee (q = i \wedge p < j)\}$, $i \in \{2, 3, 4, 5\}$ $j \in \{1, 2\}$. Let G be a resulted graph. Observe that $|V(G)| = 20$ and the partition $S_1 \cup S_2 \cup \dots \cup S_{10}$ of $V(G)$ such that $S_{p+1} = \{v_{r+1}^{s+1}, w_{r+1}^{s+1}\}$ for $p \in \{0, \dots, 9\}$, where $s = \lfloor \frac{p}{5} \rfloor$, $r \equiv p \pmod{5}$ is a $(2, 3)$ -partition of G .

For the purpose of Example 2, we assume the following 3-uniform list assignment for the graph from Fig. 3: $L(v_j^i) = \{1, 2, 3\}$, for $i \in \{1, 2, 3, 4\}$, $L(w_j^1) = \{2, 3, 4\}$, and $L(w_j^2) = \{1, 2, 4\}$, $j \in \{1, 2, 3, 4, 5\}$, while given $(2, 3)$ -partition of G is: $S_p = \{w_{r+1}^{s+1}, v_{r+1}^{s+1}\}$, where $s = \lfloor \frac{p}{5} \rfloor$, $r \equiv p \pmod{5}$, $p \in [10]$.

Thus EQUITABLE($L, \mathcal{D}_{d-1}$)-COLOURING returns equitable $(L, \mathcal{D}_2)$ -coloring of G . Note, that $20 = |V(G)| = \eta \cdot k + r_1 = 9 \cdot 2 + 2$. While on the other hand, we have $20 = |V(G)| = \beta \cdot t + r_2 = 6 \cdot 3 + 2$. Observe that $x = 0$. When we colour a vertex, we always choose the first colour on its list. The list S determined in lines 3–5 of EQUITABLE($L, \mathcal{D}_{d-1}$)-COLOURING and the colours assigned to first part of vertices of S (lines 18–24) are as follows:

$$S = \left(\begin{array}{c|c|c} \begin{array}{c} v_1^1, w_1^1 \\ r_2 \end{array} & \begin{array}{c} v_2^1, w_2^1, v_3^1, w_3^1, v_4^1, w_4^1 \\ \rho k \end{array} & \begin{array}{c} v_5^1, w_5^1, v_1^2, w_1^2, \dots, v_5^2, w_5^2 \\ \beta\gamma k \end{array} \end{array} \right)$$

$$\text{colours: } \begin{array}{cccccc} 1 & 2 & 1 & 2 & 2 & 3 \end{array}$$

*: after colouring v_3^1 with 1, $L(v_1^2) = \{2, 3\}$ - the result of line 3 in COLOUR_VERTEX.

List S_{col} after REORDER(S_{col}): $(v_2^1, w_2^1, v_3^1, w_3^1, w_4^1, v_4^1)$ with corresponding colours: $(1, 2, 1, 2, 3, 2)$.

$\bar{S} =$	$(v_5^1, w_5^1, v_1^2, w_1^2, v_2^2, w_2^2, v_3^2, w_3^2, v_4^2, w_4^2, v_5^2, w_5^2)$											
lists after	2	2	3	1	2	2	1	1	1	1	1	1
procedure	3	3		4	3	4	3	4	2	2	3	4
MODIFY_COLOURLIST		4								4		
final $c(v)$	2	3	3	1	2	4	1	4	1	2	1	4

To prove the correctness of the $\text{EQUITABLE}(L, \mathcal{D}_{d-1})$ -COLOURING algorithm, we give some observations and lemmas.

Observation 2. *The colour function c returned by the $\text{EQUITABLE}(L, \mathcal{D}_{d-1})$ -COLOURING algorithm is constructed step by step. In each step, $c(v)$ is a result of $\text{COLOUR_VERTEX}(v)$ and this value is not changed further.*

Observation 3. *The list assignment L , as a part of the input of $\text{EQUITABLE}(L, \mathcal{D}_{d-1})$ -COLOURING, is modified for a vertex v by COLOUR_LIST or by $\text{MODIFY_COLOURLISTS}$.*

Lemma 1. *Every time when $\text{EQUITABLE}(L, \mathcal{D}_{d-1})$ -COLOURING(G) calls $\text{COLOUR_VERTEX}(v)$, $L(v) \neq \emptyset$ holds, i.e. $\text{COLOUR_VERTEX}(v)$ is always executable.*

Proof. Note that COLOUR_VERTEX is called by COLOUR_LIST . Let

$R = \{v \in V(G) : \text{COLOUR_VERTEX}(v) \text{ is called when } \text{COLOUR_LIST}(L_R, r_2) \text{ in line 15 of } \text{EQUITABLE}(L, \mathcal{D}_{d-1})\text{-COLOURING is executed}\},$

$X = \{v \in V(G) : \text{COLOUR_VERTEX}(v) \text{ is called when } \text{COLOUR_LIST}(L_X, x) \text{ in line 18 of } \text{EQUITABLE}(L, \mathcal{D}_{d-1})\text{-COLOURING is executed}\}.$

Let $V_1 := S_1 \cup \dots \cup S_{\eta+1-\beta\gamma}$, $V_2 := V(G) \setminus V_1 = S_{\eta+1-(\beta\gamma-1)} \cup \dots \cup S_{\eta+1}$. Note that

$V_1 \setminus (R \cup X) = \{v \in V(G) : \text{COLOUR_VERTEX}(v) \text{ is called when } \text{COLOUR_LIST}(S', k) \text{ in line 22 of } \text{EQUITABLE}(L, \mathcal{D}_{d-1})\text{-COLOURING is executed}\}$

$V_2 = \{v \in V(G) : \text{COLOUR_VERTEX}(v) \text{ is called when } \text{COLOUR_LIST}(S, \gamma k) \text{ in line 28 of } \text{EQUITABLE}(L, \mathcal{D}_{d-1})\text{-COLOURING is executed}\}.$

Observe that $|R| = r_2, |X| = x, |V_1 \setminus (R \cup X)| = \rho k, |V_2| = \beta\gamma k$.

Case 1. $v \in R$.

In this case, the vertex v is coloured by $\text{COLOUR_LIST}(L_R, r_2)$ in line 15 of $\text{EQUITABLE}(L, \mathcal{D}_{d-1})$ -COLOURING. Since $|R| = r_2$, the **while** loop in line 2 of COLOUR_LIST is executed only once. The **while** loop in line 5 of COLOUR_LIST is executed r_2 times. Suppose, v is a vertex such that $\text{COLOUR_VERTEX}(v)$ is called in the i -th execution of the **while** loop in line 5 of COLOUR_LIST . By Observation 3, the fact that $\text{EQUITABLE}(L, \mathcal{D}_{d-1})$ -COLOURING has not called $\text{MODIFY_COLOURLISTS}$ so far, and because it is the first time when COLOUR_LIST works, we have $|C| = i - 1$, and $L(v) \setminus C$ is the current list of v . Since $t \geq r_2$, the list of v is non-empty.

Case 2. $v \in X$.

This time, the vertex v is coloured by $\text{COLOUR_LIST}(L_X, x)$ called in line 18 of $\text{EQUITABLE}(L, \mathcal{D}_{d-1})$ -COLOURING. Similarly as in *Case 1*, the **while** loop in line 2 of COLOUR_LIST is executed only once and the **while** loop in line

5 of COLOUR_LIST is executed x times. Suppose that v is a vertex such that COLOUR_VERTEX(v) is called in the i -th iteration of the **while** loop in line 5 of COLOUR_LIST. Observe that properties of the (k, d) -partition $S_1 \cup \dots \cup S_{\eta+1}$ of G (given as the input of EQUITABLE($L, \mathcal{D}_{d-1}$)-COLOURING) and the REVERSE procedure from line 4 of EQUITABLE($L, \mathcal{D}_{d-1}$)-COLOURING imply that v has at most $(x - i + 1)d - 1 + (k - x)$ neighbors w for which COLOUR_VERTEX(w) was executed earlier than COLOUR_LIST(L_X, x). More precisely, by the definition of the (k, d) -partition, v has at most $(x - i + 1)d - 1$ neighbours in $R \setminus Y$, where Y consists of the last $k - x$ vertices w for which COLOUR_VERTEX(w, d) is executed, being called by COLOUR_LIST(L_R, r_2). Thus, at most $k - i$ colours were deleted from $L(v)$ before COLOUR_LIST(L_X, x) began. If the **while** loop in line 5 of COLOUR_LIST is called for the i -th time, then $|C| = i - 1$ and so, from the current list $L(v)$ at most $i - 1$ elements were deleted. Furthermore, EQUITABLE($L, \mathcal{D}_{d-1}$)-COLOURING has not called MODIFY_COLOUR_LISTS so far. Thus the current size of $L(v)$ is at least $t - k + 1$, by Observation 3. Since $t \geq k$, the list of v is non-empty.

In a similar way we prove the remaining two cases, namely when $v \in V_1 \setminus (R \cup X)$ or $v \in V_2$. The full proof is given in [4]. $\square$

Lemma 2. *An output of EQUITABLE($L, \mathcal{D}_{d-1}$)-COLOURING(G) is an $(L, \mathcal{D}_{d-1})$ -colouring of G .*

Proof. We will show that if COLOUR_VERTEX(v) is executed, then an output $c(v)$ has always the following property. For each subgraph H of G induced by vertices x for which COLOUR_VERTEX(x) was executed so far with the output $c(x) = c(v)$, the condition $\delta(H) \leq d - 1$ holds. By Observation 2 and Lemma 1, it will imply that an output c of EQUITABLE($L, \mathcal{D}_{d-1}$)-COLOURING is an $(L, \mathcal{D}_{d-1})$ -colouring of G . Note that it is enough to show this fact for H satisfying $v \in V(H)$. By a contradiction, let v be a vertex for which the output $c(v)$ does not satisfy the condition, i.e. v has at least d neighbors in the set of vertices for which COLOUR_VERTEX was already executed with the output $c(v)$. But it is not possible because $c(v)$ was removed from $L(v)$ when the last (in the sense of the algorithm steps) of the neighbors of v , say x , obtained the colour $c(v)$ (COLOUR_VERTEX(x) removed $c(x)$ from $L(v)$ since $v \in N_G^{col}(d, x)$). $\square$

Lemma 3. *An output colour function c of EQUITABLE($L, \mathcal{D}_{d-1}$)-COLOURING(G) satisfies $|C_i| \leq \lceil |V(G)|/t \rceil$, where t is the part of the input of EQUITABLE($L, \mathcal{D}_{d-1}$)-COLOURING(G) and $C_i = \{v \in V(G) : c(v) = i\}$.*

Proof. Recall that $\lceil |V(G)|/t \rceil = \beta + 1$. We will show that there exists a partition of $V(G)$ into $\beta + 1$ sets, say $W_1 \cup \dots \cup W_{\beta+1}$, such that for each $i \in \{1, \dots, \beta + 1\}$ any two vertices x, y in W_i satisfy $c(x) \neq c(y)$. It will imply that the cardinality of every colour class in c is at most $\beta + 1$, giving the assertion.

Note that after the last, ρ -th execution of the **for** loop in line 20 of EQUITABLE($L, \mathcal{D}_{d-1}$)-COLOURING the list S_{col} consists of the coloured vertices of the set $V_1 \setminus R$ (observe that $|V_1 \setminus R| = \beta r$). The elements of S_{col} are ordered in such a way that the first x ones have different colours and for every $i \in \{1, \dots, \rho\}$ the i -th next k elements have different colours. Now the REORDER(S_{col}) procedure

in line 25 of `EQUITABLE` $(L, \mathcal{D}_{d-1})$ -COLOURING changes the ordering of elements of S_{col} in such a way that every k consecutive elements have different colours. Since $r \in \{0, \dots, k-1\}$, it follows that also every r consecutive elements of this list have different colours. The execution of `REORDER`(S_{col}) is always possible because of the previous assumptions on S_{col} .

For $i \in \{1, \dots, \beta\}$ let $H_i = S_{\eta+1-((\beta-i+1)\gamma-1)} \cup S_{\eta+1-((\beta-i+1)\gamma-2)} \cup \dots \cup S_{\eta+1-(\beta-i)\gamma}$. Thus $H_1 \cup \dots \cup H_\beta$ is a partition of V_2 into β sets, each of the cardinality γk . Note that the vertices of H_i are coloured when `COLOUR_LIST`($S, \gamma k$) in line 28 of `EQUITABLE` $(L, \mathcal{D}_{d-1})$ -COLOURING is executed. More precisely, it is during the i -th execution of the **while** loop in line 2 of `COLOUR_LIST`. It guarantees that the vertices of H_i obtain pairwise different colours. Moreover, in line 27 of `EQUITABLE` $(L, \mathcal{D}_{d-1})$ -COLOURING the lists of vertices of H_i were modified in such a way that the colours of i -th r elements from the current list S_{col} are removed from the list of each element in H_i . Hence, after the execution of `COLOUR_LIST`($S, \gamma k$) in line 28 of `EQUITABLE` $(L, \mathcal{D}_{d-1})$ -COLOURING the elements in H_i obtain colours that are pairwise different and also different from all the colours of i -th r elements from the list S_{col} (recall that S_{col} consists of the ordered vertices of $V_1 \setminus R$). Hence, for every $i \in \{1, \dots, \beta\}$ the elements of H_i and the i -th r elements of S_{col} have pairwise different colours in c and can constitute W_i . Moreover, the elements of R constitute $W_{\beta+1}$. Thus $|W_{\beta+1}| = r_2$, which finishes the proof. $\square$

Theorem 4. *For a given graph G on n vertices, a t -uniform list assignment L , a (k, d) -partition of G the `EQUITABLE` $(L, \mathcal{D}_{d-1})$ -COLOURING(G) algorithm returns $(L, \mathcal{D}_{d-1})$ -colouring of G in polynomial time.* $\square$

The full analysis of the computational complexity of the algorithm is given in [4].

3 Grids

Given two graphs G_1 and G_2 , the *Cartesian product* of G_1 and G_2 , $G_1 \square G_2$, is defined to be a graph whose the vertex set is $V(G_1) \times V(G_2)$ and the edge set consists of all edges joining vertices (x_1, y_1) and (x_2, y_2) when either $x_1 = x_2$ and $y_1 y_2 \in E(G_2)$ or $y_1 = y_2$ and $x_1 x_2 \in E(G_1)$. Note that the Cartesian product is commutative and associative. Hence the graph $G_1 \square \dots \square G_d$ is unambiguously defined for any $d \in \mathbb{N}$. Let P_n denote a path on n vertices. If each factor G_i is a path on at least two vertices then $G_1 \square \dots \square G_d$ is a *d -dimensional grid*. Note that the d -dimensional grid $P_{n_1} \square \dots \square P_{n_d}$, $d \geq 3$, may be considered as n_1 layers and each layer is the $(d-1)$ -dimensional grid $P_{n_2} \square \dots \square P_{n_d}$. We assume $n_1 \geq \dots \geq n_d$. Let $P_{n_1} \square \dots \square P_{n_d}$ denote an *incomplete d -dimensional grid*, i.e. a connected graph being a subgraph of $P_{n_1} \square \dots \square P_{n_d}$ such that its some initial layers may be empty, the first non-empty layer may be incomplete, while any next layer is complete. Note that every grid is particular incomplete grid.

In this subsection we construct a polynomial-time algorithm that for each 3-dimensional grid finds its $(3, 2)$ -partition (`PARTITION3D`(G)). Application of Theorem 1 implies the main result of this subsection.

Theorem 5. *Let $t \geq 3$ be an integer. Every 3-dimensional grid is equitably $(t, \mathcal{D}_1)$ -choosable. Moreover, there is a polynomial-time algorithm that for every t -uniform list assignment L of the 3-dimensional grid G returns an equitable $(L, \mathcal{D}_1)$ -colouring of G .* $\square$

Procedure 6: CORNER(G)

Input : Incomplete non-empty d -dimensional grid $G = P_{n_1} \square \dots \square P_{n_d}$,
 $d \geq 2$.
Output : Vertex $y = (a_1, \dots, a_d) \in V(G)$ such that $\deg_G(y) \leq d$.
1 initialization;
2 let a_1 be the number of the incomplete layer of G ;
3 **for** $i = 2$ **to** $d - 1$ **do**
4 | $a_i = \min\{x_i : \exists_{x_{i+1}, \dots, x_d} (a_1, \dots, a_{i-1}, x_i, \dots, x_d) \in V(G)\}$.
5 **end**
6 $a_d := \min\{x_d : (a_1, \dots, a_{d-1}, x_d) \in V(G)\}$.
7 **return** $(a_1, \dots, a_d)$;

Algorithm 7: PARTITION3D(G)

Input : 3-dimensional grid $G = P_{n_1} \square P_{n_2} \square P_{n_3}$.
Output : A $(3, 2)$ -partition $S_1 \cup \dots \cup S_{\alpha+1}$ of G .
1 initialization;
2 $\alpha := \lceil \frac{n_1 n_2 n_3}{3} \rceil - 1$;
3 **if** $\alpha \geq 1$ **then**
4 | **for** $j := \alpha + 1$ **downto** 2 **do**
5 | $y_1^j = (a_1, a_2, a_3) := \text{Corner}(G)$;
6 | **if** $\deg(y_1^j) = 1$ **then**
7 | | $y_2^j := \text{Corner}(G - y_1^j)$;
8 | | **if** y_1^j is the only vertex on a_1 layer **then**
9 | | | let y_3^j be any vertex on layer $a_1 + 1$ such that $y_3^j \neq y_2^j$
10 | | **else**
11 | | | let y_3^j be any vertex on layer a_1 such that $y_3^j \neq y_1^j$ and $y_3^j \neq y_2^j$,
12 | | | if exists, otherwise y_3^j is any vertex on layer $a_1 + 1$
13 | | **end**
14 | **end**
15 | **if** $\deg(y_1^j) = 2$ **then**
16 | | let y_2^j be the neighbour of y_1^j lying on the same layer as y_1^j ;
17 | | let y_3^j be any vertex on layer a_1 , if exists, otherwise, y_3^j is any
18 | | vertex on layer $a_1 + 1$
19 | **end**
20 | **if** $\deg(y_1^j) = 3$ **then**
21 | | $y_2^j := (a_1, a_2, a_3 + 1)$; $y_3^j := (a_1, a_2 + 1, a_3)$;
22 | **end**
23 | $S_j := \{y_1^j, y_2^j, y_3^j\}$; $G := G - S_j$;
24 | **end**
25 **end**
26 $S_1 := V(G)$;

Theorem 6. *For a given 3-dimensional grid G the $\text{PARTITION3D}(G)$ algorithm returns a $(3, 2)$ -partition of G in polynomial-time.*

The proof of Theorem 6 is presented in [4]. As a consequence of the above theorem and Theorem 1 we get the statement of Theorem 5.

4 Concluding Remarks

In Subsect. 2.2 we have proposed the polynomial-time algorithm that finds an equitable $(L, \mathcal{D}_{d-1})$ -colouring of a given graph G assuming that we know a (k, d) -partition of G (L is a t -uniform list assignment for G , $t \geq k$). In this context the following open question seems to be interesting: What is the complexity of recognition of graphs having a (k, d) -partition?

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