Fifth-Order Finite-Volume WENO on Cylindrical Grids

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1 Introduction

The conventional WENO scheme is specifically designed for the reconstruction in Cartesian coordinates on uniform grids [1]. The employment of Cartesian-based reconstruction scheme on a cylindrical grid suffers from a number of drawbacks [2, 3], e.g., in the original PPM paper, reconstruction was performed in volume coordinates (than the linear ones) so that algorithm for a Cartesian mesh can be used on a curvilinear mesh. However, the resulting interface states became first-order accurate even for smooth flows [2]. Another example can be the volume average assignment to the geometrical cell center of finite-volume than the centroid [2]. A breakthrough in the field of high order reconstruction in cylindrical coordinates is the application of the Vandermonde-like linear systems of equations with spatially

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varying coefficients [2]. It is reintroduced in the present work to build a basis for the derivation of the high order WENO schemes.

The motivation for the present work is to develop a fifth-order finite-volume WENO reconstruction scheme in the efficient dimension-by-dimension framework, specifically aimed at regularly-spaced and irregularly-spaced grids in cylindrical coordinates.

2 Finite-Volume Discretization in Curvilinear Coordinates

2.1 Evaluation of the Linear Weights

A non-uniform grid spacing with zone width $\Delta\xi_i = \xi_{i+\frac{1}{2}} - \xi_{i-\frac{1}{2}}$ is considered having $\xi \in (x_1, x_2, x_3)$ as the coordinate along the reconstruction direction and $\xi_{i+\frac{1}{2}}$ denoting the location of the cell interface between zones i and $i + 1$. Let $\bar{Q}_i$ be the cell average of conserved quantity Q inside zone i at some given time, which can be expressed in form of Eq. (1).

$$\bar{Q}_i = \frac{1}{\Delta\mathcal{V}_i} \int_{\xi_{i-\frac{1}{2}}}^{\xi_{i+\frac{1}{2}}} Q_i(\xi) \frac{\partial\mathcal{V}}{\partial\xi} d\xi \quad \& \quad \Delta\mathcal{V}_i = \int_{\xi_{i-\frac{1}{2}}}^{\xi_{i+\frac{1}{2}}} \frac{\partial\mathcal{V}}{\partial\xi} d\xi \quad (1)$$

where the local cell volume $\Delta\mathcal{V}_i$ of i th cell in the direction of reconstruction given in Eq. (1) and $\frac{\partial\mathcal{V}}{\partial\xi}$ is the one-dimensional Jacobian. Now, our aim is to find a p th order accurate approximation to the actual solution by constructing a $(p - 1)$ th order polynomial distribution, as given in Eq. (2).

$$Q_i(\xi) = a_{i,0} + a_{i,1}(\xi - \xi_i^c) + a_{i,2}(\xi - \xi_i^c)^2 + \dots + a_{i,p-1}(\xi - \xi_i^c)^{p-1} \quad (2)$$

where $a_{i,n}$ corresponds to a vector of the coefficients which needs to be determined and ξ_i^c can be taken as the cell centroid. However, the final values at the interface are independent of the particular choice of ξ_i^c and one may as well set $\xi_i^c = 0$ [2]. Unlike the cell center, the centroid is not equidistant from the cell interfaces in the case of cylindrical-radial coordinates, and the cell averaged values are assigned at the centroid [2]. Further, the method has to be locally conservative, i.e., the polynomial $Q_i(\xi)$ must fit the neighboring cell averages, satisfying Eq. (3).

$$\int_{\xi_{i+s-\frac{1}{2}}}^{\xi_{i+s+\frac{1}{2}}} Q_i(\xi) \frac{\partial\mathcal{V}}{\partial\xi} d\xi = \Delta\mathcal{V}_{i+s} \bar{Q}_{i+s} \quad \text{for} \quad -i_L \leq s \leq i_R \quad (3)$$

where the stencil includes i_L cells to the left and i_R cells to the right of the i th zone such that $i_L + i_R + 1 = p$. Implementing Eqs. (1)–(2) in Eq. (3) along with a simple mathematical manipulation leads to Eq. (4), which is the fundamental equation for

reconstruction in cylindrical coordinates. For the detailed derivation, kindly refer to [3].

$$\begin{pmatrix} \beta_{i-L,0} & \dots & \beta_{i-L,p-1} \\ \vdots & \ddots & \vdots \\ \beta_{i+R,0} & \dots & \beta_{i+R,p-1} \end{pmatrix}^T \begin{pmatrix} w_{i,-i_L}^\pm \\ \vdots \\ w_{i,i_R}^\pm \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ (\xi_{i \pm \frac{1}{2}} - \xi_i^c)^{p-1} \end{pmatrix} \quad (4)$$

where ‘ $\pm$ ’ represents the positive and negative weights i.e. weights for reconstructing right (+) and left (−) interface values respectively. Also, the grid dependent linear weights ($w_{i,s}^\pm$) satisfy the normalization condition [2].

2.2 Optimal Weights

For the case of fifth-order WENO interpolation, the third order interpolated variables are optimally weighed in order to achieve fifth-order accurate interpolated values as given in Eq. (5) for the case of $p = 3$ [1].

$$q_{i,0}^{(2p-1)\pm} = \sum_{l=0}^{p-1} C_{i,l}^\pm q_{i,l}^{p\pm} \quad (5)$$

where $C_{i,l}^\pm$ is the optimal weight for the positive/negative cases on the i th finite-volume. So, Eq. (4) is used again to evaluate the weights for the fifth-order ($2p-1 = 5$) interpolation ($i_L = 2, i_R = 2$).

Linear and optimal weights are independent of the mesh size for standard regularly-spaced grid cases. They can be evaluated and stored (at a nominal cost) independently before the actual computation. Also, they conform to the original WENO-JS [1] for the limiting case ($R \rightarrow \infty$). The weights required for source term and flux integration in one or more dimensions are given in [3].

2.3 Smoothness Indicators and the Nonlinear Weights

The mathematical definition of the smoothness indicator is given in Eq. (6) [1].

$$IS_{i,l} = \sum_{m=1}^{p-1} \int_{\xi_{j-\frac{1}{2}}}^{\xi_{j+\frac{1}{2}}} \left(\frac{d^m}{d\xi^m} Q_{i,l}(\xi) \right)^2 \Delta \xi_i^{2m-1} d\xi, \quad l = 0, \dots, p-1 \quad (6)$$

To evaluate the value of $IS_{i,l}$, a third order polynomial interpolation on i th cell is required using positive and negative reconstructed values by stencil S_l , as given in Eq. (2). Finally, evaluating the values of the coefficient a ’s and substituting their

values in smoothness indicator formula (6) yields the grid-independent fundamental relation (7). The nonlinear weight ($\omega_{i,l}^\pm$) for the WENO-C interpolation is defined in Eq. (8) [1], where ϵ is chosen to be 10^{-6} [1, 3].

$$IS_{i,l} = 4(39\bar{Q}_i^2 - 39\bar{Q}_i(q_{i,l}^- + q_{i,l}^+) + 10((q_{i,l}^-)^2 + (q_{i,l}^+)^2) + 19q_{i,l}^-q_{i,l}^+) \quad (7)$$

$$\omega_{i,l}^\pm = \frac{\alpha_{i,l}^\pm}{\sum_{l=0}^{p-1} \alpha_{i,l}^\pm} \quad \& \quad \alpha_{i,l}^\pm = \frac{C_{i,l}^\pm}{(\epsilon + IS_{i,l})^2} \quad l = 0, 1, 2 \quad (8)$$

The final interpolated interface values are evaluated from Eq. (9).

$$q_i^{(2p-1)\pm} = \sum_{l=0}^{p-1} \omega_{i,l}^{p\pm} q_{i,l}^{p\pm} \quad (9)$$

3 Stability Analysis of WENO-C for Hyperbolic Conservation Laws

For WENO-C to be practically useful, it is crucial that it enables a stable discretization for hyperbolic conservation laws when coupled with a proper time-integration scheme. In this section, we analyze WENO-C scheme for model problems involving smooth flow in 1D cylindrical-radial coordinates, based on a modified von Neumann stability analysis [4]. We consider scalar advection equation (10) in 1D cylindrical-radial coordinates.

$$\frac{\partial Q}{\partial t} + \frac{1}{(\partial \mathcal{V}/\partial \xi)} \frac{\partial}{\partial \xi} \left(\left(\frac{\partial \mathcal{V}}{\partial \xi} \right) Qv \right) = 0 \quad \xi \in [0, \infty], \quad t > 0 \quad (10)$$

where Q is the conserved variable, $(\partial \mathcal{V}/\partial \xi) = \xi$ is the one-dimensional Jacobian in cylindrical-radial coordinates. Boundary conditions are not considered in the present approach to reduce the complexity of the analysis. Assuming a uniform grid with $\xi_i = i \Delta \xi$ and $\xi_{i+1} - \xi_i = \Delta \xi \forall i$ and $(i \pm 1/2)$ denotes the boundaries of the finite-volume i . In the finite-volume framework, Eq. (10) transforms into the conservative scheme given in Eq. (11).

$$\frac{\partial \bar{Q}_i}{\partial t} = -\frac{1}{\Delta \mathcal{V}_i} (\hat{F}_{i+1/2} - \hat{F}_{i-1/2}) \quad (11)$$

where numerical flux $\hat{F}_{i+1/2}$ is the Lax-Friedrich flux, and $\bar{Q}_i$ and $\mathcal{V}_i$ are given in Eq. (1). For this particular problem, let $v = 1$ in Eq. (10). Therefore, only the values on the left side of the interface are considered. Based on the von Neumann stability analysis, the semi-discrete solution can be expressed as a discrete Fourier series. By the superposition principle, only one term in the series can be used for analysis, as

illustrated in Eq. (12).

$$\bar{Q}_i(t) = \hat{Q}_k(t)e^{ji\theta_k}, \quad \text{where } j = \sqrt{-1} \quad (12)$$

By substituting Eq. (12) in Eq. (11), we can separate the spatial operator L , as given in Eq. (13).

$$L = -\frac{(\hat{F}_{i+1/2} - \hat{F}_{i-1/2})}{\Delta \mathcal{V}_i} = -\frac{[Q(\partial \mathcal{V}/\partial \xi)]_{i+1/2}^- - [Q(\partial \mathcal{V}/\partial \xi)]_{i-1/2}^-}{\Delta \mathcal{V}_i} = -\frac{z(\theta_k) \bar{Q}_i}{\Delta \xi} \quad (13)$$

where the complex function $z(\theta_k)$ is the Fourier symbol. By substituting the values of $Q_{i-1/2}^-$ and $Q_{i+1/2}^-$ using fifth-order positive weights of cells $(i-1)$ and i respectively for a smooth solution, the value of $z(\theta_k)$ for WENO-C can be evaluated using Eq. (14).

$$z(\theta_k) = \frac{m+1}{i^{(m+1)} - (i-1)^{(m+1)}} \sum_{l=-2}^{+2} \left[w_{i,l}^+ i^m e^{jl\theta_k} - w_{(i-1),l}^+ (i-1)^m e^{j(l-1)\theta_k} \right] \quad (14)$$

where $m = 1$ for cylindrical-radial coordinates. Using the same approach as given in [4], we can plot the spatial spectrum $\{S : -z(\theta_k) \text{ for } \theta_k \in [0, 2\pi]\}$ and the stability domain S_I for TVD-RK order 3. The maximum stable CFL number of this scheme can be computed by finding the largest rescaling parameter $\tilde{\sigma}$, so that the rescaled spectrum still lies in the stability domain.

It can be observed from Fig. 1 that the spatial spectrums S of WENO-C differs initially with the index numbers i due to the geometrical variation of the finite-volume. However, the spectrums are the same for high index numbers (i), similar to WENO-JS, as the fifth-order interpolation weights converge. Some regions ($i = 1, 2$) require boundary conditions and thus, are not considered in the present analysis. The values of CFL number for cylindrical-radial coordinates lie in between 1.45 and 1.52. As a final remark, it can be concluded that the proposed scheme is A-stable with third or higher order of RK method with an appropriate value of CFL number for this case.

4 Numerical Tests

In this section, several tests on Euler equations are performed to analyze the performance of the WENO-C reconstruction scheme. Tests are performed on a gamma law gas ($\gamma = 1.4$) in cylindrical coordinates to investigate the essentially non-oscillatory property of WENO-C for discontinuous flows and the convex combination property for smooth flows. For first-order and second-order (MUSCL)

Fig. 1 Rescaled spectra (with maximum stable CFL number $\tilde{\sigma}$) and stability domains of fifth-order WENO-C in cylindrical-radial coordinates in a complex plane for different cell index numbers i . (a) $i = 3$, $\tilde{\sigma} = 1.45$. (b) $i = 5$, $\tilde{\sigma} = 1.52$. (c) $i = 10$, $\tilde{\sigma} = 1.50$. (d) $i = 50$, $\tilde{\sigma} = 1.46$. (e) $i = 100$, $\tilde{\sigma} = 1.45$. (f) Legend

spatial reconstructions, Euler time marching and Maccormack (predictor-corrector) schemes are respectively employed. For WENO-C, time marching is done with TVD-RK order 3 for 1D cases and RK order 5 for the 2D case.

4.1 Acoustic Wave Propagation

A smooth problem involving a nonlinear system of 1D gas dynamical equations is solved to test fifth-order accuracy of the spatial discretization scheme [3]. The Euler equations in cylindrical-radial coordinates can be written in the form of Eq. (15).

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix} + \frac{1}{R} \frac{\partial}{\partial R} \begin{pmatrix} \rho u R \\ (\rho u^2 + p) R \\ (E + p) u R \end{pmatrix} = \begin{pmatrix} 0 \\ p/R \\ 0 \end{pmatrix} \quad (15)$$

where ρ is the mass density, u is the radial velocity, p is the pressure, and E is the total energy. Equation (16) serves as the adiabatic equation of state.

$$E = \frac{p}{\gamma - 1} + \frac{1}{2} \rho u^2 \quad (16)$$

The initial conditions are provided in Eq. (17) with the perturbation given in Eq. (18). The interface flux is evaluated with Rusanov scheme [3].

$$\rho(R, 0) = 1 + \varepsilon f(R), \quad u(R, 0) = 0, \quad p(R, 0) = 1/\gamma + \varepsilon f(R) \quad (17)$$

$$f(R) = \begin{cases} \frac{\sin^4(5\pi R)}{R} & \text{if } 0.4 \leq R \leq 0.6 \\ 0 & \text{otherwise} \end{cases} \quad (18)$$

A sufficiently small perturbation with $\varepsilon = 10^{-4}$ yields a smooth solution. The interface flux is evaluated using Rusanov scheme with a CFL number of 0.3.

The initial perturbation splits into two acoustic waves traveling in opposite directions. The final time ($t = 0.3$) is set such that the waves remain in the domain and the problem is free from the boundary effects. The computational domain of unity length is uniformly divided into N different zones i.e. $N = 16, 32, 64, 128, 256$. Although an exact solution known up to $O(\varepsilon^2)$ is known, the solution on the finest mesh $N = 1024$ is taken as the reference. Figure 2 illustrate the spatial variation of density at $t = 0.3$ inside the domain. From Table 1, it clear that the scheme approaches the desired fifth-order accuracy.

Fig. 2 Spatial profiles of density at $t = 0.3$ for acoustic wave propagation test in cylindrical-radial coordinates

Table 1 L_1 norm errors and order of convergence table for acoustic wave propagation test

N	$\epsilon(\rho)$	O_{L_1}
16	1.01E-05	—
32	4.91E-06	1.036
64	6.74E-07	2.865
128	3.24E-08	4.380
256	1.27E-09	4.670

4.2 Sedov Explosion Test

Sedov explosion test is performed to investigate code’s ability to deal with strong shocks and non-planar symmetry [3]. The problem involves a self-similar evolution of a cylindrical blastwave in a uniform grid ($N = 100$) from a localized initial pressure perturbation (delta-function) in an otherwise homogeneous medium. Governing equations are given in Eq. (15) and the fluxes are evaluated with Rusanov scheme and GKS [5]. For the code initialization, dimensionless energy $\epsilon = 1$

is deposited into a small region of radius $\delta = 3\Delta R$. Inside this region, the dimensionless pressure P'_0 is given by Eq. (19).

$$P'_0 = \frac{3(\gamma - 1)\epsilon}{(m + 2)\pi\delta^{(m+1)}} \tag{19}$$

where $m = 1$ for cylindrical geometry. Reflecting boundary condition is employed at the center ($R = 0$), whereas boundary condition at $R = 1$ is not required for this problem. The initial velocity and density inside the domain are 0 and 1 respectively and the initial pressure everywhere except the kernel is 10^{-5} . As the source term is very stiff, the CFL number is set to be 0.1. The final time is $t = 0.05$.

Figure 3 shows that the peak for WENO-C is higher for density and is closest to the analytical value, similar to fifth-order finite difference version [3], but MUSCL has higher offset peaks for pressure and velocity. GKS performs slightly better than RS, as the peaks are slightly higher for all the cases.

Fig. 3 Variation of density, velocity, and pressure with the radius for Sedov explosion test in cylindrical-radial coordinates. Domain is restricted to $R = 0.4$ for the sake of clarity

4.3 Modified 2D Riemann Problem in $(R - z)$ Coordinates

The final test for the present scheme involves a modified 2D Riemann problem in cylindrical $(R - z)$ coordinates, as illustrated in Fig. 4 (top left). The problem involves 2 contact discontinuities and 2 shocks as the initial condition, resulting in the formation of a self-similar structure propagating towards the low density-low pressure region (region 3). The governing equations in cylindrical $(R - z)$ coordinates are provided in Eq. (20).

The computations are performed until $t = 0.2$ with a CFL number of 0.5 on a domain $(R, z)=[0,1] \times [0,1]$ divided into 500×500 zones. The boundary conditions are symmetry at the center (except for the antisymmetric radial velocity) and outflow elsewhere. HLL Riemann solver is used for flux evaluations. Rich small-scale structures in the contact-contact region (region 1) can be observed from

Fig. 4 Modified 2D Riemann problem in cylindrical $(r - z)$ coordinates: schematic (top left), density contours at $t = 0.2$ with first-order (top right), second-order MUSCL (bottom left), and WENO-C (bottom right) reconstruction schemes

Fig. 4 for WENO-C reconstruction, when compared with first and second-order MUSCL reconstruction. Structures are highly smeared for the case of first-order reconstruction.

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho v_R \\ \rho v_z \\ \rho e \end{pmatrix} + \frac{1}{R} \frac{\partial}{\partial R} \begin{pmatrix} \rho v_R R \\ (\rho v_R^2 + p)R \\ \rho v_R v_z R \\ (\rho e + p)v_R R \end{pmatrix} + \frac{\partial}{\partial z} \begin{pmatrix} \rho v_z \\ \rho v_R v_z \\ \rho v_z^2 + p \\ (\rho e + p)v_z \end{pmatrix} = \begin{pmatrix} 0 \\ p/R \\ 0 \\ 0 \end{pmatrix} \quad (20)$$

5 Conclusions

The fifth-order finite-volume WENO-C reconstruction scheme is proposed for structured grids in cylindrical coordinates to achieve high order spatial accuracy along with ENO transition. A grid independent smoothness indicator is derived for this scheme. For uniform grids, the analytical values in cylindrical-radial coordinates for the limiting case ($R \rightarrow \infty$) conform to WENO-JS. Linear stability analysis of the present scheme is performed using a scalar advection equation in radial coordinates. Several tests involving smooth and discontinuous flows are performed, which testify for the fifth-order accuracy and ENO property of the scheme.

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