Spectral Galerkin Method for Solving Helmholtz and Laplace Dirichlet Problems on Multiple Open Arcs

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1 Introduction

We seek solutions of Helmholtz and Laplace equations in a two-dimensional plane after removing a finite collection of open finite curves—also called arcs. This setting can be found in areas such as structural and mechanical engineering [2], or biomedical imaging [11] to name a few. Such problems pose the following challenges: (1) *unbounded domains*, which call for boundary integral methods with carefully chosen radiation conditions; (2) *singular behaviors* of solutions near arc endpoints; and (3) *large number of degrees of freedom* when the wavenumber or number of arcs increase.

Our approach is to recast the problem as a system of boundary integral equations defined on the arcs, so as to obtain an integral representation of the volume solution. Well-posedness for a single arc was proven in [9], with an extension to the multiple arcs case given in [5]. We will consider numerical approximations of the resulting surface densities based on Galerkin-Bubnov discretizations of the corresponding system of boundary integral equations.

In the present note, we start by briefly introducing a spectral scheme to account for general arcs as well as for a wide wavenumber range. We show that significant reduction in both memory consumption and computational work can be achieved by an *ad hoc* matrix compression algorithm. Moreover, we establish detailed interde-

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dependencies between compression parameters and accuracy. Numerical experiments validate our claims and point out further improvements.

2 Continuous Model Problem

Let the canonical domain $(-1, 1) \times \{0\}$ be denoted by $\widehat{\Gamma}$. We say that $g : \widehat{\Gamma} \rightarrow \mathbb{C}$ is ρ -analytic if the function $t \mapsto g(t, 0)$ can be extended to an analytic function on the Bernstein ellipse of parameter $\rho > 1$ (cf. [10, Chapter 8]). We say that $\Lambda \subset \mathbb{R}^2$ is a regular Jordan arc of class C^m , for $m \in \mathbb{N}$, if it is the image of a bijective parametrization, denoted by $\mathbf{r} = (r_1, r_2)$, such that its components are $C^m(\widehat{\Gamma})$ -functions, $\mathbf{r} : \widehat{\Gamma} \rightarrow \overline{\Lambda}$ and $\|\mathbf{r}'(t)\|_2 > 0$, $\forall t \in \widehat{\Gamma}$, where $\|\cdot\|_2$ is the Euclidean norm. Similarly, we define ρ -analytic arcs as those whose components are ρ -analytic. Throughout, we will assume that for any Λ regular Jordan arc, there exists an extension of Λ to $\overline{\Lambda}$, which is a closed and keep the same regularity.

Consider a finite number $M \in \mathbb{N}$ of at least C^1 -arcs, written $\{\Gamma_i\}_{i=1}^M$, such that their closures are mutually disjoint. Moreover, we assume that there are disjoint domains Ω_i whose boundaries are given by extensions $\partial\Omega_i = \overline{\Gamma}_i$, for $i = 1, \dots, M$. Let us define

$$\Gamma := \bigcup_{i=1}^M \Gamma_i \quad \text{and} \quad \Omega := \mathbb{R}^2 \setminus \overline{\Gamma}.$$

We say that Γ is of class C^m , $m \in \mathbb{N}$, if each arc Γ_i is of class C^m and analogously for the ρ -analytic case. For $i \in \{1, \dots, M\}$, let $\mathbf{r}_i : \widehat{\Gamma} \rightarrow \Gamma_i$ and $g_i : \overline{\Gamma}_i \rightarrow \mathbb{C}$. We claim that $\mathbf{g} = (g_1, \dots, g_M)$ is of class $C^m(\Gamma)$ if $g_i \circ \mathbf{r}_i \in C^m(\widehat{\Gamma})$, for $i \in \{1, \dots, M\}$. A similar definition holds for the analytic case.

Let $G \subseteq \mathbb{R}^d$, $d = 1, 2$, be an open domain. For $s \in \mathbb{R}$, we denote by $H^s(G)$ the standard Sobolev spaces, by $H_{loc}^s(G)$ their locally integrable counterparts [8, Section 2.3], and by $\widetilde{H}^{-s}(G)$ the corresponding dual spaces. The corresponding duality product (when the dual space of $L^2(G)$ is identified with itself) is denoted $\langle \cdot, \cdot \rangle_G$. Finally, $\widetilde{H}_{(0)}^s(G)$ refers to mean-zero spaces [5, Section 2.3]. We will also make use of the following Hilbert space in $\mathbb{R}^2$:

$$W(G) := \left\{ U \in \mathcal{D}^*(G) : \frac{U(\mathbf{x})}{\sqrt{1 + \|\mathbf{x}\|_2^2} \log(2 + \|\mathbf{x}\|_2^2)} \in L^2(G), \nabla U \in L^2(G) \right\},$$

where $\mathcal{D}^*(G)$ is the dual space of $C^\infty(G) = \cap_{n \geq 1} C^n(G)$. For $s \in \mathbb{R}$ and for the finite union of disjoint open arcs Γ , we define Cartesian product spaces as

$$\mathbb{H}^s(\Gamma) := H^s(\Gamma_1) \times H^s(\Gamma_2) \times \dots \times H^s(\Gamma_M).$$

Spaces $\widetilde{\mathbb{H}}^s(\Gamma)$ and $\widetilde{\mathbb{H}}_{(0)}^s(\Gamma)$ are defined similarly. Also, $\mathbb{H}^s(\widehat{\Gamma})$ is to be understood as the Cartesian product $\prod_{i=1}^M H^s(\widehat{\Gamma}_i)$. Finally, given an open bounded neighborhood G_i such that $\Gamma_i \subset \partial G_i$, Dirichlet traces are defined as extensions to $H^s(G_i)$, for $s \geq 1/2$, of the following operator (applied to smooth functions):

$$\gamma_i^\pm u(\mathbf{y}) := \lim_{\epsilon \downarrow 0} u(\mathbf{y} \pm \epsilon \mathbf{n}_i(\mathbf{y})),$$

where $\mathbf{n}_i(\mathbf{y})$ is the unitary vector with direction $(r'_{i,2}(t), -r'_{i,1}(t))$ and t such that $\mathbf{r}(t) = \mathbf{y}$. For a function u defined in an open neighborhood of Γ_i such that $\gamma_i^+ u = \gamma_i^- u$, we denote $\gamma_i u := \gamma_i^\pm u$.

Problem 1 (Volume Problem) Let $\mathbf{g} \in \mathbb{H}^{\frac{1}{2}}(\Gamma)$ and $\kappa \geq 0$. We seek $U \in H_{loc}^1(\Omega)$ such that

$$-\Delta U - \kappa^2 U = 0 \quad \text{in } \Omega, \quad (1)$$

$$\gamma_i^\pm U = g_i \quad \text{for } i = 1, \dots, M, \quad (2)$$

$$\text{Condition at infinity}(\kappa). \quad (3)$$

The behavior at infinity (3) depends on κ in the following way: if $\kappa > 0$, we employ the classical Sommerfeld condition [8, Section 3.9]. If $\kappa = 0$, we seek for solutions $U \in W(\Omega)$. This last condition was discussed in detail in [5, Remarks 3.9, 4.2 and 4.5] with uniqueness proofs for $\kappa \geq 0$ provided in [5, Propositions 3.8 and 3.10].

For $\kappa \geq 0$, we can express U solution of Problem 1 as

$$U(\mathbf{x}) = \sum_{i=1}^M (\text{SL}_i[\kappa] \lambda_i)(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega, \quad (4)$$

where

$$(\text{SL}_i[\kappa] \lambda_i)(\mathbf{x}) := \int_{\Gamma_i} G_\kappa(\mathbf{x}, \mathbf{y}) \lambda_i(\mathbf{y}) d\Gamma_i(\mathbf{y}), \quad \forall \mathbf{x} \in \Omega,$$

denotes the single layer potential generated at a curve Γ_i with G_κ the corresponding fundamental solution, defined as in [8, Section 3.1]. It is direct from (4) that U solves (1)–(2) in Ω (see [8, Theorem 3.1.1]). Also, it displays the desired behavior at infinity as long as each λ_i lies in the right functional space [5, Section 4]. In order to find the surface densities λ_i , we take Dirichlet traces $\gamma_i^\pm$ of the SL_j and impose boundary conditions (2). This naturally defines of weakly singular boundary integral operators:

$$\mathcal{L}_{ij}[\kappa] := \frac{1}{2} \left(\gamma_i^+ \text{SL}_j[\kappa] + \gamma_i^- \text{SL}_j[\kappa] \right) = \gamma_i \text{SL}_j[\kappa],$$

and an equivalent boundary integral equation problem to Problem 1.

Problem 2 (Boundary Integral Problem) Let $\mathbf{g} \in \mathbb{H}^{\frac{1}{2}}(\Gamma)$. For $\kappa > 0$, we seek $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_M) \in \tilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)$ such that

$$\mathcal{L}[\kappa]\boldsymbol{\lambda} = \mathbf{g},$$

where $\mathcal{L}[\kappa] : \tilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma) \rightarrow \mathbb{H}^{\frac{1}{2}}(\Gamma)$ is a matrix operator with entries $\mathcal{L}[\kappa]_{ij} = \mathcal{L}_{ij}[\kappa]$, for $i, j \in \{1, \dots, M\}$. If $\kappa = 0$, we seek $\boldsymbol{\lambda} \in \tilde{\mathbb{H}}_{(0)}^{-\frac{1}{2}}(\Gamma)$, given $\mathbf{g}$ in the dual space of the aforementioned space.

Theorem 1 (Theorem 4.13 in [5]) For $\kappa > 0$, Problem 2 has a unique solution $\boldsymbol{\lambda} \in \tilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)$, whereas for $\kappa = 0$ a unique solution exists in the subspace $\tilde{\mathbb{H}}_{(0)}^{-\frac{1}{2}}(\Gamma)$. Also, the following continuity estimate holds

$$\|\boldsymbol{\lambda}\|_{\tilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)} \leq C(\Gamma, \kappa) \|\mathbf{g}\|_{\mathbb{H}^{\frac{1}{2}}(\Gamma)}.$$

3 Spectral Discretization

We present a family of finite dimensional subspaces in $\tilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)$ that can be used to approximate the solution of Problem 2 (cf. [4, 6]). Let $\mathbb{T}_N(\widehat{\Gamma})$ denote the space spanned by first kind Chebyshev polynomials, denoted by $\{T_n\}_{n=0}^N$, of degree lower or equal than N on $\widehat{\Gamma}$, orthogonal with the $L^2(-1, 1)$ inner product, under the weight w^{-1} with $w(t) := \sqrt{1-t^2}$. Now, let us construct elements $p_n^i = T_n \circ \mathbf{r}_i^{-1}$ over each arc Γ_i spanning the space $\mathbb{T}_N(\Gamma_i)$. For practical reasons, we define the normalized space:

$$\overline{\mathbb{T}}_N(\Gamma_i) := \left\{ \bar{p}^i \in C(\Gamma_i) : \bar{p}_n^i := \frac{p_n^i}{\|\mathbf{r}'_i \circ \mathbf{r}_i^{-1}\|_2}, \quad p_n^i \in \mathbb{T}_N(\Gamma_i) \right\}.$$

We account for edge singularities by multiplying the basis $\{\bar{p}_n^i\}_{n=0}^N$ by a suitable weight:

$$\mathbb{Q}_N(\Gamma_i) := \left\{ q_n^i := w_i^{-1} \bar{p}_n^i : \bar{p}_n^i \in \overline{\mathbb{T}}_N(\Gamma_i) \right\},$$

wherein $w_i := w \circ \mathbf{r}_i^{-1}$. The corresponding basis for $\mathbb{Q}_N(\Gamma_i)$ will be denoted $\{q_n^i\}_{n=0}^N$. By Chebyshev orthogonality, we can easily define the mean-zero subspace $\mathbb{Q}_{N,(0)}(\Gamma_i) := \mathbb{Q}_N(\Gamma_i) \setminus \mathbb{Q}_0(\Gamma_i)$, spanned by $\{q_n^i\}_{n=1}^N$. With these definitions, we

set the discretization space for a Galerkin-Bubnov solution of Problem 2 as

$$\mathbb{H}_N[\kappa] := \begin{cases} \prod_{i=1}^M \mathbb{Q}_{N, \langle 0 \rangle}(\Gamma_i) & \text{for } \kappa = 0, \\ \prod_{i=1}^M \mathbb{Q}_N(\Gamma_i) & \text{for } \kappa > 0. \end{cases}$$

Problem 3 (Linear System) For $\kappa > 0$, let $N \in \mathbb{N}$ and $\mathbf{g} \in \mathbb{H}^{\frac{1}{2}}(\Gamma)$ be the same as in Problem 2. Then, we seek coefficients $\mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_M) \in \mathbb{C}^{M(N+1)}$, such that

$$\mathbf{L}[\kappa]\mathbf{u} = \mathbf{g}.$$

Therein, we have defined the Galerkin matrix $\mathbf{L}[\kappa] \in \mathbb{C}^{M(N+1) \times M(N+1)}$ composed of matrix blocks $\mathbf{L}_{ij}[\kappa] \in \mathbb{C}^{(N+1) \times (N+1)}$ whose entries are

$$(\mathbf{L}_{ij}[\kappa])_{lm} = \left\langle \mathcal{L}_{ij}[\kappa] q_m^j, q_l^i \right\rangle_{\Gamma_i} = \left\langle \widehat{\mathcal{L}}_{ij}[\kappa] w^{-1} T_m, w^{-1} T_l \right\rangle_{\widehat{\Gamma}}.$$

There, $\widehat{\mathcal{L}}_{ij}[\kappa]$ is the weakly-singular operator whose kernel is parametrized by $\mathbf{r}_i, \mathbf{r}_j$ and right-hand $\mathbf{g} = (\mathbf{g}_1, \dots, \mathbf{g}_M) \in \mathbb{C}^{M(N+1)}$ with components

$$(\mathbf{g}_i)_l = \left\langle g_i, q_l^i \right\rangle_{\Gamma_i} = \left\langle \widehat{g}_i, w^{-1} T_l \right\rangle_{\widehat{\Gamma}},$$

where $\widehat{g}_i = g_i \circ \mathbf{r}_i$. The approximation $\lambda_N \in \mathbb{H}_N[\kappa]$ is constructed as

$$(\lambda_N)_i = \sum_{m=0}^N (\mathbf{u}_i)_m q_m^i \text{ in } \Gamma_i, \quad \text{for all } i \in \{1, \dots, M\}.$$

For $k = 0$ we need $\mathbf{g}$ as in Problem 2; we also have $\mathbf{u} \in \mathbb{C}^{MN}$, and $\mathbf{L}[0] \in \mathbb{C}^{MN \times MN}$ since the approximation space is $\mathbb{H}_N[0]$. By conformity and density of these spaces in $\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)$, one derives the following result:

Theorem 2 (Theorem 4.23 [4]) *Let $\kappa \geq 0$, $m \in \mathbb{N}$ with $m > 2$, $\Gamma \in C^m$, $\mathbf{g} \in C^m(\Gamma)$, and λ be the only solution of Problem 2. Then, there exists $N_0 \in \mathbb{N}$ such that for every $N > N_0 \in \mathbb{N}$ there is a unique $\lambda_N \in \mathbb{H}_N[\kappa]$ solution of Problem 3. Moreover, the following error convergence rates hold*

$$\|\lambda - \lambda_N\|_{\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)} \leq C(\Gamma, \kappa) N^{-m+1}.$$

Moreover, if Γ and $\mathbf{g}$ are ρ -analytic with $\rho > 1$, we have the following super-algebraic convergence rates

$$\|\lambda - \lambda_N\|_{\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)} \leq C(\Gamma, \kappa) \rho^{-N+2} \sqrt{N},$$

where $C(\Gamma, \kappa)$ is a positive constant, which does not depend on N .

Remark 1 Observe that the constants $C(\Gamma, \kappa)$ and N_0 depend on the geometry and frequency. To the best of our knowledge previous convergence results for 2D arcs are somehow limited. For intervals, the result was established in [6] whereas for more general arc results are only obtained for the Laplace case [1]. Super-algebraic convergence rates can be achieved by the method detailed in [3], though their scheme is limited to intervals and to the case of elliptic problems ($N_0 = 0$). More complex cases are still an open problem.

4 Numerical Implementation and Compression Algorithm

Before fleshing out our proposed compression technique, we explain how $\mathbf{L}[\kappa]$ and $\mathbf{g}$ of Problem 3 are computed. For the right-hand side, one must compute integrals of the form:

$$\int_{-1}^1 \widehat{g}(t) w^{-1}(t) T_l(t) dt, \quad \forall l \in \mathbb{N}_0,$$

which corresponds to Fourier-Chebyshev coefficients of $\widehat{g}(t)$ and can be approximated using the Fast Fourier Transform [10]. Computations for matrix terms $\mathbf{L}_{ij}[\kappa]$ are split into two groups: (a) *cross-interactions*, where test and trial functions supports lie along curves Γ_i, Γ_j with $i \neq j$; and (b) *self-interactions*, where both trial and test functions are defined on the same curve. As for cross-interactions the integral kernel is smooth, we use the same computational procedure for the right-hand side.

For self-interactions, the kernel function has a singularity that can be characterized as

$$G_k(\mathbf{r}(t), \mathbf{r}(s)) = (2\pi)^{-1} \log |t - s| J_0(k \|\mathbf{r}(t) - \mathbf{r}(s)\|_2) + G_r(t, s), \quad t \neq s,$$

for $t, s \in \widehat{\Gamma}$, where J_0 is the zeroth-order first kind Bessel function, and G_r is a regular function. Thus, integration for the regular part is done as in the cross-interaction case, while integrals with the first term as kernel are obtained by convolution as integrals for $\log |t - s|$ are known (see [6, Remark 4.2]).

Yet, as κ increases, larger values of N will be required, and thus, the need to compress the resulting matrix terms. As stated in [10, Chapters 7 and 8], the regularity of a function controls the decay of its Fourier-Chebyshev coefficients. Hence, as the entries of the matrix $\mathbf{L}[\kappa]$ are precisely such coefficients, for a smooth kernel one observes fast decaying terms. This implies that we can select small blocks to approximate the matrix and obtain a sparse approximation by discarding the remaining entries, based on a predetermined tolerance $\epsilon > 0$. Specifically, the kernel function is smooth when we compute cross-interactions. Let the routine $\text{Quadrature}(l, m)$ compute the term (l, m) of this interaction matrix using a 2D

Gauss-Chebyshev quadrature. Given a tolerance $\epsilon > 0$, we minimize the number of computations needed by performing the following binary search:

Matrix Compression Algorithm

```

INPUT: Tolerance (Tol), Max level of search (Lmax)
OUTPUT: Number of columns to use (Ncols)
INITIALIZE: Ncols = N, level = 0, a = 0, b = N
While{level < Lmax}
    m = (a+b)/2
    Tleft = m-1
    Tcenter = m
    Tright = m+1
    Vleft = abs(Quadrature(0,Tleft))
    Vcenter = abs(Quadrature(0,Tcenter))
    Vright = abs(Quadrature(0,Tright))
    If{Vright & Vcenter < 0.5*Tol} or {Vleft & Vcenter < 0.5*Tol}
        b = m
    Else
        a = m
    EndIF
    level++
EndWhile
Ncols = b

```

The algorithm returns the minimum number of columns required, N_{cols} , by searching in the first row the minimum index such that the matrix entries' absolute value is lower than ϵ . The binary search is restricted to a depth $L_{\max} \in \mathbb{N}$. The same procedure is used to estimate the number of rows, N_{rows} , by executing a binary search in the first column. Once N_{cols} and N_{rows} are selected, we define $N_\epsilon := \max\{N_{rows}, N_{cols}\}$ and compute the block of size $N_\epsilon \times N_\epsilon$ as in the full matrix implementation.

The matrix compression percentage will strongly depend on the regularity of the arcs involved. For ρ -analytic arcs, using [10, Theorem 8.1] we can prove the lower bound:

$$N_\epsilon \geq \frac{-\log \epsilon}{2\Upsilon \log \rho},$$

where Υ is an upper bound for the absolute value of the kernel in the corresponding Bernstein ellipse. However, since compression is done by a binary search, the bound for the compression rate depends on $L_{\max}$ as

$$N_\epsilon \geq \frac{N}{2^{L_{\max}}}.$$

Compression of self-interaction blocks does not follow the same ideas. In fact, these blocks can be characterized as two perturbations over the canonical case, $\Gamma = \widehat{\Gamma}$ for $\kappa = 0$, leading to a diagonal matrix. Namely, these are

1. A low frequency perturbation caused by the mapping $r_i : \widehat{\Gamma} \mapsto \Gamma$, similar to the cross-interaction case.
2. A frequency perturbation that creates banded matrices.

In order to reduce memory consumption—though not computational time—we discard the entries of the self-interaction matrices lower than the given tolerance.

As expected, matrix compression induces an extra error as it perturbs the original linear system solved by λ_N in Problem 3. We denote by $\mathbf{L}_\epsilon[k]$ the matrix generated by the compression algorithm with tolerance ϵ , and define the matrix difference $\Delta\mathbf{L}_\epsilon[k] := \mathbf{L}_\epsilon[k] - \mathbf{L}[k]$. We seek to control the solution $\mathbf{u}^\epsilon = \mathbf{u} + \Delta\mathbf{u}$ of

$$(\mathbf{L}[k] + \Delta\mathbf{L}_\epsilon[k])\mathbf{u}^\epsilon = \mathbf{g},$$

where $\mathbf{u}$ and $\mathbf{g}$ are the same as in Problem 3. In order to bound this error, we will assume that, for every pair of indices (i, j) in the matrix $\mathbf{L}[k]$, we have,

$$|(\Delta\mathbf{L}_\epsilon[k])_{ij}| < \epsilon. \quad (5)$$

Theorem 3 *Let $N \in \mathbb{N}$ be such there is only one λ_N solution of Problem 3. Then, there is a constant $C(\Gamma, \kappa) > 0$, not depending on N , such that*

$$\frac{\|\Delta\mathbf{u}\|_2}{\|\mathbf{u}\|_2} \leq \left| \frac{N\epsilon}{C(\kappa, \Gamma) - N\epsilon} \right|.$$

Proof By [7, Section 1.13.2] we have that

$$\frac{\|\Delta\mathbf{u}\|_2}{\|\mathbf{u}\|_2} \leq \frac{\|\Delta\mathbf{L}_\epsilon[k]\|_2}{\|(\mathbf{L}[k])^{-1}\|_2 - \|\Delta\mathbf{L}_\epsilon[k]\|_2},$$

and thus, we need to estimate $\|\Delta\mathbf{L}_\epsilon[k]\|_2$ and $\|(\mathbf{L}[k])^{-1}\|_2$. The bound for the first term is direct from (5) and matrix norm definitions. By the classical bound of a matrix inverse and the continuity of the associated boundary integral operator, it holds that

$$\|\mathbf{L}[k]^{-1}\mathbf{g}\|_2 \geq \|\mathbf{L}[k]\mathbf{g}\|_2^{-1} \geq C(\kappa, \Gamma),$$

from where the result follows directly. □

We can also estimate the error introduced by the compression algorithm in terms of the energy norm. In order to do so, define $(\lambda_N^\epsilon)_i := \sum_{m=0}^N (\mathbf{u}_i^\epsilon)_m q_m^i$ in Γ_i . By the same arguments in the above proof, we obtain

$$\|\lambda_N - \lambda_N^\epsilon\|_{\tilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)} \leq C_1(\kappa, \Gamma) \|\mathbf{g}\|_{\mathbb{H}(\Gamma)^{\frac{1}{2}}} \frac{\epsilon N^{3/2}}{C_2(\kappa, \Gamma) - \epsilon N},$$

where $\mathbf{g}$ is the same that in Problem 2 and $C_1(\kappa, \Gamma)$, $C_2(\kappa, \Gamma)$ are two different constants.

Remark 2 Our compression algorithm produces a faster and less memory demanding implementation of the spectral Galerkin method at the cost of accuracy loss, similar to fast multipole or hierarchical matrices methods. Moreover, once we have compressed the matrix, we can implement a fast matrixvector product.

5 Numerical Results

To illustrate the above claims, Fig. 1 presents convergence results for different wavenumbers, $\kappa = 0, 25, 50, 100$ for a configuration of $M = 28$ arcs. As the chosen geometry and excitation are given by analytic functions, Theorem 2 predicts exponential rate of convergence as observed numerically.

Table 1 provides matrix compression results for $\kappa = 100$ and for the same geometry of Fig. 1. It presents the percentage of non-zero entries (%NNZ) and relative errors as bounded in Theorem 3 as functions of the maximum level of binary

Fig. 1 (a) Smooth geometry with $M = 28$ open arcs parametrized as $\mathbf{r}_i(t) = (a_i t, c_i \sin(b_i t) + d_i)$, with $a_i \in [0.14, 0.25]$, $b_i \in [0, 0.2]$, $c_i \in [1, 2]$, $d_i \in [0, 20]$, $t \in [-1, 1]$. (b) Convergence results for different wavenumbers and a planewave excitation along $(1, 1)$. Errors computed against an overkill solution using $N = 660$ per arc

Table 1 Compression performance for $\kappa = 100$

Order	$L_{\max} = 2$		$L_{\max} = 3$		$L_{\max} = 4$	
	% NNZ	Rel. error	% NNZ	Rel. error	% NNZ	Rel. error
	$\epsilon = 1e-6$					
5	65.24	5.05e−01	65.24	5.05e−01	65.24	5.05e−01
10	81.62	5.32e−01	81.62	5.32e−01	81.62	5.32e−01
20	89.41	2.33e−01	88.62	2.33e−01	88.31	2.33e−01
40	77.63	9.10e−04	70.63	9.10e−04	67.11	9.10e−04
60	45.25	2.02e−07	36.68	2.76e−07	33.36	3.31e−07
80	27.20	1.97e−07	21.97	3.17e−07	19.50	3.35e−07
	$\epsilon = 1e-10$					
5	65.29	5.05e−01	65.29	5.05e−01	65.29	5.05e−01
10	81.68	5.32e−01	81.68	5.32e−01	81.68	5.32e−01
20	89.86	2.33e−01	89.59	2.33e−01	89.44	2.33e−01
40	83.46	9.10e−04	78.70	9.10e−04	76.28	9.10e−04
60	51.94	2.14e−09	44.87	3.19e−09	40.70	3.89e−09
80	33.86	2.31e−09	26.89	1.73e−08	23.78	1.73e−10

search ($L_{\max}$), tolerances (ϵ), and polynomial order per arc (Order). For low orders (Order < 60), relative errors are quite large, and therefore, most of the matrix terms are kept. This is due to an insufficient number of matrix entries to solve the problem with good accuracy (see Fig. 1), rendering compression pointless. On the other hand, once convergence is achieved, the compression error drastically decreases along with the percentage of matrix terms stored.

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