On the Order Reduction of Entropy Stable DGSEM for the Compressible Euler Equations

Florian J. Hindenlang and Gregor J. Gassner

1 Introduction

Discontinuous Galerkin spectral element collocation method (DGSEM) with either Legendre-Gauss or Legendre-Gauss-Lobatto (LGL) nodes (see e.g. [14]) are among the most efficient variants in the class of element based high order methods, such as e.g. discontinuous Galerkin, flux reconstruction, or summation-by-parts (SBP) finite differences. In particular, the LGL variant, starting in [9], turned out to be similar to a SBP finite difference approximation with simultaneous-approximate-term technique (SAT). This relationship allowed to construct conservative skew-symmetric approximations, e.g. [9, 10, 21], and later enabled DGSEM-LGL approximations that are discretely entropy stable, e.g. [1, 3, 6, 8, 13, 17, 19, 20], and/or kinetic energy preserving [12]. These novel variants of nodal split form DG methods feature drastically increased non-linear robustness towards aliasing induced instabilities and favourable properties regarding the simulation of unresolved turbulence, e.g. [7, 23].

In addition to the very robust dissipative entropy stable versions, it is also possible to construct virtually dissipation free variants by choosing appropriate element interface numerical fluxes. These entropy conserving variants all show an odd-even behavior when experimentally testing the order of convergence, e.g. [9, 21], where the observed convergence order for even polynomial degrees

F. J. Hindenlang

Max Planck Institute for Plasma Physics, Garching, Germany

e-mail: florian.hindenlang@ipp.mpg.de

G. J. Gassner (✉)

Department for Mathematics and Computer Science, Center for Data and Simulation Science,
University of Cologne, Cologne, Germany

e-mail: ggassner@math.uni-koeln.de

© The Author(s) 2020

S. J. Sherwin et al. (eds.), *Spectral and High Order Methods for Partial Differential Equations ICOSAHOM 2018*, Lecture Notes in Computational Science and Engineering 134, https://doi.org/10.1007/978-3-030-39647-3_2

N is N and for odd N is $N + 1$. Lately, a discussion emerged in the community, with interesting debates during the recent ICOSAHOM conference in London, where researchers reported non-optimal convergence behavior of the entropy stable DGSEM-LGL even with dissipative numerical surface fluxes, e.g. [6].

This paper contributes to this discussion and presents results of an experimental convergence order study for the compressible Euler equations with (1) the standard DGSEM with either Gauss and LGL nodes, (2) the entropy stable DGSEM with LGL nodes. For these nodal schemes, we test the convergence order with different numerical surface fluxes and report the results depending on the Mach number of the test case. The remainder of the paper is organized as follows: in the next section we describe the numerical model for our numerical experiments, in Sect. 3 we present our observed experimental convergence orders for different configurations and draw our conclusion in Sect. 4.

2 Numerical Model

We consider the compressible Euler equations defined in the domain $\Omega \subset \mathbb{R}^3$

$$\mathbf{u}_t + \sum_{i=1}^3 \frac{\partial \mathbf{f}_i}{\partial x_i} = \mathbf{0}. \quad (1)$$

The state vector contains the conservative variables and the advective flux components are

$$\mathbf{u} = \begin{bmatrix} \varrho \\ \varrho \vec{v} \\ E \end{bmatrix} = \begin{bmatrix} \varrho \\ \varrho v_1 \\ \varrho v_2 \\ \varrho v_3 \\ E \end{bmatrix}, \quad \mathbf{f}_1 = \begin{bmatrix} \varrho v_1 \\ \varrho v_1^2 + p \\ \varrho v_1 v_2 \\ \varrho v_1 v_3 \\ (E + p)v_1 \end{bmatrix}, \quad \mathbf{f}_2 = \begin{bmatrix} \varrho v_2 \\ \varrho v_2 v_1 \\ \varrho v_2^2 + p \\ \varrho v_2 v_3 \\ (E + p)v_2 \end{bmatrix}, \quad \mathbf{f}_3 = \begin{bmatrix} \varrho v_3 \\ \varrho v_3 v_1 \\ \varrho v_3 v_2 \\ \varrho v_3^2 + p \\ (E + p)v_3 \end{bmatrix}. \quad (2)$$

Here, ϱ , $\vec{v} = (v_1, v_2, v_3)^T$, p , E are the mass density, fluid velocities, pressure and total energy. We close the system with the ideal gas assumption, which relates the total energy and pressure

$$p = (\gamma - 1) \left(E - \frac{1}{2} \varrho \|\vec{v}\|^2 \right), \quad (3)$$

where γ denotes the adiabatic coefficient.

For our discretization, we subdivide the domain into non-overlapping hexahedral elements. For each element, we define a transfinite mapping to a unit reference space and use this mapping to transform the Eq. (1) from physical to reference space. A weak form is created by taking the inner product of the transformed equation with a test function. We use integration-by-parts for the flux term and approximate the resulting weak form as follows: the conservative variables are approximated by a polynomial in reference space with degree N , interpolated at the Gauss or LGL nodes. The volume fluxes are replaced by a standard interpolation of the non-linear flux function at the same Gauss/LGL nodes (standard DGSEM-Gauss or DGSEM-LGL), see e.g. [14]. For the LGL variant, we are also able to introduce the split form volume integral based on entropy conserving and kinetic energy preserving numerical volume fluxes (Split-DGSEM), e.g. [12] and [22], resulting in either the entropy conserving or entropy stable DGSEM variants, depending on the choice of numerical surface flux.

3 Convergence Results

In this section, we compare the convergence of the standard DGSEM and the entropy conservative and entropy stable discretization for different choices of the numerical flux and polynomial degrees $N = 2, 3, 4, 5$.

We choose the test case of a two-dimensional density wave, with a constant pressure and transported with a constant velocity, which was proposed for one-dimensional convergence tests in [4]. The density evolves as

$$\varrho(x_1, x_2, t) = 1 + 0.1 \sin \left(\pi \left((x_1 - v_1 t) + (x_2 - v_2 t) \right) \right) \quad (4)$$

with a prescribed velocity (v_1, v_2) . The pressure is chosen as $p = 1/\gamma$ with $\gamma = 1.4$, so that the sound speed ranges between $c = 0.95 \dots 1.05$. Thus, by changing the velocity, we change the Mach number of the flow $\text{Ma} = |\vec{v}|/c$. Three Mach numbers are chosen: $\text{Ma} \approx 0.2$ with $(v_1, v_2) = (0.1, 0.15)$, $\text{Ma} \approx 1.0$ with $(v_1, v_2) = (0.7, 0.65)$ and $\text{Ma} \approx 3.5$ with $(v_1, v_2) = (2.5, 2.4)$. The experimental order of convergence (EOC) is computed with the L_2 error of the density at $t = 1$.

The convergence study is performed with the open source, three-dimensional curvilinear split-form DG framework FLUXO (www.github.com/project-fluxo). As the test case is two-dimensional, we use fully periodic cartesian meshes of the domain $[-1, 1]^3$ with an equal number of elements in x- and y-directions and always one element in z-direction. Note that h_0 in the convergence tables refers to the coarsest mesh level, which is 4^2 elements for $N = 2, 3$ ($h_0 = 1/2$) and 2^2 elements for $N = 4, 5$ ($h_0 = 1$).

All simulation results are obtained with an explicit five stage, fourth order accurate low storage Runge–Kutta scheme [2], where a stable time step is computed according to the adjustable coefficient $CFL \in (0, 1]$ the local maximum wave speed, and the relative grid size, e.g. [11]. We made sure that the time integrator did not influence the spatial convergence order, by adjusting the CFL number accordingly.

3.1 *Standard DGSEM*

The convergence of the standard DGSEM with Gauss-Legendre nodes (DGSEM-Gauss) and with Legendre-Gauss-Lobatto (DGSEM-LGL) is shown in Tables 1 and 2, for the three Mach numbers and two choices of the numerical flux, namely the HLL (Harten, Lax, van Leer) flux and the Roe flux. The results of the LLF (local Lax-Friedrichs) flux and the HLLC flux (HLL variant with three waves, C for ‘contact’ wave) are reported in the Appendix, as the HLL results are similar to LLF, and HLLC behaves exactly the same as Roe, see Tables 4 and 5. Details on the properties and the implementation of the LLF, HLL, HLLC, and Roe fluxes are found in the book of Toro [18] and the references therein.

For the HLL flux and the low Mach number $Ma = 0.2$, we observe an odd-even behavior with an *order reduction* for even polynomial degrees $N = 2, 4$. Also for $Ma = 1.0$, the convergence for even degrees is slightly affected, whereas for the high Mach number, all fluxes converge with full order. Comparing the L_2 errors of the finest mesh for HLL and Roe for the low Mach number, HLL is less accurate for $N = 2, 4$ and more accurate for $N = 3, 5$.

All numerical fluxes are approximate Riemann solvers, but the LLF and HLL only use the maximum wave speeds, whereas the HLLC and Roe also take the contact wave into account, and therefore keep the full order of the scheme for all Mach numbers for this test case.

3.2 *Entropy Conservative and Entropy Stable DGSEM*

Now, we investigate the order reduction of the entropy conservative and entropy stable discretizations. Here, the standard DGSEM volume integral is replaced by split-form formulation (Split-DGSEM) using a two-point entropy conservative and kinetic energy preserving flux (ECKEP). If we choose the ECKEP flux at the surface, we get an entropy-conserving scheme. For entropy stability, we can use the LLF or HLL flux directly at the surface, or use the ECKEP flux and add a dissipation term, which must still satisfy the entropy inequality condition. In Winters et al. [22], such dissipation terms are carefully derived, using either only the maximum wave

Table 1 Experimental order of convergence of L_2 error to the exact density (4), using the standard DGSEM-Gauss with HLL and Roe fluxes

Volume disc. + Surface flux DGSEM-Gauss + HLL	Mach	Mesh level	N = 2		N = 3		N = 4		N = 5	
			$L_2(\rho)$	EOC	$L_2(\rho)$	EOC	$L_2(\rho)$	EOC	$L_2(\rho)$	EOC
	3.5	$h_0/2$	1.87e-04	3.34	8.57e-06	4.02	1.03e-05	5.02	6.76e-07	6.07
		$h_0/4$	2.27e-05	3.04	5.35e-07	4.00	3.30e-07	4.96	1.07e-08	5.99
		$h_0/8$	2.82e-06	3.00	3.34e-08	4.00	1.02e-08	5.01	1.66e-10	6.01
		$h_0/16$	3.53e-07	3.00	2.09e-09	4.00	3.22e-10	4.99	2.60e-12	5.99
	1.0	$h_0/2$	2.60e-04	2.55	5.92e-06	4.55	1.15e-05	4.39	6.74e-07	6.84
		$h_0/4$	3.74e-05	2.80	3.27e-07	4.18	4.94e-07	4.54	7.23e-09	6.54
		$h_0/8$	4.88e-06	2.94	1.95e-08	4.07	1.73e-08	4.84	9.98e-11	6.18
		$h_0/16$	6.18e-07	2.98	1.20e-09	4.03	5.60e-10	4.95	1.52e-12	6.04
	0.2	$h_0/2$	4.87e-04	1.70	4.36e-06	5.05	1.57e-05	2.56	9.80e-07	6.66
		$h_0/4$	1.08e-04	2.18	1.10e-07	5.31	9.90e-07	3.99	4.47e-09	7.77
		$h_0/8$	1.95e-05	2.46	5.84e-09	4.23	5.06e-08	4.29	3.96e-11	6.82
		$h_0/16$	3.06e-06	2.67	2.34e-10	4.64	2.26e-09	4.49	4.57e-13	6.44
DGSEM-Gauss + Roe	3.5	$h_0/2$	1.87e-04	3.34	8.57e-06	4.02	1.03e-05	5.02	6.76e-07	6.07
		$h_0/4$	2.27e-05	3.04	5.35e-07	4.00	3.30e-07	4.96	1.07e-08	5.99
		$h_0/8$	2.82e-06	3.00	3.34e-08	4.00	1.02e-08	5.01	1.66e-10	6.01
		$h_0/16$	3.53e-07	3.00	2.09e-09	4.00	3.22e-10	4.99	2.60e-12	5.99
	1.0	$h_0/2$	1.82e-04	3.07	8.76e-06	4.04	1.11e-05	5.07	6.95e-07	6.08
		$h_0/4$	2.26e-05	3.00	5.40e-07	4.02	3.44e-07	5.00	1.07e-08	6.01
		$h_0/8$	2.82e-06	3.00	3.35e-08	4.01	1.05e-08	5.04	1.67e-10	6.01
		$h_0/16$	3.53e-07	3.00	2.09e-09	4.00	3.25e-10	5.01	2.61e-12	6.00

(continued)

Table 1 (continued)

Volume disc. + Surface flux	Mach	Mesh level	N = 2		N = 3		N = 4		N = 5	
			$L_2(\varrho)$	EOC	$L_2(\varrho)$	EOC	$L_2(\varrho)$	EOC	$L_2(\varrho)$	EOC
	0.2	$h_0/2$	2.14e-04	2.65	1.04e-05	3.78	1.16e-05	4.53	7.76e-07	5.78
		$h_0/4$	2.22e-05	3.26	5.49e-07	4.25	3.49e-07	5.05	1.08e-08	6.17
		$h_0/8$	2.82e-06	2.98	3.76e-08	3.87	1.04e-08	5.07	1.70e-10	5.99
		$h_0/16$	3.53e-07	3.00 ■	2.07e-09	4.19 ■	3.38e-10	4.94 ■	2.64e-12	6.01 ■

Full order is marked with ■ ($\gtrsim N + 1$) and an order reduction with ☒

Table 2 Experimental order of convergence of L_2 error to the exact density (4), using standard DGSEM-LGL with HLL and Roe fluxes

Volume disc. + Surface flux DGSEM-LGL + HLL	Mach	Mesh level	N = 2		N = 3		N = 4		N = 5	
			$L_2(\rho)$	EOC	$L_2(\rho)$	EOC	$L_2(\rho)$	EOC	$L_2(\rho)$	EOC
	3.5	$h_0/2$	1.22e-03	3.38	3.85e-05	4.03	4.34e-05	4.93	2.70e-06	5.93
		$h_0/4$	1.26e-04	3.27	2.41e-06	4.00	1.39e-06	4.96	4.33e-08	5.96
		$h_0/8$	1.48e-05	3.10	1.51e-07	4.00	4.34e-08	5.01	6.67e-10	6.02
		$h_0/16$	1.81e-06	3.03	9.42e-09	4.00	1.36e-09	4.99	1.05e-11	5.98
	1.0	$h_0/2$	1.04e-03	2.30	3.44e-05	4.44	4.11e-05	4.89	2.89e-06	6.33
		$h_0/4$	1.58e-04	2.72	1.88e-06	4.20	1.77e-06	4.54	3.85e-08	6.23
		$h_0/8$	2.11e-05	2.91	1.16e-07	4.02	6.25e-08	4.82	5.37e-10	6.16
		$h_0/16$	2.69e-06	2.97	7.18e-09	4.01	2.05e-09	4.93	8.36e-12	6.00
	0.2	$h_0/2$	1.20e-03	1.96	4.86e-05	3.51	5.16e-05	3.91	3.85e-06	5.31
		$h_0/4$	2.72e-04	2.14	1.85e-06	4.71	2.99e-06	4.11	4.25e-08	6.50
		$h_0/8$	5.57e-05	2.29	1.21e-07	3.94	1.56e-07	4.26	5.45e-10	6.28
		$h_0/16$	1.01e-05	2.47	5.95e-09	4.34	7.19e-09	4.44	7.11e-12	6.26
DGSEM-LGL + Roe	3.5	$h_0/2$	1.22e-03	3.38	3.85e-05	4.03	4.34e-05	4.93	2.70e-06	5.93
		$h_0/4$	1.26e-04	3.27	2.41e-06	4.00	1.39e-06	4.96	4.33e-08	5.96
		$h_0/8$	1.48e-05	3.10	1.51e-07	4.00	4.34e-08	5.01	6.67e-10	6.02
		$h_0/16$	1.81e-06	3.03	9.42e-09	4.00	1.36e-09	4.99	1.05e-11	5.98
	1.0	$h_0/2$	9.17e-04	2.86	3.96e-05	3.94	4.41e-05	4.94	2.76e-06	6.02
		$h_0/4$	1.15e-04	2.99	2.41e-06	4.04	1.47e-06	4.90	4.44e-08	5.96
		$h_0/8$	1.44e-05	3.00	1.51e-07	4.00	4.38e-08	5.07	6.84e-10	6.02
		$h_0/16$	1.80e-06	3.00	9.42e-09	4.00	1.35e-09	5.02	1.08e-11	5.99

(continued)

Table 2 (continued)

Volume disc. + Surface flux	Mach	Mesh level	N = 2		N = 3		N = 4		N = 5	
			$L_2(\varrho)$	EOC	$L_2(\varrho)$	EOC	$L_2(\varrho)$	EOC	$L_2(\varrho)$	EOC
	0.2	$h_0/2$	9.26e-04	2.35	4.63e-05	3.45	4.26e-05	4.27	2.97e-06	5.37
		$h_0/4$	1.19e-04	2.96	2.40e-06	4.27	1.59e-06	4.74	4.48e-08	6.05
		$h_0/8$	1.43e-05	3.06	1.57e-07	3.93	4.34e-08	5.19	6.83e-10	6.04
		$h_0/16$	1.80e-06	2.99 ■	9.37e-09	4.07 ■	1.49e-09	4.86 ■	1.09e-11	5.97 ■

Full order is marked with ■ ($\gtrsim N + 1$) and an order reduction with ☒

Table 3 Experimental order of convergence of L_2 error to the exact density (4), using entropy conservative ECKEP flux and entropy stable HLL and ECKEP-Roe fluxes

Volume disc. + Surface flux Split-DGSEM + ECKEP	Mach	Mesh level	N = 2		N = 3		N = 4		N = 5	
			$L_2(\rho)$	EOC	$L_2(\rho)$	EOC	$L_2(\rho)$	EOC	$L_2(\rho)$	EOC
Split-DGSEM + HLL	3.5	$h_0/2$	1.62e-03	4.23	8.45e-05	2.77	5.80e-05	5.88	4.63e-06	5.63
		$h_0/4$	1.30e-04	3.64	7.14e-06	3.56	1.60e-06	5.18	8.38e-08	5.79
		$h_0/8$	1.05e-05	3.62	7.90e-07	3.18	4.56e-08	5.13	3.59e-09	4.54
		$h_0/16$	1.69e-06	2.64	9.58e-08	3.04	1.23e-09	5.21	1.18e-10	4.93
	1.0	$h_0/2$	1.41e-03	3.89	9.45e-05	2.34	7.71e-05	5.86	3.49e-06	6.00
		$h_0/4$	1.25e-04	3.49	1.26e-05	2.90	1.98e-06	5.28	5.00e-08	6.12
		$h_0/8$	1.48e-05	3.09	1.60e-06	2.98	4.06e-08	5.61	1.22e-09	5.36
		$h_0/16$	1.32e-06	3.48	2.01e-07	3.00	1.13e-09	5.17	3.85e-11	4.98
	0.2	$h_0/2$	1.13e-03	2.26	8.03e-05	2.98	5.95e-05	4.14	4.11e-06	5.14
		$h_0/4$	1.13e-04	3.32	1.02e-05	2.97	1.85e-06	5.01	1.33e-07	4.95
		$h_0/8$	1.12e-05	3.34	1.29e-06	2.99	4.02e-08	5.52	4.21e-09	4.98
		$h_0/16$	1.98e-06	2.50	1.61e-07	3.00	1.43e-09	4.81	1.32e-10	5.00
Split-DGSEM + ECKEP	3.5	$h_0/2$	1.23e-03	3.36	3.88e-05	4.06	4.49e-05	4.94	3.01e-06	5.97
		$h_0/4$	1.27e-04	3.27	2.42e-06	4.00	1.43e-06	4.97	4.84e-08	5.96
		$h_0/8$	1.48e-05	3.10	1.51e-07	4.00	4.44e-08	5.01	7.48e-10	6.02
		$h_0/16$	1.81e-06	3.03	9.46e-09	4.00	1.40e-09	4.99	1.18e-11	5.98
	1.0	$h_0/2$	1.04e-03	2.30	3.48e-05	4.44	4.16e-05	4.90	3.48e-06	6.10
		$h_0/4$	1.58e-04	2.72	1.88e-06	4.21	1.79e-06	4.54	4.66e-08	6.23
		$h_0/8$	2.11e-05	2.90	1.16e-07	4.02	6.39e-08	4.81	6.08e-10	6.26
		$h_0/16$	2.69e-06	2.97	7.20e-09	4.01	2.11e-09	4.92	9.28e-12	6.03

(continued)

Table 3 (continued)

Volume disc. + Surface flux	Mach	Mesh level	$N = 2$		$N = 3$		$N = 4$		$N = 5$	
			$L_2(q)$	EOC	$L_2(q)$	EOC	$L_2(q)$	EOC	$L_2(q)$	EOC
0.2		$h_0/2$	1.20e-03	1.96	4.90e-05	3.51	5.23e-05	3.92	4.47e-06	5.16
		$h_0/4$	2.72e-04	2.14	1.87e-06	4.71	3.02e-06	4.12	6.14e-08	6.19
		$h_0/8$	5.58e-05	2.29	1.21e-07	3.95	1.57e-07	4.26	6.18e-10	6.64
		$h_0/16$	1.01e-05	2.47	5.98e-09	4.34	7.28e-09	4.43	8.20e-12	6.23
3.5	Split-DGSEM + ECKEP-Roe	$h_0/2$	1.23e-03	3.36	3.88e-05	4.06	4.49e-05	4.94	3.01e-06	5.97
		$h_0/4$	1.27e-04	3.27	2.42e-06	4.00	1.43e-06	4.97	4.84e-08	5.96
		$h_0/8$	1.48e-05	3.10	1.51e-07	4.00	4.44e-08	5.01	7.48e-10	6.02
		$h_0/16$	1.81e-06	3.03	9.46e-09	4.00	1.40e-09	4.99	1.18e-11	5.98
1.0		$h_0/2$	9.18e-04	2.86	3.98e-05	3.94	4.51e-05	4.92	3.08e-06	5.95
		$h_0/4$	1.15e-04	2.99	2.42e-06	4.04	1.51e-06	4.90	4.97e-08	5.95
		$h_0/8$	1.44e-05	3.00	1.51e-07	4.00	4.49e-08	5.07	7.66e-10	6.02
		$h_0/16$	1.80e-06	3.00	9.46e-09	4.00	1.38e-09	5.02	1.21e-11	5.99
0.2		$h_0/2$	9.26e-04	2.35	4.65e-05	3.45	4.37e-05	4.26	3.26e-06	5.34
		$h_0/4$	1.19e-04	2.96	2.40e-06	4.27	1.63e-06	4.74	5.03e-08	6.02
		$h_0/8$	1.43e-05	3.06	1.58e-07	3.93	4.45e-08	5.19	7.66e-10	6.04
		$h_0/16$	1.80e-06	2.99	9.40e-09	4.07	1.53e-09	4.86	1.22e-11	5.97

Full order is marked with ■ ($\geq N + 1$) and an order reduction with ⊠

speed (LLF-type) or incorporating all waves (Roe-type), which we will refer to as ECKEP-LLF and ECKEP-Roe fluxes.

In Table 3, we summarize the convergence of the dissipation-free ECKEP flux, the HLL and ECKEP-Roe flux. The results for LLF and ECKEP-LLF fluxes are found in the Appendix in Table 6, as they have the same convergence and error levels as the HLL flux. As expected, the dissipation-free surface flux (ECKEP) produces an order reduction for all Mach numbers for $N = 3, 5$, and for $N = 2$ full order is not kept in the last refinement step.

If we simply use the HLL flux, we have an entropy stable scheme, but an order reduction for $N = 2, 4$ can be observed for the low Mach number flow, analogously to the standard DGSEM-LGL scheme. Interestingly, the odd-even behavior switches between entropy conserving and entropy stable fluxes.

The ECKEP-Roe entropy stable flux accounts for all waves of the Riemann problem and adjusts the dissipation for each wave accordingly, which gives full order convergence for all Mach numbers.

4 Conclusions

In this work, we report the convergence of standard DGSEM Gauss and Gauss-Lobatto schemes to entropy conservative (EC) and entropy stable (ES) DGSEM schemes for the Euler equations, as there have been findings of order reduction for EC and ES schemes. We choose a simple density transport test case on a periodic domain and investigate the influence of the Mach number of the transport velocity.

The EC scheme is dissipation free and an order reduction is observed by the convergence study presented here, confirming many similar observations found in literature. We also confirm that the ES scheme can have an order reduction for low Mach numbers, but only if the entropy stable numerical flux relies on simple approximate Riemann solvers such as local Lax-Friedrichs or HLL. If all waves are accounted for in the dissipation term of the entropy stable flux as presented in [22], the full order is observed for all Mach numbers. In addition, we reproduce the same behavior for the standard DGSEM Gauss and Gauss-Lobatto schemes, where the LLF and HLL fluxes suffer from order reduction at low Mach number, and HLLC and Roe fluxes have full order for all Mach numbers.

We want to emphasize that the present convergence study should be seen merely as an observation, confirming that the numerical flux can have strong influence on the convergence order for both the standard DGSEM and the entropy stable DGSEM. Also, we stress that in our tests the order reduction is related to the form of the dissipation term in the numerical surface flux and is not related to the insufficient integration precision of the LGL-quadrature.

Based on the observations presented in this work, a possible explanation for the loss of convergence for the density transport at low Mach numbers when using LLF and HLL fluxes is the form of dissipation from the approximate Riemann solver. In the case of the density transport, the exact solution follows the characteristic

with velocity $\vec{v}$. However, the approximate Riemann solver LLF and HLL consider only two waves with maximum velocity $\sim (|\vec{v}| + c)$ and do not consider the contact wave with velocity $\vec{v}$. Thus, the contact wave is dissipated proportional to $\sim (|\vec{v}| + c)$ and not to $|\vec{v}|$. For low Mach numbers, where $c > |\vec{v}|$, this causes over-upwinding. Over-upwinding was discussed in [5, 15]. It is not intuitive at first, but over-upwinding (over-penalization) can lead to a reduction of the in-built dissipation of the DG scheme, getting wave-propagation characteristics similar to a continuous Galerkin method [16]. This loss of in-built dissipation could be an explanation for the even-odd behavior we observed. However, it is still unclear why numerical surface fluxes with no in-built dissipation that are symmetric, e.g. EC flux, lead to an odd-even behavior in the convergence order and why numerical surface fluxes with over-upwinding, i.e. reduced dissipation due to over-penalization, cause an opposite even-odd behavior. What supports the explanation is the recovery of full convergence order for LLF and HLL when the difference in wave speed becomes smaller for higher Mach numbers, i.e. no over-upwinding. In contrast to LLF and HLL, the HLLC and Roe solvers take specifically the contact wave into account and adjust the dissipation accordingly and thus avoid strong over-upwinding by construction. In our tests, we always observe full convergence order for all Mach numbers for HLLC and Roe.

Lastly we note that a convergence study using a manufactured solution technique can be misleading, as full convergence order is found independent of the choice of numerical flux. Hence, the introduction of a source term to balance the prescribed solution overcomes possible deficiencies of the surface fluxes, showing the limit of the manufactured solution technique in this context. In the Appendix, the convergence results of a manufactured solution are reported.

Acknowledgements Gregor Gassner thanks the European Research Council for funding through the ERC Starting Grant “An Exascale aware and Un-crashable Space-Time-Adaptive Discontinuous Spectral Element Solver for Non-Linear Conservation Laws” (Extreme), ERC grant agreement no. 714487. Florian Hindenlang thanks Eric Sonnendrücker and the Max-Planck Institute for Plasma Physics in Garching for their constant support. We would also like to thank all participants of the ICOSAHOM 2018 for the valuable discussions on the topic of entropy stable schemes, which motivated this work.

Appendix

Additional Convergence Results

In this section, we present additional convergence results of the density wave test case for the DGSEM-Gauss and DGSEM-LGL with LLF and HLLC fluxes in Table 4 and Table 5, and also the entropy stable schemes with LLF and ECKEP-LLF fluxes in Table 6. The results for LLF-type fluxes behave like the HLL flux, and for the HLLC flux like the Roe-type fluxes presented in Table 3.

Table 4 Experimental order of convergence of L_2 error to the exact density (4), using DGSEM-Gauss with LLF and HLLC fluxes

Volume disc. + Surface flux	Mach	Mesh level	N = 2		N = 3		N = 4		N = 5	
			$L_2(\varrho)$	EOC	$L_2(\varrho)$	EOC	$L_2(\varrho)$	EOC	$L_2(\varrho)$	EOC
DGSEM-Gauss + LLF	3.5	$h_0/2$	2.42e-04	2.97	6.43e-06	4.41	1.05e-05	4.51	6.68e-07	6.60
		$h_0/4$	3.24e-05	2.90	3.71e-07	4.11	4.32e-07	4.60	8.04e-09	6.38
		$h_0/8$	4.15e-06	2.96	2.27e-08	4.03	1.47e-08	4.87	1.15e-10	6.13
		$h_0/16$	5.22e-07	2.99	1.41e-09	4.01	4.73e-10	4.96	1.77e-12	6.02
	1.0	$h_0/2$	3.13e-04	2.29	4.59e-06	4.84	1.18e-05	3.90	6.69e-07	7.40
		$h_0/4$	5.30e-05	2.56	2.25e-07	4.35	6.08e-07	4.28	5.56e-09	6.91
		$h_0/8$	7.43e-06	2.83	1.29e-08	4.12	2.47e-08	4.62	6.79e-11	6.35
		$h_0/16$	9.61e-07	2.95	7.65e-10	4.07	8.53e-10	4.85	9.97e-13	6.09
	0.2	$h_0/2$	4.95e-04	1.69	4.33e-06	5.07	1.58e-05	2.53	9.88e-07	6.68
		$h_0/4$	1.12e-04	2.15	1.06e-07	5.35	1.01e-06	3.97	4.46e-09	7.79
		$h_0/8$	2.06e-05	2.44	5.47e-09	4.28	5.23e-08	4.27	3.86e-11	6.85
		$h_0/16$	3.29e-06	2.65	2.15e-10	4.67	2.38e-09	4.46	4.35e-13	6.47
DGSEM-Gauss + HLLC	3.5	$h_0/2$	1.87e-04	3.34	8.57e-06	4.02	1.03e-05	5.02	6.76e-07	6.07
		$h_0/4$	2.27e-05	3.04	5.35e-07	4.00	3.30e-07	4.96	1.07e-08	5.99
		$h_0/8$	2.82e-06	3.00	3.34e-08	4.00	1.02e-08	5.01	1.66e-10	6.01
		$h_0/16$	3.53e-07	3.00	2.09e-09	4.00	3.22e-10	4.99	2.60e-12	5.99
	1.0	$h_0/2$	1.82e-04	3.07	8.76e-06	4.04	1.11e-05	5.07	6.95e-07	6.08
		$h_0/4$	2.26e-05	3.00	5.40e-07	4.02	3.44e-07	5.00	1.07e-08	6.01
		$h_0/8$	2.82e-06	3.00	3.35e-08	4.01	1.05e-08	5.04	1.67e-10	6.01
		$h_0/16$	3.53e-07	3.00	2.09e-09	4.00	3.25e-10	5.01	2.61e-12	6.00

(continued)

Table 4 (continued)

Volume disc. + Surface flux	Mach	Mesh level	N = 2		N = 3		N = 4		N = 5	
			$L_2(\varrho)$	EOC	$L_2(\varrho)$	EOC	$L_2(\varrho)$	EOC	$L_2(\varrho)$	EOC
	0.2	$h_0/2$	2.14e-04	2.65	1.04e-05	3.78	1.16e-05	4.53	7.76e-07	5.78
		$h_0/4$	2.22e-05	3.26	5.49e-07	4.25	3.49e-07	5.05	1.08e-08	6.17
		$h_0/8$	2.82e-06	2.98	3.76e-08	3.87	1.04e-08	5.07	1.70e-10	5.99
		$h_0/16$	3.53e-07	3.00 ■	2.07e-09	4.19 ■	3.38e-10	4.94 ■	2.64e-12	6.01 ■

Full order is marked with ■ ($\gtrsim N + 1$) and an order reduction with ☒

Table 5 Experimental order of convergence of L_2 error to the exact density (4), using DGSEM-GL with LLF and HLLC fluxes

Volume disc. + Surface flux	Mach	Mesh level	N = 2		N = 3		N = 4		N = 5	
			$L_2(\varrho)$	EOC	$L_2(\varrho)$	EOC	$L_2(\varrho)$	EOC	$L_2(\varrho)$	EOC
DGSEM-LGL + LLF	3.5	$h_0/2$	1.23e-03	3.11	3.44e-05	4.36	4.02e-05	4.82	2.91e-06	6.14
		$h_0/4$	1.51e-04	3.02	2.00e-06	4.10	1.61e-06	4.64	3.80e-08	6.26
		$h_0/8$	1.89e-05	3.00	1.23e-07	4.02	5.55e-08	4.86	5.46e-10	6.12
		$h_0/16$	2.36e-06	3.00	7.66e-09	4.01	1.79e-09	4.95	8.54e-12	6.00
	1.0	$h_0/2$	1.10e-03	1.97	3.15e-05	4.83	4.02e-05	4.93	2.85e-06	6.73
		$h_0/4$	2.03e-04	2.43	1.68e-06	4.23	1.98e-06	4.34	3.68e-08	6.28
		$h_0/8$	2.99e-05	2.76	1.03e-07	4.02	8.35e-08	4.57	4.86e-10	6.24
		$h_0/16$	3.94e-06	2.93	6.39e-09	4.01	2.96e-09	4.82	7.53e-12	6.01
	0.2	$h_0/2$	1.21e-03	1.95	4.88e-05	3.51	5.20e-05	3.90	3.89e-06	5.31
		$h_0/4$	2.77e-04	2.12	1.86e-06	4.72	3.05e-06	4.09	4.27e-08	6.51
DGSEM-LGL + HLLC	3.5	$h_0/8$	5.78e-05	2.26	1.21e-07	3.94	1.61e-07	4.24	5.47e-10	6.29
		$h_0/16$	1.06e-05	2.44	5.98e-09	4.34	7.57e-09	4.41	7.16e-12	6.26
		$h_0/2$	1.22e-03	3.38	3.85e-05	4.03	4.34e-05	4.93	2.70e-06	5.93
		$h_0/4$	1.26e-04	3.27	2.41e-06	4.00	1.39e-06	4.96	4.33e-08	5.96
	1.0	$h_0/8$	1.48e-05	3.10	1.51e-07	4.00	4.34e-08	5.01	6.67e-10	6.02
		$h_0/16$	1.81e-06	3.03	9.42e-09	4.00	1.36e-09	4.99	1.05e-11	5.98
		$h_0/2$	9.17e-04	2.86	3.96e-05	3.94	4.41e-05	4.94	2.76e-06	6.02
		$h_0/4$	1.15e-04	2.99	2.41e-06	4.04	1.47e-06	4.90	4.44e-08	5.96
	0.2	$h_0/8$	1.44e-05	3.00	1.51e-07	4.00	4.38e-08	5.07	6.84e-10	6.02
		$h_0/16$	1.80e-06	3.00	9.42e-09	4.00	1.35e-09	5.02	1.08e-11	5.99

(continued)

Table 5 (continued)

Volume disc. + Surface flux	Mach	Mesh level	N = 2		N = 3		N = 4		N = 5	
			$L_2(\varrho)$	EOC	$L_2(\varrho)$	EOC	$L_2(\varrho)$	EOC	$L_2(\varrho)$	EOC
	0.2	$h_0/2$	9.26e-04	2.35	4.63e-05	3.45	4.26e-05	4.27	2.97e-06	5.37
		$h_0/4$	1.19e-04	2.96	2.40e-06	4.27	1.59e-06	4.74	4.48e-08	6.05
		$h_0/8$	1.43e-05	3.06	1.57e-07	3.93	4.34e-08	5.19	6.83e-10	6.04
		$h_0/16$	1.80e-06	2.99 ■	9.37e-09	4.07 ■	1.49e-09	4.86 ■	1.09e-11	5.97 ■

Full order is marked with ■ ($\gtrsim N + 1$) and an order reduction with ☒

Table 6 Experimental order of convergence of L_2 error to the exact density (4), using entropy stable LLF and ECKEP-LLF flux

Volume disc. + Surface flux	Mach	Mesh level	N = 2		N = 3		N = 4		N = 5	
			$L_2(q)$	EOC	$L_2(q)$	EOC	$L_2(q)$	EOC	$L_2(q)$	EOC
Split-DGSEM + LLF	3.5	$h_0/2$	1.24e-03	3.11	3.49e-05	4.37	4.16e-05	4.85	3.37e-06	6.05
		$h_0/4$	1.52e-04	3.03	2.01e-06	4.12	1.64e-06	4.67	4.49e-08	6.23
		$h_0/8$	1.89e-05	3.00	1.23e-07	4.03	5.67e-08	4.85	6.21e-10	6.18
		$h_0/16$	2.36e-06	3.00	7.68e-09	4.01	1.84e-09	4.95	9.58e-12	6.02
	1.0	$h_0/2$	1.09e-03	1.97	3.21e-05	4.82	4.07e-05	4.94	3.77e-06	6.34
		$h_0/4$	2.03e-04	2.43	1.69e-06	4.25	1.99e-06	4.35	4.69e-08	6.33
		$h_0/8$	2.99e-05	2.76	1.04e-07	4.03	8.46e-08	4.56	5.57e-10	6.39
		$h_0/16$	3.94e-06	2.93	6.41e-09	4.02	3.03e-09	4.80	8.38e-12	6.05
	0.2	$h_0/2$	1.21e-03	1.95	4.92e-05	3.50	5.26e-05	3.91	4.52e-06	5.15
		$h_0/4$	2.77e-04	2.12	1.88e-06	4.71	3.07e-06	4.10	6.21e-08	6.19
Split-DGSEM + ECKEP-LLF	3.5	$h_0/8$	5.79e-05	2.26	1.22e-07	3.95	1.63e-07	4.24	6.21e-10	6.64
		$h_0/16$	1.06e-05	2.44	6.01e-09	4.34	7.66e-09	4.41	8.26e-12	6.23
		$h_0/2$	1.24e-03	3.10	3.49e-05	4.37	4.16e-05	4.84	3.38e-06	6.06
		$h_0/4$	1.52e-04	3.03	2.01e-06	4.12	1.64e-06	4.67	4.49e-08	6.23
	1.0	$h_0/8$	1.89e-05	3.01	1.23e-07	4.03	5.68e-08	4.85	6.21e-10	6.18
		$h_0/16$	2.36e-06	3.00	7.68e-09	4.01	1.84e-09	4.95	9.58e-12	6.02
		$h_0/2$	1.10e-03	1.97	3.20e-05	4.83	4.08e-05	4.94	3.77e-06	6.36
		$h_0/4$	2.04e-04	2.43	1.69e-06	4.25	2.00e-06	4.35	4.69e-08	6.33
	0.2	$h_0/8$	3.00e-05	2.76	1.04e-07	4.03	8.49e-08	4.56	5.57e-10	6.40
		$h_0/16$	3.95e-06	2.93	6.41e-09	4.02	3.04e-09	4.81	8.38e-12	6.05

(continued)

Table 6 (continued)

Volume disc. + Surface flux	Mach	Mesh level	N = 2		N = 3		N = 4		N = 5	
			$L_2(\varrho)$	EOC	$L_2(\varrho)$	EOC	$L_2(\varrho)$	EOC	$L_2(\varrho)$	EOC
	0.2	$h_0/2$	1.21e-03	1.95	4.93e-05	3.51	5.27e-05	3.91	4.53e-06	5.15
		$h_0/4$	2.78e-04	2.12	1.88e-06	4.71	3.08e-06	4.10	6.21e-08	6.19
		$h_0/8$	5.79e-05	2.26	1.22e-07	3.95	1.63e-07	4.24	6.21e-10	6.65
		$h_0/16$	1.07e-05	2.44	6.01e-09	4.34	7.67e-09	4.41	8.26e-12	6.23

Full order is marked with ■ ($\gtrsim N + 1$) and an order reduction with ⊠

Manufactured Solution with Source Term

Here, we run a convergence test with the method of manufactured solutions. To do so, we assume a two-dimensional solution of the form

$$\mathbf{u} = [\varrho, \varrho v_1, \varrho v_2, \varrho v_3, E]^T = [g, g, g, 0, g^2]^T \quad (5)$$

with $g = g(x_1, x_2, t) = 0.5 \sin(2\pi(x_1 + x_2 - t)) + 2$.

Note that the average Mach number in the domain is $\text{Ma} = 0.8$. Inserting (5) into the Euler equations, and using the fact that spatial and time derivatives are $g' = \partial_{x_1} g = \partial_{x_2} g = -\partial_t g$, we get an additional residual

$$\mathbf{u}_t + \sum_{i=1}^3 \frac{\partial \mathbf{f}_i}{\partial x_i} = \begin{pmatrix} g' \\ (3\gamma - 2)g' + 2(\gamma - 1)gg' \\ (3\gamma - 2)g' + 2(\gamma - 1)gg' \\ 0 \\ (6\gamma - 2)g' + 2(2\gamma - 1)gg' \end{pmatrix} \quad (6)$$

To solve the inhomogeneous problem, we subtract the residual from the approximate solution in each Runge–Kutta step. Moreover, we run the test case up to the final time $t = 1.0$.

In the convergence results for the standard DGSEM Gauss and Gauss-Lobatto, we see that the LLF flux still leads to an order reduction for $N = 2, 4$, whereas full order is found for the HLL, HLLC and Roe fluxes, see Tables 7 and 8.

In Table 9 the entropy conservative scheme shows again an order reduction for $N = 3, 5$, and the LLF-Type dissipation too, for $N = 2, 4$, and for this test case, all entropy stable schemes exhibit full order.

Table 7 Experimental order of convergence of L_2 error of density for the manufactured solution (5), using DGSEM-Gauss with LLF, HLL, HLLC and Roe fluxes

	Mesh level	$N = 2$		$N = 3$		$N = 4$		$N = 5$	
		$L_2(q)$	EOC	$L_2(q)$	EOC	$L_2(q)$	EOC	$L_2(q)$	EOC
Volume disc. + Surface flux DGSEM-Gauss + LLF	$h_0/2$	2.30e-03	2.20	4.54e-05	5.34	1.13e-04	6.35	4.52e-05	4.38
	$h_0/4$	4.81e-04	2.25	1.99e-06	4.52	4.78e-06	4.56	2.37e-07	7.58
	$h_0/8$	9.48e-05	2.34	1.02e-07	4.28	2.88e-07	4.05	1.53e-09	7.28
	$h_0/16$	1.57e-05	2.60	6.25e-09	4.03	1.73e-08	4.05	1.57e-11	6.61
DGSEM-Gauss + HLL	$h_0/2$	1.24e-03	2.84	5.46e-05	4.35	1.32e-04	5.22	1.47e-05	6.30
	$h_0/4$	1.17e-04	3.41	3.36e-06	4.02	2.89e-06	5.51	1.44e-07	6.67
	$h_0/8$	1.41e-05	3.04	1.85e-07	4.18	7.14e-08	5.34	1.74e-09	6.37
	$h_0/16$	1.76e-06	3.00	1.07e-08	4.11	2.15e-09	5.05	2.24e-11	6.28
DGSEM-Gauss + HLLC	$h_0/2$	1.24e-03	2.84	5.46e-05	4.35	1.32e-04	5.22	1.47e-05	6.30
	$h_0/4$	1.17e-04	3.41	3.36e-06	4.02	2.89e-06	5.51	1.44e-07	6.67
	$h_0/8$	1.41e-05	3.04	1.85e-07	4.18	7.14e-08	5.34	1.74e-09	6.37
	$h_0/16$	1.76e-06	3.00	1.07e-08	4.11	2.15e-09	5.05	2.24e-11	6.28
DGSEM-Gauss + Roe	$h_0/2$	1.24e-03	2.84	5.46e-05	4.35	1.32e-04	5.22	1.47e-05	6.30
	$h_0/4$	1.17e-04	3.41	3.36e-06	4.02	2.89e-06	5.51	1.44e-07	6.67
	$h_0/8$	1.41e-05	3.04	1.85e-07	4.18	7.14e-08	5.34	1.74e-09	6.37
	$h_0/16$	1.76e-06	3.00	1.07e-08	4.11	2.15e-09	5.05	2.24e-11	6.28

Full order is marked with ($\geq N + 1$) and an order reduction with

Table 8 Experimental order of convergence of L_2 error of density for the manufactured solution (5), using DGSEM-LGL with LLF, HLL, HLLC and Roe fluxes

Volume disc. + Surface flux	Mesh level	$N = 2$		$N = 3$		$N = 4$		$N = 5$	
		$L_2(q)$	EOC	$L_2(q)$	EOC	$L_2(q)$	EOC	$L_2(q)$	EOC
DGSEM-LGL + LLF	$h_0/2$	7.36e-03	2.85	3.15e-04	4.38	5.69e-04	5.78	9.33e-05	5.34
	$h_0/4$	1.33e-03	2.47	1.43e-05	4.46	2.04e-05	4.80	9.27e-07	6.65
	$h_0/8$	2.79e-04	2.25	7.99e-07	4.16	8.81e-07	4.54	9.34e-09	6.63
	$h_0/16$	5.31e-05	2.39	4.72e-08	4.08	5.94e-08	3.89	1.37e-10	6.09
DGSEM-LGL + HLL	$h_0/2$	5.32e-03	3.16	2.52e-04	4.00	3.84e-04	5.36	4.33e-05	6.24
	$h_0/4$	5.99e-04	3.15	1.38e-05	4.19	1.43e-05	4.75	4.58e-07	6.56
	$h_0/8$	7.25e-05	3.05	7.69e-07	4.17	2.92e-07	5.61	7.08e-09	6.02
	$h_0/16$	9.02e-06	3.01	4.74e-08	4.02	7.77e-09	5.23	1.10e-10	6.01
DGSEM-LGL + HLLC	$h_0/2$	5.32e-03	3.16	2.52e-04	4.00	3.84e-04	5.36	4.33e-05	6.24
	$h_0/4$	5.99e-04	3.15	1.38e-05	4.19	1.43e-05	4.75	4.58e-07	6.56
	$h_0/8$	7.25e-05	3.05	7.69e-07	4.17	2.92e-07	5.61	7.08e-09	6.02
	$h_0/16$	9.02e-06	3.01	4.74e-08	4.02	7.77e-09	5.23	1.10e-10	6.01
DGSEM-LGL + Roe	$h_0/2$	5.32e-03	3.16	2.52e-04	4.00	3.84e-04	5.36	4.33e-05	6.24
	$h_0/4$	5.99e-04	3.15	1.38e-05	4.19	1.43e-05	4.75	4.58e-07	6.56
	$h_0/8$	7.25e-05	3.05	7.69e-07	4.17	2.92e-07	5.61	7.08e-09	6.02
	$h_0/16$	9.02e-06	3.01	4.74e-08	4.02	7.77e-09	5.23	1.10e-10	6.01

Full order is marked with ■ ($\geq N + 1$) and an order reduction with ✗

Table 9 Experimental order of convergence of L_2 error of density for the manufactured solution (5), using entropy conservative and entropy stable schemes

Volume disc. + Surface flux	Mesh level	$N = 2$			$N = 3$			$N = 4$			$N = 5$		
		$L_2(\rho)$	EOC		$L_2(\rho)$	EOC		$L_2(\rho)$	EOC		$L_2(\rho)$	EOC	
Split-DGSEM + ECKEP	$h_0/2$	1.31e-02	3.53		1.28e-03	2.88		5.62e-03	2.49		1.12e-03	4.05	
	$h_0/4$	1.30e-03	3.34		1.13e-04	3.49		3.12e-04	4.17		5.97e-06	7.56	
	$h_0/8$	1.23e-04	3.40		1.24e-05	3.20		2.00e-06	7.29		9.45e-08	5.98	
	$h_0/16$	1.76e-05	2.80 ■		1.67e-06	2.89 ☒		3.41e-08	5.87 ■		3.17e-09	4.90 ☒	
Split-DGSEM + LLF	$h_0/2$	7.60e-03	3.13		3.70e-04	3.91		6.56e-04	4.44		1.28e-04	4.75	
	$h_0/4$	1.63e-03	2.22		1.90e-05	4.29		3.09e-05	4.41		2.01e-06	6.00	
	$h_0/8$	3.40e-04	2.26		9.89e-07	4.26		1.83e-06	4.08		1.80e-08	6.80	
	$h_0/16$	6.14e-05	2.47 ☒		6.41e-08	3.95 ■		9.15e-08	4.32 ☒		2.17e-10	6.37 ■	
Split-DGSEM + ECKEP-LLF	$h_0/2$	7.66e-03	3.21		3.74e-04	3.98		6.79e-04	4.45		1.33e-04	4.75	
	$h_0/4$	1.64e-03	2.23		1.90e-05	4.30		3.08e-05	4.46		2.06e-06	6.01	
	$h_0/8$	3.41e-04	2.26		9.88e-07	4.26		1.83e-06	4.07		1.79e-08	6.85	
	$h_0/16$	6.16e-05	2.47 ☒		6.40e-08	3.95 ■		9.17e-08	4.32 ☒		2.17e-10	6.37 ■	
Split-DGSEM + HLL	$h_0/2$	5.82e-03	3.10		3.01e-04	3.85		5.11e-04	4.35		7.06e-05	5.27	
	$h_0/4$	7.06e-04	3.04		2.04e-05	3.88		1.67e-05	4.94		1.08e-06	6.03	
	$h_0/8$	8.63e-05	3.03		1.16e-06	4.14		5.08e-07	5.04		1.67e-08	6.02	
	$h_0/16$	1.08e-05	3.00 ■		7.20e-08	4.01 ■		1.62e-08	4.97 ■		2.64e-10	5.98 ■	
Split-DGSEM + ECKEP-Roe	$h_0/2$	5.81e-03	3.11		3.01e-04	3.85		5.11e-04	4.35		7.06e-05	5.27	
	$h_0/4$	7.06e-04	3.04		2.04e-05	3.88		1.67e-05	4.94		1.08e-06	6.03	
	$h_0/8$	8.63e-05	3.03		1.16e-06	4.14		5.08e-07	5.04		1.67e-08	6.02	
	$h_0/16$	1.08e-05	3.00 ■		7.20e-08	4.01 ■		1.62e-08	4.97 ■		2.64e-10	5.98 ■	

Full order is marked with ■ ($\geq N + 1$) and an order reduction with ☒

References

1. Bohm, M., Winters, A.R., Gassner, G.J., Derigs, D., Hindenlang, F., Saur, J.: An entropy stable nodal discontinuous Galerkin method for the resistive MHD equations. Part I: Theory and numerical verification. *J. Comput. Phys.* 108076 (2018). <https://doi.org/10.1016/j.jcp.2018.06.027>
2. Carpenter, M., Kennedy, C.: Fourth-order 2N-storage Runge-Kutta schemes. Technical Report NASA TM 109111 (1994)
3. Carpenter, M., Fisher, T., Nielsen, E., Frankel, S.: Entropy stable spectral collocation schemes for the Navier–Stokes equations: discontinuous interfaces. *SIAM J. Sci. Comput.* **36**(5), B835–B867 (2014)
4. Chan, J.: On discretely entropy conservative and entropy stable discontinuous Galerkin methods. *J. Comput. Phys.* **362**, 346–374 (2018)
5. Chan, J., Warburton, T.: On the penalty stabilization mechanism for upwind discontinuous Galerkin formulations of first order hyperbolic systems. *Comput. Math. Appl.* **74**(12), 3099–3110 (2017)
6. Chen, T., Shu, C.-W.: Entropy stable high order discontinuous Galerkin methods with suitable quadrature rules for hyperbolic conservation laws. *J. Comput. Phys.* **345**, 427–461 (2017)
7. Flad, D., Gassner, G.: On the use of kinetic energy preserving DG-schemes for large eddy simulation. *J. Comput. Phys.* **350**, 782–795 (2017)
8. Friedrich, L., Winters, A.R., Del Rey Fernández, D.C., Gassner, G.J., Parsani, M., Carpenter, M.H.: An entropy stable h/p non-conforming discontinuous Galerkin method with the summation-by-parts property. *J. Sci. Comput.* **77**, 689–725 (2018)
9. Gassner, G.: A skew-symmetric discontinuous Galerkin spectral element discretization and its relation to SBP-SAT finite difference methods. *SIAM J. Sci. Comput.* **35**(3), A1233–A1253 (2013)
10. Gassner, G.J.: A kinetic energy preserving nodal discontinuous Galerkin spectral element method. *Int. J. Numer. Methods Fluids* **76**(1), 28–50 (2014)
11. Gassner, G., Kopriva, D.A.: A comparison of the dispersion and dissipation errors of Gauss and Gauss–Lobatto discontinuous Galerkin spectral element methods. *SIAM J. Sci. Comput.* **33**, 2560–2579 (2011)
12. Gassner, G.J., Winters, A.R., Kopriva, D.A.: Split form nodal discontinuous Galerkin schemes with summation-by-parts property for the compressible Euler equations. *J. Comput. Phys.* **327**, 39–66 (2016)
13. Gassner, G.J., Winters, A.R., Hindenlang, F.J., Kopriva, D.A.: The BR1 scheme is stable for the compressible Navier–Stokes equations. *J. Sci. Comput.* **77**, 154–200 (2018)
14. Kopriva, D.A.: Implementing Spectral Methods for Partial Differential Equations. Scientific Computation. Springer, Berlin (2009)
15. Manzanero, J., Ferrer, E., Rubio, G., Valero, E.: On the role of numerical dissipation in stabilising under-resolved turbulent simulations using discontinuous Galerkin methods. Preprint arXiv:1805.10519 (2018)
16. Moura, R., Sherwin, S., Peirò, J.: Eigensolution analysis of spectral/hp continuous Galerkin approximations to advection-diffusion problems: insights into spectral vanishing viscosity. *J. Comput. Phys.* **307**, 401–422 (2016)
17. Parsani, M., Carpenter, M.H., Nielsen, E.J.: Entropy stable wall boundary conditions for the three-dimensional compressible Navier–Stokes equations. *J. Comput. Phys.* **292**, 88–113 (2015)
18. Toro, E.: Riemann Solvers and Numerical Methods for Fluid Dynamics. Springer, Berlin (2009)
19. Wintermeyer, N., Winters, A.R., Gassner, G.J., Kopriva, D.A.: An entropy stable nodal discontinuous Galerkin method for the two dimensional shallow water equations on unstructured curvilinear meshes with discontinuous bathymetry. *J. Comput. Phys.* (2016, submitted). arXiv:1509.07096 [math.NA]

20. Wintermeyer, N., Winters, A.R., Gassner, G.J., Warburton, T.: An entropy stable discontinuous Galerkin method for the shallow water equations on curvilinear meshes with wet/dry fronts accelerated by GPUs. *J. Comput. Phys.* **375**, 447–480 (2018)
21. Winters, A.R., Gassner, G.J.: A comparison of two entropy stable discontinuous Galerkin spectral element approximations for the shallow water equations with non-constant topography. *J. Comput. Phys.* **301**, 357–376 (2015)
22. Winters, A.R., Derigs, D., Gassner, G.J., Walch, S.: A uniquely defined entropy stable matrix dissipation operator for high Mach number ideal MHD and compressible Euler simulations. *J. Comput. Phys.* **332**, 274–289 (2017)
23. Winters, A.R., Moura, R.C., Mengaldo, G., Gassner, G.J., Walch, S., Peiro, J., Sherwin, S.J.: A comparative study on polynomial dealiasing and split form discontinuous Galerkin schemes for under-resolved turbulence computations. *J. Comput. Phys.* **372**, 1–21 (2018)

Open Access This chapter is licensed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence and indicate if changes were made.

The images or other third party material in this chapter are included in the chapter's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the chapter's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder.