Chapter 2

Basic Concepts of Stokes Flows

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2.1 Introduction

Stokes flows have many applications in both physical theory and practice. For example, they have been used to describe dynamics of complex fluids in microfluidics, lab-on-chip technologies [1], medical applications [2, 3], design of innovative materials [4–6] and micro-devices—e.g. to carry drugs [7, 8] or act as fuel cells [9]—and in biological systems [10–15].

In this chapter, we discuss some fundamental properties of Stokes flows, namely: negligibility of inertial forces, reversibility and the minimum energy dissipation theorem. First we will briefly discuss how the neglecting of inertial forces simplifies the nonlinear Navier–Stokes equations to the linear Stokes equations. We then discuss two basic aspects of Stokes flows: reversibility and the minimum energy dissipation theorem. In order to bring out the nature of the three principles, we will demonstrate by example how these properties can be used to obtain conclusions about investigated fluid systems without laborious construction of analytical solutions. We then move beyond the Stokes approximation in various ways in order to see how the principles work in a general context. Finally, we conclude by discussing the logical structure of the principles as revealed by the examples considered.

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2.2 Navier–Stokes and Stokes Equations

2.2.1 Navier–Stokes Equations

We start with the general Navier–Stokes equations for an incompressible fluid. These are [16–18]

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = \mu \nabla^2 \mathbf{u} - \nabla p + \mathbf{F} \quad (2.1)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (2.2)$$

where ρ is the density of the fluid, $\mathbf{u}$ is the velocity field of a fluid, μ is the dynamic viscosity of the fluid, p is the fluid pressure field,¹ and $\mathbf{F}$ captures the effects of external forces. The left-hand side of this equation is the inertial forces, that is, the acceleration of a fluid element with unit volume. The right-hand side is sum of the viscous and pressure forces, $\mu \nabla^2 \mathbf{u}$ and ∇p , respectively, exerted on surfaces of this fluid element, and any external body forces $\mathbf{F}$ acting on the fluid element. The second equation is based on the conservation of mass of a fluid element and achieves its simple form because of incompressibility of the fluid element.

We use non-dimensionalisation in order to capture the relative scale of the forces. Define U to be a characteristic velocity of the fluid and L to be a characteristic length scale. Other characteristic dimensional scales of the flow, for instance, a time scale $T = L/U$, can be defined implicitly from these scales. There is still some freedom when normalising pressure p and body forces $\mathbf{F}$. We choose to normalise pressure by a characteristic viscous force per unit area and $\mathbf{F}$ by a characteristic viscous force per unit volume as in [18]. Using a star to denote non-dimensionalised objects, this results in the following definitions:

$$\begin{aligned} \mathbf{u}^* &= \frac{\mathbf{u}}{U} \\ \nabla^* &= L \nabla \\ \frac{\partial}{\partial t^*} &= \frac{L}{U} \frac{\partial}{\partial t} \\ p^* &= \frac{L}{\mu U} p \\ \mathbf{F}^* &= \frac{L^2}{\mu U} \mathbf{F} \end{aligned} \quad (2.3)$$

¹In the presence of a gravitational field, p is the so-called modified pressure, which takes into account also gravitational potential energy per unit fluid volume.

With the above characteristic dimensional scales, the inertial force per unit volume is estimated by $\frac{\rho U^2}{L}$ and the scale of the viscous force per unit volume is $\frac{\mu U}{L^2}$. The Reynolds number, Re , a non-dimensional number defined as the ratio of inertial and viscous forces in a fluid, takes the form

$$Re = \frac{(\rho U^2)/L}{(\mu U)/L^2} = \frac{\rho U L}{\mu} \quad (2.4)$$

The end result is the following non-dimensional version of the Navier–Stokes equation (2.1):

$$Re \left(\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* \right) = \nabla^{*2} \mathbf{u}^* - \nabla^* p^* + \mathbf{F}^* \quad (2.5)$$

The left-hand side is the inertial force and the right-hand side is the viscous, pressure and body forces. Flows with the same Re are hydrodynamically similar [18].

A difficulty to using Eqs. (2.1) and (2.2) (or their non-dimensional form) in the analysis of fluids is that the inertial forces are nonlinear in $\mathbf{u}$. In terms of forces, the so-called Stokes approximation can be understood as when the viscous and pressure forces dominate the inertial forces absolutely. The Reynolds number allows one to test the applicability of Stokes approximation to fluids. The Stokes approximation holds exactly in the limit as this ratio goes to zero [17–22]. For this reason, Stokes flows are often called low Reynolds number, non-inertial or viscous flows.

2.2.2 Stokes Flows

Taking the limit $Re \rightarrow 0$ in Eq. (2.5) one obtains the non-dimensional steady Stokes equations. In dimensional form, without external body forces, sources or sinks, they read

$$\mu \nabla^2 \mathbf{u} - \nabla p = 0 \quad (2.6)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (2.7)$$

The first equation states the balance of forces in a non-accelerating fluid. The second equation is, as in Eq. (2.2), the conservation of mass for incompressible fluids.

The Stokes equations (2.6) and (2.7) must be combined with boundary conditions appropriate to the physical situation. The so-called stick or no-slip boundaries for rigid walls and at the surfaces of particles are important examples. Consider a surface S moving with local velocity $\mathbf{w}$. It has no-slip boundary condition for the fluid velocity $\mathbf{u}$ if on S one has

$$\mathbf{u}(\mathbf{r}) = \mathbf{w}(\mathbf{r}) \quad \text{for } \mathbf{r} \in S \quad (2.8)$$

There are many other important examples of boundary conditions, such as the boundaries for a free surface [18], but we will focus on the stick boundary conditions, which are sufficient for considering inertial forces, reversibility and the minimum energy dissipation. When considering reversibility especially, one must remember that boundaries can be time dependent. This means that the boundaries move, such as in the classical Taylor–Couette experiment involving a fluid between two rotating cylinders [23]. The Stokes equations (2.6) and (2.7) can also apply to unbounded flow problems by the selection of an appropriate boundary at infinity. For instance, a fluid can be constrained to be at rest at infinity, as in case of particles settling in a quiescent fluid.

Equations (2.6)–(2.7) are linear, so that any linear combination of solutions $(\mathbf{u}_1, p_1)$ and $(\mathbf{u}_2, p_2)$ is also a solution $(\mathbf{u}_1 + \mathbf{u}_2, p_1 + p_2)$. Linearity allows for classes of solutions to be constructed. One example is the case of flow around a rigid sphere, where a complete set of elementary solutions to Eqs. (2.6) can be constructed, as done by Lamb [16]. In his families of elementary solutions, the pressure p is expanded in spherical harmonics and the velocity field $\mathbf{u}$ is written as an infinite series of solid harmonics. This concept is used in the multipole method of solving the Stokes equations for systems of particles moving in fluids [20, 24–30].

2.3 Reversibility of Fluid Flows

Because Stokes equations (2.6)–(2.7) are steady and linear, the motion they predict is reversible in time. Mathematically, it means that the reversibility transformation of any solution, that is, $(\mathbf{u}(\mathbf{x}, t), p(\mathbf{x}, t)) \mapsto -(\mathbf{u}(\mathbf{x}, t), p(\mathbf{x}, t))$, will also give a solution. This can be checked by simple algebraic manipulation of the governing equations. G.I. Taylor explained in his film *Low Reynolds Number Flows* [31] the physical meaning of reversibility—“low Reynolds number flows are reversible when the direction of motion of the boundaries which gave rise to the flow is reversed”. Actually, a reversed fluid flow can result from reversing velocity of the boundary (equal to the fluid velocity at the surfaces of particles or walls) or from reversing directions of the external and the opposite hydrodynamic forces. In the following, we will show how reversibility allows to predict symmetries of fluid flows and motion of particles in fluids.

2.3.1 Examples of Reversibility

One of the most dramatic presentations of reversibility is seen in the film mentioned above [31]. In this experiment, the volume between two transparent cylinders is filled with glycerine. Dyes are injected which form a compact coloured volume into the glycerine to help visualise the flow. The inner cylinder is rotated causing the dyes to stir and apparently mix. The inner cylinder is then rotated in the

opposite direction and one sees the seemingly mixed fluids unstir themselves. This experiment demonstrates the difficulty of mixing low Reynolds number fluids, an important problem for microfluidics.

G.I. Taylor used this experiment to explain the concept of reversibility in the following way: “On reversal of the motion of the boundary, every particle retracts exactly the same path on its return journey as on the outward journey, and at every point its speed is the same fraction of the boundary speed as it was at the same point on its outward journey, so that when the boundary has returned to its original position every particle in the fluid has also done so and the original pattern of dye is reproduced” [32].

A very important consequence of reversibility in biology is that the ordinary swimming motion done by an idealised swimmer with a rigid tail could not produce forward motion in a non-inertial fluid, since any propulsion created by the swimmer when the tail moves left is exactly cancelled when it moves right, as demonstrated in [31]. This is a consequence of the “Scallop theorem” fundamental to the study of the locomotion of microscale organisms [11].

One can use reversibility to derive basic properties of solutions to the Stokes equations without finding the solutions explicitly. Take the case of a rotating, but not translating, sphere immersed in fluid governed by Eqs. (2.6) and (2.7). This situation is illustrated in Fig. 2.1. We might ask how much force such a sphere would feel. Here we mean the force exerted by the fluid on the sphere owing to stick boundary conditions on its surface. This hydrodynamic force needs to be balanced by the opposite external (non-hydrodynamic) force acting on the sphere. Through the use

Fig. 2.1 A solid sphere rotating without translations near a solid wall. Reversibility implies that the sphere does not feel any external force perpendicular to the wall

of superposed reversibility and symmetry transformations, we can discover that in this situation the answer is that the sphere would not feel any hydrodynamic force in the direction perpendicular to the wall [33]. This can be proven by contradiction. Suppose $F_x \neq 0$. Notice that if we put the origin at the centre of the sphere, the system is symmetric for the transformation $y \mapsto -y$. This reflection reverses the rotation of the sphere, but leaves the x -component of the force the same. Now apply the reversibility transformation. The rotation is now reversed back to the original sense. The force vector should have the opposite direction. The result is that the sphere is at the same position, has the same physical rotation, but opposite F_x . This is a contradiction. This argument shows how reversibility and symmetry arguments can be combined to put strong restrictions on Stokes flow [33, 34].

We can also apply reversibility arguments again to the case of a sphere which moves under a constant gravitational force parallel to a solid wall. Applying the time reversal, we now reverse also the direction of the sphere velocity and force. By the same argument above, i.e. by combining the time reversal with the reflection with respect to the plane $y = 0$, there will be no velocity in the direction perpendicular to the wall; the sphere will keep translating parallel to the wall [34]. This reasoning applies to the study of sedimentation of a slowly moving particle of any shape and material symmetric with respect to reflection in the plane $y = 0$ [33, 34]. We have demonstrated that reversibility has observable consequences which do not require elaborate constructions.

2.3.2 *Irreversible Trajectories in Stokes Flow*

Applying reversibility, one must take care that reversibility applies to time and forces. In particular, the paths that particles take need not be reversible in time even though the Stokes equation is reversible in time. As an example, consider the system shown in Fig. 2.2: two spheres of the same radii—one fixed and another one settling from above under gravity. For non-touching spheres, trajectories of the moving sphere centre are symmetric with respect to reflection in the plane $z = 0$. Under the time reversal, the gravitational force is reversed and the sphere centre moves backwards along the same trajectory. However, reversibility of the trajectories is broken when two spheres come so close to each other that their surfaces interact by direct forces, such as van der Waals attraction or mechanical reaction of rough surfaces at the contact [35–39]. The reason is that central direct forces are not symmetric with respect to superposition of the time reversal with reflection in the horizontal plane $z = 0$.

Fig. 2.2 Experimentally observed trajectories of the centre of sphere settling under gravity in a silicon oil towards another fixed sphere of the same radius. Top: reprinted by permission from Ref. [36]. Copyright Kluwer Academic Publisher (2002). Bottom: reprinted with permission from Ref. [37]. Initially, the line of the sphere centres is inclined with respect to gravity. For a large inclination, the surfaces of the spheres are always separated by a fluid, and the trajectories are reversible. However, if the initial inclination is small enough, after some time the surfaces come into contact and the resulting direct forces break the reversibility of the trajectories

2.4 Minimum Energy Dissipation Theorem

We will now give a “variational” view of Stokes flow. A solution to Stokes equations (2.6) and (2.7) is the unique divergence-free vector field that minimises the extensive energy dissipation rate (that is, the energy dissipated by the bulk of the fluid) [20]. In this section, we will state this minimum energy dissipation theorem precisely and sketch a proof. After that, we will apply it to derive “inclusion monotonicity”, a principle about particles moving through Stokes flows.

2.4.1 Statement

Consider a fluid filling a volume V with an impermeable boundary $\partial V = S$. Let $\mathbf{u}$ be the velocity of a Stokes flow defined by Eqs. (2.6) and (2.7). Let $\mathbf{v}$ be a divergence-free vector field describing a flow in V with the same boundary conditions as $\mathbf{u}$. The minimum energy dissipation theorem is

$$\epsilon^{\mathbf{u}} \leq \epsilon^{\mathbf{v}} \quad (2.9)$$

where $\epsilon^{\mathbf{u}}$ is the extensive energy dissipation rate of the Stokes flow and $\epsilon^{\mathbf{v}}$ is the extensive energy dissipation rate of the other flow.

For an excellent discussion of how these relations for the change of internal energy over time are established physically, see section 3.4 of [18]. For now we will simply use the fact that for an incompressible fluid, the intensive energy dissipation rate (i.e. the energy dissipated per unit volume) Φ is

$$\Phi^{\mathbf{u}} = 2\mu \mathbf{e}^{\mathbf{u}} : \mathbf{e}^{\mathbf{u}} \quad (2.10)$$

$$\Phi^{\mathbf{v}} = 2\mu \mathbf{e}^{\mathbf{v}} : \mathbf{e}^{\mathbf{v}} \quad (2.11)$$

where $\mathbf{e}$ is the rate of strain tensor for the Stokes flow $\mathbf{u}$ given component-wise as $e_{ij}^{\mathbf{u}} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ (and similarly for $e_{ij}^{\mathbf{v}}$) and $:$ is the double dot product. Integrating Φ over V gives the extensive energy dissipation rates ϵ , so that

$$\epsilon^{\mathbf{u}} = \int \Phi^{\mathbf{u}} dV \quad (2.12)$$

$$\epsilon^{\mathbf{v}} = \int \Phi^{\mathbf{v}} dV \quad (2.13)$$

Having thus connected the energy dissipation rate to the mechanical properties of the flow, we can now discuss the proof of Eq. (2.9). Because the minimum energy dissipation theorem is proven and discussed in many textbooks, such as [20], we will only give a brief outline. One starts by demonstrating

$$\int (e_{ij}^{\mathbf{v}} - e_{ij}^{\mathbf{u}}) e_{ij}^{\mathbf{u}} dV = 0 \quad (2.14)$$

from Green's theorem, the divergence theorem and Stokes equations (2.6) and (2.7). Then one subtracts Eq. (2.14) from the extensive energy dissipation rate for $\mathbf{v}$ and rearranges

$$2\mu \int e_{ij}^{\mathbf{v}} e_{ij}^{\mathbf{v}} dV = 2\mu \int (e_{ij}^{\mathbf{v}} e_{ij}^{\mathbf{v}} - (e_{ij}^{\mathbf{v}} - e_{ij}^{\mathbf{u}}) e_{ij}^{\mathbf{u}}) dV \quad (2.15)$$

$$= 2\mu \int (e_{ij}^{\mathbf{u}} e_{ij}^{\mathbf{u}} + (e_{ij}^{\mathbf{v}} - e_{ij}^{\mathbf{u}}) e_{ij}^{\mathbf{v}}) dV \quad (2.16)$$

$$= 2\mu \int (e_{ij}^{\mathbf{u}} e_{ij}^{\mathbf{u}} + (e_{ij}^{\mathbf{v}} - e_{ij}^{\mathbf{u}}) e_{ij}^{\mathbf{v}} - (e_{ij}^{\mathbf{v}} - e_{ij}^{\mathbf{u}}) e_{ij}^{\mathbf{u}}) dV \quad (2.17)$$

$$= 2\mu \int (e_{ij}^{\mathbf{u}} e_{ij}^{\mathbf{u}} + (e_{ij}^{\mathbf{v}} - e_{ij}^{\mathbf{u}})^2) dV \quad (2.18)$$

Which shows that $2\mu \int (e_{ij}^{\mathbf{v}} e_{ij}^{\mathbf{v}} - e_{ij}^{\mathbf{u}} e_{ij}^{\mathbf{u}}) dV \geq 0$, which by Eqs. (2.10) and (2.11) is the same as

$$\int (\Phi_{\mathbf{v}} - \Phi_{\mathbf{u}}) dV \geq 0 \quad (2.19)$$

By Eqs. (2.12) and (2.13), one sees that Eq. (2.19) is the same as Eq. (2.9), the minimum energy dissipation theorem.

2.4.2 An Application of the Minimum Energy Dissipation Theorem

One advantage of variational principles such as the minimum energy dissipation theorem is that they can be used to describe the behaviour of general rigid bodies in a Stokes flow. We will give an example through the principle of “inclusion monotonicity”. If one particle is large enough to completely contain another particle, then we can compare the magnitude of the so-called drag force resulting from a Stokes flow. Inclusion monotonicity follows from the minimum energy dissipation theorem, which we will now show in a manner following [20].

Let a rigid particle 1 take up a volume V_1 with surface $\partial V_1 = S_1$ and compare with the flow around rigid particle 2 taking up a volume V_2 with a surface $\partial V_2 = S_2$. They are undergoing the same translational motion with velocity $\mathbf{w}$ without rotation. The fluid is described by Stokes equations (2.6) and (2.7). Further, the particles have no-slip boundary conditions on their surfaces. The forces the fluid flow exerts on these particles are

$$\mathbf{f}_1 = \oint \boldsymbol{\sigma}_1 \cdot \mathbf{n}_1 dS_1 \quad (2.20)$$

$$\mathbf{f}_2 = \oint \boldsymbol{\sigma}_2 \cdot \mathbf{n}_2 dS_2 \quad (2.21)$$

where σ_i is the fluid stress tensor and $\mathbf{n}_i$ is the normal coming out of surface of particle i . The force of the fluid on the particle has the same magnitude but opposite direction.

Inclusion monotonicity principle: If $V_2 \subset V_1$, then $\mathbf{f}_2 \cdot \mathbf{w} \leq \mathbf{f}_1 \cdot \mathbf{w}$ (2.22)

The drag is the component of the fluid force on the particle in the direction of $\mathbf{w}$ [17]. One can see by dividing through by $|\mathbf{w}|$ that inclusion monotonicity relation Eq. (2.22) gives that the magnitude of the drag force on particle 1 is greater than magnitude of the drag force on particle 2. Proof of inclusion monotonicity principle Eq. (2.22) is illustrated in Fig. 2.3 and given below.

Let $\mathbf{u}_1$ be the Stokes flow around the larger particle 1 and $\mathbf{u}_2$ be the Stokes flow around the smaller particle 2. The energy dissipation rate per unit time in the fluid is proportional to the drag [20]

$$\epsilon^{\mathbf{u}_1} = \mathbf{f}_1 \cdot \mathbf{w} \quad (2.23)$$

$$\epsilon^{\mathbf{u}_2} = \mathbf{f}_2 \cdot \mathbf{w} \quad (2.24)$$

Fig. 2.3 Proof of inclusion monotonicity principle, illustrated. Three panels are drawn with particles in grey and fluid in white. In panel 1 and 2 particle 1 and particle 2 displace volumes such that $V_2 \subset V_1$. The particles are moving with the same velocity $\mathbf{w}$ —shown with white tipped arrows—creating fluid velocity fields $\mathbf{u}_1$ and $\mathbf{u}_2$ shown with black tipped arrows. The last panel depicts a non-physical velocity field $\mathbf{v}$ which is equal to $\mathbf{u}_1$ outside of V_1 and $\mathbf{w}$ in $V_1 - V_2$

From the above equations it is easily seen that Eq. (2.22) is equivalent to $\epsilon^{\mathbf{u}_2} \leq \epsilon^{\mathbf{u}_1}$. However because these are Stokes flows for different geometries, the energy dissipation rates cannot be directly compared. Therefore, we will construct a (non-physical) vector field $\mathbf{v}$ in order to compare the energy dissipated by the motion of the two particles. Define $\mathbf{v}$ piecewise to be $\mathbf{u}_1$ outside of V_1 and $\mathbf{v} = \mathbf{w}$, the translational velocity, inside of $V_1 - V_2$. The vector field $\mathbf{v}$ is continuous because of the no-slip boundary condition. Now we will compare the energy dissipation rates of the different vector fields $\mathbf{u}_2$, $\mathbf{u}_1$ and $\mathbf{v}$.

We start by comparing $\mathbf{u}_1$ and $\mathbf{v}$. Because $\mathbf{v}$ is rigid body motion on $V_1 - V_2$, $\mathbf{v}$ does not dissipate any energy there. Outside that set, $\mathbf{v} = \mathbf{u}_1$. Therefore

$$\epsilon^{\mathbf{v}} = \epsilon^{\mathbf{u}_1} \quad (2.25)$$

We now move on to the comparison between $\mathbf{u}_2$ and $\mathbf{v}$. By definition, outside of V_1 , $\mathbf{v} = \mathbf{u}_1$ which is a divergence-less vector field. On $V_1 - V_2$, $\mathbf{v}$ is constant, so it is automatically divergence free there. Therefore $\mathbf{v}$ is a divergence-free vector field defined on the same volume of fluid as $\mathbf{u}_2$. Therefore, by minimum energy dissipation theorem we have that $\mathbf{v}$ cannot dissipate less energy than $\mathbf{u}_2$, i.e.

$$\epsilon^{\mathbf{u}_2} \leq \epsilon^{\mathbf{v}} \quad (2.26)$$

Substituting the formulas for the energy dissipation rates Eqs. (2.23) and (2.24) into the above gives inclusion monotonicity Eq. (2.22).

2.5 Limits of the Stokes Approximation

2.5.1 Example of a System Where the Stokes Approximation Does Not Work

The examination of the validity of the Stokes approximation is very revealing of the logical structure of the features of Stokes flow (negligibility of inertial forces, reversibility and the minimum energy dissipation theorem). The most dramatic setting to consider is the famous “Stokes paradox”. This paradox arises in the uniform Stokes flow past an infinite rigid cylinder. Suppose that such a cylinder is translating through a fluid with constant non-zero velocity $\mathbf{u}_0$ and has “no slip” on its surface. We suppose that very far from the cylinder, the fluid is at rest: $\mathbf{u}(\mathbf{x}) \rightarrow 0$ as $\mathbf{x} \rightarrow \infty$. Unfortunately, there is no solution to Eqs. (2.6) and (2.7) consistent with these boundary conditions [16, 19]. In a more general context, Stokes paradox occurs when a non-trivial two-dimensional solution of the Stokes equations (2.6) and (2.7) has no-slip boundary conditions on an object whose surface is a simple closed curve. The velocity is then necessarily logarithmically unbounded as one gets far from the object [18, 40]. More physically, Stokes paradox occurs because

the energy dissipated by the cylinder does not decline far from the particle—in other words, due to the minimum dissipation principle.

Other Linear Flow Equations

Because the Stokes approximation is not always justified and Navier–Stokes equations (2.1)–(2.2) are mathematically complicated, it is desirable to have other linear equation systems for fluid flow. We will very briefly give two such example systems in which Stokes paradox demonstrably does not occur but are still tractable: the Oseen and Brinkman equations.

We start with the well-known Oseen equations [19]. Let there be some constant background flow $\mathbf{u}_\infty$ imposed on the fluid. As mentioned before, the Navier–Stokes equations give that the inertial force have the nonlinear form $\rho(\mathbf{u} \cdot \nabla)\mathbf{u}$. If the characteristic velocity of the flow is much less than $|\mathbf{u}_\infty|$, then the main component of inertia is the resistance of the fluid flow against the background flow. We can decompose the local flow as $\mathbf{u} = \mathbf{u}_\infty + \mathbf{u}^O$ and call $\mathbf{u}^O$ the Oseen flow. The inertial force has therefore $\rho[(\mathbf{u}_\infty + \mathbf{u}^O) \cdot \nabla](\mathbf{u}^O + \mathbf{u}_\infty) = \rho(\mathbf{u}_\infty \cdot \nabla)\mathbf{u}^O + \rho(\mathbf{u}_\infty \cdot \nabla)\mathbf{u}_\infty + \rho(\mathbf{u}^O \cdot \nabla)\mathbf{u}_\infty + \rho(\mathbf{u}^O \cdot \nabla)\mathbf{u}^O$. Because $\mathbf{u}_\infty$ is constant, the middle terms are zero. Furthermore, we are looking for a linear equation, we assume that $|\mathbf{u}^O| \ll |\mathbf{u}_\infty|$. Therefore, we can neglect the nonlinear term. We use the term $\rho(\mathbf{u}_\infty \cdot \nabla)\mathbf{u}^O$ to incorporate inertial forces into linear equations. The equations resulting from the addition of this term to Eq. (2.6) are termed the Oseen equations [41, 42]. The Oseen equations for a steady, incompressible fluid have the form

$$\begin{aligned} \rho(\mathbf{u}_\infty \cdot \nabla)\mathbf{u}^O &= \mu \nabla^2 \mathbf{u}^O - \nabla p^O \\ \nabla \cdot \mathbf{u}^O &= 0 \end{aligned} \tag{2.27}$$

where p^O is the pressure associated with such a flow.

There are considerations other than inertial forces that one can take into account for fluid motion in systems described by linear equations. For example, fluid flows in porous media can be described by linear equations. The solid skeleton causes an additional hydrodynamic resistance, which in the Brinkman model of porous media is introduced as a new term. This results in the following equations for fluid velocity $\mathbf{u}^B$ and fluid pressure p^B :

$$\begin{aligned} \mu \nabla^2 \mathbf{u}^B - \nabla p^B &= c \mathbf{u}^B \\ \nabla \cdot \mathbf{u}^B &= 0 \end{aligned} \tag{2.28}$$

where c is the ratio of the fluid dynamic viscosity and the permeability of the porous media.

2.5.2 Departures from Reversibility Caused by Inertia

The Stokes approximation—which involves the deliberate neglecting of inertia—cannot be applied to systems in which inertial forces materially contribute to motion. This can be seen in flow visualisation. In symmetric environments, reversibility implies that the flow will also be symmetric [33]. For non-Stokes flows (i.e. $Re \gg 0$), the symmetry in the flow lines breaks down [43]. This departure from reversibility grows with the Reynolds number [44].

Reversible flow was shown in Sect. 2.3.1 to have the interesting property that a spherical particle under an external force parallel to a wall would not experience any lateral motion. In an inertial flow, however, a spherical particle tends to drift away from walls, breaking the reversibility, and causing the “tubular pinch effect” [44], with a different pattern of fluid streamlines.

In the analysis given in Sect. 2.3.1, a sphere rotating in a non-inertial fluid was considered. This leads to a reversible, time symmetric fluid flow [33]. However, a sphere (or a cylinder) which experiences inertial effects while rotating will create an irreversible flow. The inertial forces will cause the cylinder to irreversibly create vortices, which then interact with the rotation of the cylinder in a complex, non-time symmetric way, as shown in Ref. [45].

2.5.3 Accelerating Fluid Example

Even when the Stokes approximation is mathematically coherent, one should think with care how to interpret their results. As an illustrative example, consider a fluid contained within an infinite, impenetrable cylinder with radius R rotating with angular velocity Ω and with no-slip boundary conditions at its surface. The explicit solution of the Stokes equations has the form,

$$\mathbf{u}^S = \Omega r \hat{\boldsymbol{\theta}} \quad (2.29)$$

$$p^S = c \quad (2.30)$$

where c is a constant and $\hat{\boldsymbol{\theta}}$ is the unit vector in the azimuthal direction of the corresponding cylindrical coordinates. The flow velocity is, effectively, rigid body rotation. Pressure is constant in space and therefore there are clearly no centrifugal forces in the radial direction.

Moving on to consider the Navier–Stokes equations, we find that the solution becomes

$$\mathbf{u}^{NS} = \Omega r \hat{\boldsymbol{\theta}} \quad (2.31)$$

$$p^{NS} = \frac{1}{2} \rho \Omega^2 r^2 + c \quad (2.32)$$

Like the Stokes case, the fluid is undergoing rigid body rotation. But now a centrifugal force appears in the form of a pressure gradient in the radial direction. The Stokes solution $(\mathbf{u}^S, p^S)$ has no forces in the radial direction, but in practice we would expect a centrifugal force in the presence of rotation. In the steady Navier–Stokes case, the centrifugal force per unit fluid volume is pressure gradient. Therefore, the centrifugal term p^{NS} is much more realistic than the constant term p^S .

2.6 Conclusions

Summarising, various properties essential to the understanding of Stokes flow, have been discussed, including negligibility of inertial forces, reversibility and the minimum energy dissipation theorem. Illustrative examples related to these properties have been provided: irreversible trajectories in Stokes flow, inertial terms for the fluid flow generated by a rotating cylinder, force on a rotating sphere close to a solid plane wall, Stokes paradox, energy dissipation for particles of different shapes. The meaning and the limits of the Stokes approximation have been discussed in the context of more general equations.

We will conclude with some analysis of the logical relationship between the assumption of the negligibility of inertial forces, the assumption of reversibility and the minimum energy dissipation theorem.

The assumption that a flow minimises the energy dissipation rate entails that the flow satisfies the Stokes equations. This means that minimum energy dissipation implies both reversibility and the negligibility of inertial forces. Stated contrapositively, irreversible flows or flows with inertial forces dissipate more energy than Stokes flows.

Furthermore, reversibility implies the negligibility of inertial forces. This is equivalent to saying that the presence of inertial forces implies irreversibility. Any term proportional to $\rho(\mathbf{u} \cdot \nabla)\mathbf{u}$, the inertial force term in the Navier–Stokes equation, will make a flow irreversible.

However, neither negligibility of inertial forces nor reversibility does not imply the minimum energy dissipation theorem. Like the Stokes equations (2.6) and (2.7), the Brinkman equations (2.28) and (2.29) are reversible and do not contain inertial terms. But one can now simply apply the proof in Sect. 2.4.1 by substituting $\mathbf{u}^B$ for the general solenoidal vector field $\mathbf{v}$ to find that the Brinkman flow dissipates more energy than the Stokes one. This also shows that, counterintuitively, reversibility is not sufficient to achieve the minimum energy dissipation achieved by Stokes flows.

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