Chapter 6
Learning to Look at the World Through Mathematical Spectacles—A Personal Tribute to Realistic Mathematics Education

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To Joop van Dormolen,
Who patiently introduced me to the depth, nuances and scope of Realistic Mathematics Education.

Abstract As professionals of mathematics education, we seldom offer personal accounts of our own learning and development. Although such subjective experiences may be idiosyncratic and hardly generalisable, a brief racconto of what and how one came to know may be useful—firstly, to those from whom we learned (maybe what we learned from them is not what they intended to teach us, and this is worth explicating), secondly, to those whom we teach (for them to know who we are and some of the sources of our learning), and, thirdly, to some colleagues willing to start conversations and to share experiences. This essay subjectively describes aspects of the inspiration generated by the insightful, applicable and effective principles of mathematics instruction that Realistic Mathematics Education has offered to us all, influencing the approaches to teaching and learning and the doing in mathematics education.

Keywords Realistic Mathematics Education · Mathematical gaze · Procedural and conceptual · Modelling

6.1 At the Beginning It Was Symbol Crunching, but with a Bit of Spice

For as long as I can remember, I have enjoyed mathematics. However, the mathematics I enjoyed so much was the only one I was then offered: highly procedural and

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rule-oriented, especially in Grade 8–9 algebra. I was happy to 'understand' the rules of the game, to be able to eschew the most common calculation mistakes, to play with complicated algebraic expressions and to enjoy the gratification provided by a correct result or the checking and redoing when it was incorrect. Such enjoyment was even greater when the procedural was spiced with even a minimal degree of exploration. For example, I remember myself liking very much the exercises devoted to the rationalisation of denominators (i.e., eliminating irrational terms from the denominator without altering the value of the fraction), both with numbers only and with algebraic expressions. One can hardly say that producing an expression to multiply both numerator and denominator such that the denominator becomes a rational term or a number requires much creativity. Nevertheless, I liked the 'freedom' to search for and to create an appropriate expression (especially when based on the rule $(a - b)(a + b) = a^2 - b^2$). I saw these tasks as requiring some ingenuity and thus they were gratifying. The challenge and the satisfaction of playing with these kinds of tasks led me to try my hand at inventing rules beyond those taught and allowed. I will never forget my elation when, in eighth grade, I realised that $\frac{a+a}{b+b} = \frac{a}{b}$, I was able to explain why and then I proposed the following 'simplification rule' $\frac{\cancel{a}\cancel{+}a}{\cancel{b}\cancel{+}b}$. I became even more excited when I noticed that this is true for any number of a's and b's, provided their number is the same. I showed this to my mother (who at the time was also the mathematics teacher of the class) and she smiled; "nice," she said. Later on in life, I learned that many teachers use this 'nice' reaction either for lack of a better one or as a harmless euphemism to avoid discouragement; for other teachers such a 'nice' reaction was perceived by many of my fellow students as worse than a scolding.

I relish these memories, as do many of us with pleasant learning events from childhood. When I was 18, I started to teach. Neither my mathematics nor my didactics were solid, to say the least, but I firmly believed that if I knew something (or at least I thought I did) and I liked it, it was enough to teach it and to teach it well. I had a sense of the satisfaction that students (or at least some of them who were like me) may experience when they solve procedural exercises. I became very skilled at producing loads of such exercises on the spot, and sometimes I even tested the idea that these procedural repetitions may yield interesting 'discoveries'. I remember vividly that while preparing for a lesson, I was amazed to 'discover' (possibly rediscover since I must have seen this in high school) that the results of $a - b$ and $b - a$ were the same number (in absolute value) with opposite signs. I was able to justify this 'insight' formally without too much trouble, but I think this was one of the first opportunities in which the algebraic game-playing did not provide me a satisfactory answer as to the question of why that spoke to my inner sense of intuition. By playing with the number line representation, this discomfort was somehow dispelled. Unfortunately, at the time, as a young teacher I had no mentors who could legitimise the asking of questions, the search for answers and the attempts to reconcile between formal arguments and the inner (highly subjective) conviction about their validity. Also, at the time I disregarded the fact that game-playing which occasionally yields insights for some, may have adverse effects for many other students, and I ignored (double entendre intended) the many other rich facets of mathematics.

With hindsight and many (!) years later, these early mathematical experiences lead me to reflect on three non-trivial issues related to the learning and teaching of rules and procedures in mathematics. Firstly, whereas we often hear (and agree with the claim) that procedural tasks are boring and turn so many students off, and also that procedures are not necessarily at the core of the significant mathematics one wants them to learn, still there are students out there who may find it enjoyable to undertake procedural tasks. As a teacher, I should have been very aware of both types of students. Secondly, there is a widespread tendency to identify the performing of procedures with ritualised mathematics (in the sense of Sfard, 2016), namely tasks that are undertaken by students for social reasons ("we have to", "to please the teacher"), mostly by imitation and usually scaffolded by teachers who point out mistakes and correct them. My own brief stories above can be considered as a counterexample. Creating an expression, inventing a new formula or noticing a new result were undertaken by me to pose new challenges to myself and I was not always scaffolded by others. This insight led me to observe students and teachers over many years, to collect instances of and to elaborate the notion of symbol sense (Arcavi, 1994, 2005a). This was also the main motivation to join my colleague Alex Friedlander, an amazingly creative task designer, in the non-trivial and enjoyable activity of producing tasks to address rules and procedures, which nudge students to de-ritualise their practices and turn them into more explorative ones (Friedlander & Arcavi, 2017). Thirdly, after encountering Realistic Mathematics Education (RME), I was pleased to realise the scope of the main term in its name:

> the adjective 'realistic' is definitely in agreement with how the teaching and learning of mathematics is seen within RME, but on the other hand this term is also confusing. In Dutch, the verb 'zich realiseren' means 'to imagine'. In other words, the term 'realistic' refers more to the intention that students should be offered problem situations which they can imagine … than that it refers to the 'realness' or authenticity of problems. However, the latter does not mean that the connection to real life is not important. It only implies that the contexts are not necessarily restricted to real-world situations. The fantasy world of fairy tales and *even the formal world of mathematics can be very suitable contexts for problems*, as long as they are 'real' in the students' minds. (Van den Heuvel-Panhuizen, 2003, p. 9–10; emphasis added)

6.2 Starting to Look at the World with Mathematical Spectacles

As mentioned above, during my high school years, most of the mathematics I encountered was oriented towards solving 'pure' mathematical exercises by means of formal procedures. I do not remember many occasions of using mathematics in high school in order to solve a problem from outside mathematics, except perhaps for a few instances of problems in kinematics, but that was in the physics class. To my shame, I heard the term 'mathematical model' for the first time during my university studies. At university, and in many cases supported by the way the mathematics was taught, I still retained my fondness for handling symbols and solving procedural exercises, sometimes pushing them beyond what I was strictly required to do. However, in parallel I started to acknowledge and appreciate the many opportunities to invoke mathematics, especially those arising in everyday life, as demonstrated by the following story (see also Arcavi, 2002).

When I was a first-year undergraduate student in Argentina, I collected bus tickets with palindromes for a friend. The bus tickets bore five-digit numbers (see Fig. 6.1). At that time, it occurred to me to think about the odds of getting a palindrome, and for that I needed to know how many different five-digit palindromes there are. Had this problem been assigned in class, the bus and the tickets would have been omitted as irrelevant frills (if provided at all) and the solution would have looked something like this: in a palindrome, the units, the tens, and the hundreds digits can vary freely, the thousands are the same as the tens, and the ten thousands are the same as the units. There are ten possibilities for each of the varying digits (if numbers like 00100 are considered as a five-digit number), therefore there are 1000 palindromes. My solution during my bus ride was very different from this one, precisely because of the context. In the bus, the tickets were torn from a roll with consecutive numbers. The device from which the bus driver tore the numbers is shown in Fig. 6.1.

It was precisely the sequential appearance of consecutive numbers that led me in producing a solution. If I got, say, 04369, I knew that I had missed a palindrome (04340) by 29 tickets, and that the next would appear only when the 3 changes into 4 (04440), which is 71 tickets away. Similarly, if I got 34221, the previous and next palindromes would be 34143 and 34243 respectively. Thus, I concluded that the palindrome density should be 1 per 100, and since there are 100,000 five digit numbers there must be 1000 palindromes in all. However, I was uncomfortable with this density argument. It took me a while to figure out the source of my doubts. Consider 19991, the next palindrome is 20,002, definitely less than 100 numbers away, or 10901 and its next palindrome 11011 which are more than 100 numbers

Fig. 6.1 Bus tickets (used in Argentina in the 1970s) (Photo by A. Arcavi)

apart! But in analysing the problem further, I found that this does not at all affect the density property; there still is one palindrome for every 100 numbers (namely one palindrome for each change in the middle), although they are not always evenly distributed.

This problem and my proposed solution were a revelation for me. Firstly, it was me who posed a mathematical problem out of the mathematics class and this problem arose from a 'real-world' personal situation: the chances of getting a palindrome. This was one of the first times I experienced that the world out there can be mathematically poor or mathematically rich depending on the ways one looks at it and the questions it may inspire us to pose and solve (I develop this point further in Sect. 6.4). Much later, I also realised that this story has yet another important moral. The context of a problem is not just a mere excuse or a frill to engage students in doing mathematics, the problem can arise from a genuine situation awaiting to be solved. The particular features of a context can also inspire and lead to a solution approach which may be rather different than the method to solve the same problem when presented in 'pure' mathematical terms. Moreover, a context-oriented solution approach can uncover a characteristic of the solution (the irregular distribution of the palindromes although their density is constant). This characteristic of five-digit palindromes emerged as an incidental by-product of a solution method that relied on the contextual features of the problem and which is not at all obvious from the combinatorial solution.

Thus, in my university years, I began to enlarge my mathematical horizons and the types of mathematics that I might encounter, learn and teach.

6.3 Meeting RME

Towards the end of the 1980s and the beginning of the 1990s, I was introduced to RME. At the time, I started my academic career after a post-doctoral fellowship in mathematics education, and my experience with 'applied mathematics' (in curriculum development, teacher education, research on learning) was much richer than in my high school and early university years. Yet, even though procedures and rules were not the only focus of my professional interests (Arcavi, 1994, 2005a), my focus was still on pure mathematics. My acquaintance with Joop van Dormolen started with occasional meetings, which intensified through our collaboration within the International Group for the Psychology of Mathematics Education (PME), when I became Treasurer and Joop was Executive Secretary. Our collaboration grew far beyond mere administrative issues and I started to learn from him about RME. Learning 'live' from one of the persons so deeply involved with all the aspects of RME (curriculum design, teacher development and policy making) was a real treat. Reading and learning from texts is essential to get acquainted with a worldview, but to have an expert nearby to whom one can address questions and discuss answers leads to a faster and deeper learning. These exchanges led us to embark in the joint adventure of co-authoring two volumes of an elementary geometry book which we entitled *Seeing and Doing Geometry* (in Hebrew) (Halevy, Bouhadana, Van Dormolen, &

Arcavi, 1997; Bouhadana, Van Dormolen, & Arcavi, 2000) with its corresponding teacher's guide. This was for me the ultimate experience to learn from within and to enact the RME worldview.

Since I became more knowledgeable about mathematical modelling, and I was better acquainted with its potential to motivate students by appreciating the utility of mathematics, what was there for me to learn from RME? Well, a lot! Mathematical modelling, as I knew it from my own studies and from most school curricula I worked with, presupposed knowledge of pure mathematics which must be properly invoked and applied in order to make a model of a given real situation, solve it mathematically and reinterpret the results in terms of the situation being modelled. This implies that knowing pure mathematics comes before its applications, which seemed to me a natural chronology, and thus it did not occur to me to contest it. RME, as I understood it, inverted the order: the real world and intriguing situations can and should be a springboard to mathematise, first 'horizontally' and then 'vertically' (Treffers & Goffree, 1985; Freudenthal, 1991; Hershkowitz, Parzysz, & Van Dormolen, 1996). In other words, the foundational mathematics education tenet of RME was "to let that rich context of reality serve as a source for learning mathematics." (Treffers, 1993, p. 89)

This key idea was a real eye-opener, because it builds on students' knowledge from outside mathematics as a main resource and it relies on their common sense and their capacity to harness ad hoc intuitive and non-formal strategies as the main stepping stones upon which to build further. Even before Joop and I co-authored the geometry book, a good exercise for me was to co-author with him an article for teachers applying this insightful principle to the nature of two geometrical concepts: circumference and circle. What we did was to gather various real-life appearances that can be a resource for defining these concepts mathematically (Van Dormolen & Arcavi, 2000, 2001). RME proposes such an approach for teaching most topics in mathematics to both the less mathematically-oriented students as well as to the more advanced.

RME had extraordinary achievements both within the Netherlands and abroad. It was founded and directed for many years by Hans Freudenthal, a leading mathematician who dedicated much of his work to mathematics education, its possibilities, as well as its enormous challenges. Freudenthal took special care in understanding the characteristics of the learners, their potential and their difficulties. Not very often do mathematicians engage as deeply with the complexities of mathematics education as Freudenthal did, and thus their contributions tend to remain local or transient. RME under Freudenthal and his many distinguished collaborators and followers covered all levels of school mathematics in a gigantic effort to design, field test and re-design learning and teaching materials taking 'realistic' situations as departure points. RME also developed a systemic implementation that involved a whole country and influenced, directly or indirectly, many curriculum development projects all over the world.

RME's perspective also influenced the world-wide discussions around mathematical literacy and it even inspired the design of items for the PISA examination. The PISA examination can be contentious regarding the negative effects of the test preparation spree it provoked in many countries and the controversial claims regarding correlations between the examination grades and the economic growth of a country (e.g., *The Guardian*, 2014). However, in many cases, due to the prominence and visibility of the examination, it encouraged countries to rethink and to revise their curricula, since it was realised that the kinds of knowledge required for the examination items (mostly inspired by RME) were not appropriately supported or emphasised at school. There are only a few other mathematics education projects that have so wide a scope and that have so strongly influenced mathematics education in the world. There is much to learn from its principles, its implementation and hopefully its lasting effects, even if one does not fully adopt RME.

6.4 Developing a 'Mathematical Gaze'—From Instructional Design to a Learning Goal

The design principles of RME and the heuristics to enact them have been described at length in many sources (e.g., De Lange, 2015), and there is much to be learned from them. In this section, I would like to briefly focus on just one of these heuristics that had an influence on me and which can be described as follows: "I first look for 'images' which can stimulate associative thinking, an approach that has led to many new and surprising discoveries" (De Lange, 2015, pp. 290–291). In other words, looking around for visually salient cues can provide inspiring raw materials from which tasks and problems can be designed in order to lead students to horizontal and vertical mathematisation. This heuristic implies a highly developed eye capable of a 'mathematical gaze'. In my view, such a mathematical gaze would include:

> […] the predisposition, ability and trust in the usefulness of seeing the non-evident mathematics behind many daily life situations. It also would mean meaningfully imposing the mathematics on these situations by creating and posing problems and questions to which only mathematics can provide an in-depth answer. (Arcavi, 2004, pp. 234–235)

It only takes a quick browsing of the materials of RME to realise that their instructional designers developed a mathematical gaze in amazingly rich ways. It occurred to me that the development of such a gaze could shift its function from a resource for instructional design to a learning goal for students. I witnessed this with myself in the bus story that having a goal in mind (collecting tickets with palindromes on them) may induce the self-posing of some mathematical questions worth pursuing. Why not introduce students to such practices already in elementary school? Consider, for example, the following images and the ensuing questions which can be the object of a classroom discussion (Arcavi, 2016).

Unlike the problem posed about palindromes on the bus tickets, in this case pausing and reflecting on what is the role of the numbers requires more than just pursuing a goal. It requires a certain habit of mind to look at what we take so much for granted and start to ask questions. This is not a trivial switch in the attention (or the lack of it) that we pay to our surroundings, as acknowledged in the following:

> Sources of insight can be clogged by automatisms… the question of how and why is not asked any more, cannot be asked any more, and is not even understood any more as a meaningful and relevant question. (Freudenthal, 1983, p. 469)

A number on a license plate or a number indicating a certain building on a street are clearly a way of identification and thus each object is assigned one and only one number. But is this not the basic principle underlying counting? In what sense can we switch from looking at the numbers in a license plate as identifiers, to using them for counting? One way is to realise that one can place an upper bound on the number of cars using the same type of license plates (within a city or a whole country). In the first case, the picture on the left in Fig. 6.2, there are at most ten million cars with these types of license plates. This information was implicit in the license plate number, and a 'mathematical gaze' helped to unfold it. In the second case, we can even ponder about the reason to use letters as well as digits. This may lead to discussions about elementary combinatorics (which we do not need when we use numbers), for example, the English alphabet consists of 26 letters, thus in the position for a letter there are 26 possibilities, and license plates of the type of the picture on the right in Fig. 6.2 can thus accommodate $26 \times 26 \times 26 \times 10 \times 10 \times 10$ cars (much more than with just six digits). This kind of discussion can be scaffolded and supported in upper classes of elementary school and they may even be of interest to junior high school mathematics classes, or teachers in a professional development course, especially if one relates to CCC as a number in base 26.

When we look at numbers that identify addresses in different cities in the world, we may notice that in some of them the numbers are smaller (1–2 digits) than in others (4–5 digits). What could be the reason for that? The urbanisation of many cities adopts the form of a square grid, in which streets are either parallel or perpendicular and they surround 'blocks'. In many cities in Argentina, for example, these blocks are of 100 m by 100 m. Thus a number such as 1363 (see Fig. 6.3) indicates that the address identified is located at 63 m (the ten and the unit digits) from the beginning of the fourteenth block (the thousand and the hundred digits). With such a convention, it is almost impossible to have two neighbouring addresses with consecutive numbers. In other countries, the addresses just use consecutive numbers regardless of the distances and the blocks, and this indeed results in addresses with much smaller numbers. In

Fig. 6.2 What do these numbers indicate? (Photos by A. Arcavi)

Fig. 6.3 What do these numbers indicate? (Photos by A. Arcavi)

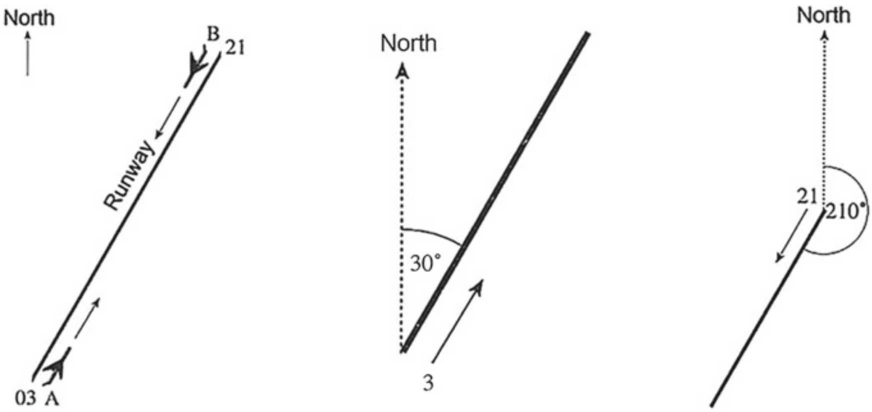

Fig. 6.4 Numbering convention for airport runways

this case, asking a question may lead to looking for information outside mathematics, but which is reflected in the way mathematics is used.

Numbers are everywhere and the unpacking of the question "What does a number indicate?" may yield to practical as well as to mathematical insights. Consider, for example, the numbers at both ends of an airport runway anywhere in the world which express the clockwise measurements of the angle between the runway and the North. Figure 6.4 shows a runway and its two ends which clarify the numbering convention (using two digits only).

This can be the source for a few lessons on the concept of angle, its uses, its measures and more. Consider the road sign in Fig. 6.5.

This sign indicates the steepness of the upcoming part of the road, and can be used to discuss the mathematical concept of slope and the many ways to measure it, including the uses of percentages.

Pairs of numbers seen in elevators around the world (indicating the maximum load and the maximum number of persons allowed) may lead to discussing mathematical concepts such as proportions and averages, and one may even end up suggesting

Fig. 6.5 Road sign indicating the steepness of a road (Photo by A. Arcavi)

Fig. 6.6 Elevator signs in different countries (Photos by A. Arcavi)

non-mathematical issues like comparing the assumed physical characteristics for the population served by these elevators (Fig. 6.6).

The world out there can provide many opportunities to observe, question, calculate, answer and make conclusions about situations to which many people are blind. Such opportunities may include situations like the following two (Arcavi, 2005b) (Figs. 6.7 and 6.8).

These are just some possible examples which can be used for engaging children in both problem solving around real-life situations and mostly for supporting the development of a mathematical gaze which implies:

Fig. 6.7 How long would it take to you to run the distance that you can walk in 10 min? (Photo by A. Arcavi)

Fig. 6.8 Why are the fish arranged in a circle? (Photo by A. Arcavi)

- A fresh look at situations which are usually taken for granted
- Identifying the 'mathematisable' within situations
- Pondering and asking questions about these situations without expecting them to be given by the teacher or the textbook
- Relating natural language to formal and symbolic language
- Proposing ideas, solutions and conclusions
- Posing problems.

6.5 Coda

By providing an inclusive, educationally sound definition of 'realistic', RME allowed me to look with new eyes at my initial fondness for the procedural and to consider it as an integral part of what can be mathematically valuable. Moreover, it legitimised my perception that the procedural can have a respectable place in what can be 'realistic' for many students. With hindsight, and under the conviction that the procedural and the conceptual (in the sense of Hiebert, 1986) should be deeply interwoven, these were the roots of my work on sense making both with symbols and with images (Arcavi, 1994, 2003).

By re-positioning modelling not only as a way to use mathematics to solve applied problems, but also as a way of using authentic real-life situations as departure points for launching the teaching and learning of mathematical concepts and strategies, RME convinced me of the immense potential of these ideas. Moreover, it inspired me to think that the development of a 'mathematical gaze' is not only a powerful tool for task design, but also an implementable goal even with elementary school students.

By placing the idea of mathematical literacy on the public agenda and by influencing the design of many items in the PISA examination, RME provided me with broader spectacles through which to look at school curricula and caused me to realise what may be missing in them.

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References

Arcavi, A. (1994). Symbol sense: Informal sense-making in formal mathematics. *For the Learning of Mathematics, 14*(3), 24–35.

Arcavi, A. (2002). The everyday and the academic in mathematics. M. Brenner & J. Moschkovich (Eds.), *Everyday and academic mathematics in the classroom. A Monograph of the Journal for Research in Mathematics Education*. (pp. 12–29). Reston, VA: National Council of Teachers of Mathematics.

Arcavi, A. (2003). The role of visual representations in the teaching and learning of mathematics. *Educational Studies in Mathematics, 52*(3), 215–241.

Arcavi, A. (2004). Education of mathematics teachers. In R. Strässer, G. Brandell, B. Grevholm, & O. Helenius (Eds.), *Educating for the future. Proceedings of an International Symposium on Mathematics Teacher Education* (pp. 227–234). Göteborg, Sweden: The Royal Swedish Academy of Sciences.

Arcavi, A. (2005a). Developing and using symbol sense in mathematics. *For the Learning of Mathematics, 25*(2), 50–55.

Arcavi, A. (2005b). Some Japanese mathematical landscapes. http://math-info.criced.tsukuba.ac.jp/museum/arcavi/arcavi_english/. Accessed March 8, 2016.

Arcavi, A. (2016). Miradas matemáticas y pensamiento numérico. *Avances de Investigación en Educación Matemática, 9,* 11–19.

Bouhadana, R., Van Dormolen, J., & Arcavi, A. (2000). *Seeing and doing geometry II (in Hebrew)*. Rehovot, Israel: The Weizmann Institute of Science.

De Lange, J. (2015). There is, probably, no need for this presentation. In A. Watson & M. Ohtani (Eds.), *Task design in mathematics education. ICMI study 22* (pp. 287–308). New York, NY: Springer.

ECNAIS. (2014). OECD and Pisa tests are damaging education worldwide—academics. Open letter to Dr. Andreas Schleicher. *The Guardian*. http://www.theguardian.com/education/2014/may/06/oecd-pisa-tests-damaging-education-academics. Accessed March 8, 2016.

Freudenthal, H. (1983). *The didactical phenomenology of mathematical structures*. Dordrecht, the Netherlands: Reidel.

Freudenthal, H. (1991). *Revisiting mathematics education. China lectures*. Dordrecht, the Netherlands: Kluwer Academic Publishers.

Friedlander, A., & Arcavi, A. (2017). *Tasks & competencies in the teaching and learning of algebra*. Reston, VA: National Council of Teachers of Mathematics.

Halevy, T., Bouhadana, R., Van Dormolen, J., & Arcavi, A. (1997). *Seeing and doing geometry (in Hebrew)*. Rehovot, Israel: The Weizmann Institute of Science.

Hershkowitz, R., Parzysz, B., & Van Dormolen, J. (1996). Shape and space. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 161–204). Dordrecht, the Netherlands: Kluwer Academic Publishers.

Hiebert, J. (Ed.). (1986). *Conceptual and procedural knowledge: The case of mathematics*. Hillsdale, NJ: Lawrence Erlbaum.

Sfard, A. (2016). When words get in your eyes: On challenges of investigating mathematics-in-teaching and on the importance of paying attention to words. https://www.msri.org/workshops/793/schedules/20595. Accessed March 8, 2016.

Treffers, A. (1993). Wiskobas and Freudenthal—Realistic mathematics education. *Educational Studies in Mathematics, 25*(1), 89–108.

Treffers, A., & Goffree, F. (1985). Rational analysis of realistic mathematics education. In L. Streefland (Ed.), *Proceedings of the 9th International Conference for the Psychology of Mathematics Education* (Vol. II, pp. 97–123). Utrecht, the Netherlands: OW&OC.

Van den Heuvel-Panhuizen, M. (2003). The didactical use of models in Realistic Mathematics Education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics, 54*(1), 9–35.

Van Dormolen, J., & Arcavi, A. (2000). What is a circle? *Mathematics in School, 29*(5), 15–19.

Van Dormolen, J., & Arcavi, A. (2001). Wat is een cirkel? *Euclides, 76*(5), 182–186.