Chapter 11
The Enrichment of Belgian Secondary School Mathematics with Elements of the Dutch Model of Realistic Mathematics Education Since the 1980s

Dirk De Bock, Johan Deprez and Dirk Janssens

Abstract In search for alternatives for the failed New Math movement of the 1960s and 1970s, Belgian mathematics educators looked with great interest to the Dutch model of Realistic Mathematics Education (RME), developed by Hans Freudenthal (1905–1990) and his team at the University of Utrecht. In this chapter, we primarily focus on how, from the mid 1980s until the mid 1990s, valuable elements of that model were integrated in Belgian secondary school mathematics. At that time, the influence of Dutch mathematics education on Belgian curricula was quite substantial, but some form of collaboration between the communities of mathematics teachers in both countries already existed since the early 1950s. However, from the 1950s until the 1970s, school mathematics in both countries evolved largely independent of each other. In Belgium, the structural New Math approach, with Georges Papy (1920–2011) as the main figurehead, became dominant in school mathematics, while the modernisation of school mathematics in the Netherlands was strongly inspired by Freudenthal's RME model emphasising the role of applications and modelling.

Keywords Georges Papy · Hans Freudenthal · Modelling and applications · New math · Realistic mathematics education · Uitwiskeling

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11.1 Papy and Freudenthal: Opposite Views on Mathematics Education in Neighbouring Countries

With the creation of the International Commission for the Study and Improvement of Mathematics Teaching (CIEAEM) in the early 1950s, international, or at least Western European, collaboration in mathematics education was launched. According to Caleb Gattegno (1911–1988), the founder of the CIEAEM, national organisations of mathematics teachers had to take a leading role in international exchange. Gattegno stimulated the creation of such organisations and set a good example by creating the Association for Teaching Aids in Mathematics, now the Association of Teachers of Mathematics (ATM) in the United Kingdom in 1952. Following Gattegno's call, the Société Belge de Professeurs de Mathématiques (SBPM)/Belgische Vereniging van Wiskundeleraren (BVW) was founded in 1953 and Willy Servais (1913–1979), one of the main CIEAEM personalities of the time, became its first president (Miewis, 2003; Vanpaemel, De Bock, & Verschaffel, 2012). The SBPM/BVW brought together a few hundred mathematics teachers from both linguistic communities (Dutch and French) and from all school networks (state schools and Catholic schools). It started its own (also bilingual) professional journal *Mathematica & Paedagogia* (*M&P*) and in his first editorial, Servais held a strong plea for international openness and exchange. He wrote: "Mathematics as a truly universal language has, by its nature, an international vocation; we will open the columns of our journal to colleagues in other countries" (Servais, 1953, p. 4).[1] Servais' plea was received favourably by the international mathematics and mathematics education communities of the time. Several famous authors, most of them members of the CIEAEM, submitted contributions to *M&P*. The Belgian journal rapidly became a forum for national and international exchange in mathematics education (De Bock & Vanpaemel, 2015b).

This spirit of internationalisation was also present in the articles of the SBPM/BVW, in which the establishment of relationships with foreign associations of mathematics teachers and other international organisations sharing similar goals, was included as an important objective. In *M&P* 1, Servais presented the international network of the SBPM/BVW, including professional organisations of mathematics teachers in France, United Kingdom, Switzerland, Italy and Germany. Later, as communicated in *M&P* 5, this network was expanded through collaboration with the National Council of Teachers of Mathematics in the United States and with LIWENAGEL and WIMECOS, two associations of mathematics teachers in the Netherlands at the time (see, e.g., Maassen, 2000). The collaboration with the Dutch was realised in three ways: the associations exchanged and reviewed each other's journals and entrance exams (for future students of civil and military engineering), board meetings were mutually attended and mathematics educators of both countries gave lectures at conferences of the fellow associations. We do not claim there was a real interaction or mutual influence between Belgian and Dutch mathematics education at that time, but at the professional organisations' level, both communities

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[1]In this chapter, all translations into English are by the authors.

of mathematics teachers were regularly informed about what happened in the other country.

During the 1950s, several leading mathematics educators from the Netherlands (Freudenthal, Van Hiele, Vredenduin, …) contributed to *M&P*. Of particular interest was a contribution by Luke N. H. Bunt (1905–1984) who wrote about an interesting development in the Netherlands. Bunt was invited at the SBPM/BVW conference of 1959 to report about an introductory course on probability and statistics he had developed with a team of six mathematics teachers and with which he had experimented in the alpha streams of Dutch secondary schools (Bunt, 1959). Statistics was a blind spot in Belgian mathematics education at that time and Bunt's main target group, future students in economics, psychology and other social sciences, was mostly neglected. Bunt presented a pragmatic approach, which was also in contrast with the more systematic and rigorous approaches that were generally applied in Belgium. He deliberately started with provisional definitions, definitions that are incomplete from a scientific point of view. For example, he first defined the probability of an event as the ratio between the number of favourable and the total number of outcomes in the case of equally likely events. Based on that definition, he proved the main calculation rules for probabilities. Later on in his course, when the need arose to cover more situations, Bunt presented a new definition, based on the limit of relative frequencies and without further explanation, he stated that "for probabilities based on this new definition, the previously proven calculation rules remain valid" (Bunt, 1959, p. 38). As a consequence of his pragmatism, Bunt was able to arrive at the basic ideas of hypotheses testing in a limited number of lessons.

By the end of the 1950s and in the 1960s, Belgian school mathematics was gripped by New Math or 'modern mathematics', a structural approach to mathematics teaching that was officially launched at the Royaumont Seminar (1959) and then spread worldwide (De Bock & Vanpaemel, 2015a). One of the main objectives was narrowing the gap between school mathematics and mathematics as a scientific discipline. New Math not only led to new mathematical content—sets, relations, logic, mathematical structures (groups, rings, …), linear algebra and topology—but also to a modernisation of teaching aids, for example, the use of Venn diagrams, arrow-graphs and colour conventions. Proper notations and symbols, the use of the right jargon and theory development received a lot of attention. Barriers between mathematical subdomains (algebra, geometry, trigonometry, …) were largely eliminated and geometry education was redirected towards transformation and vector geometry. The main architect and uncontested leader of Belgian New Math was Georges Papy, professor of mathematics at the Université libre de Bruxelles, who was, at that time, also influential at the international level, for example, as president of the CIEAEM in the 1960s (Bernet & Jaquet, 1998). After some years of experimentation (Fig. 11.1), coordinated by the Centre Belge de Pédagogie de la Mathématique/Belgisch Centrum voor Methodiek van de Wiskunde, from 1968 on, New Math became compulsory in the first year of all secondary schools in Belgium (and from then on gradually in the subsequent years). It was one of the most drastic educational reforms that Belgium had ever seen. A few years later, New Math was also introduced at the primary level. For

Fig. 11.1 Papy and one of
his students experimenting
with New Math, 1960
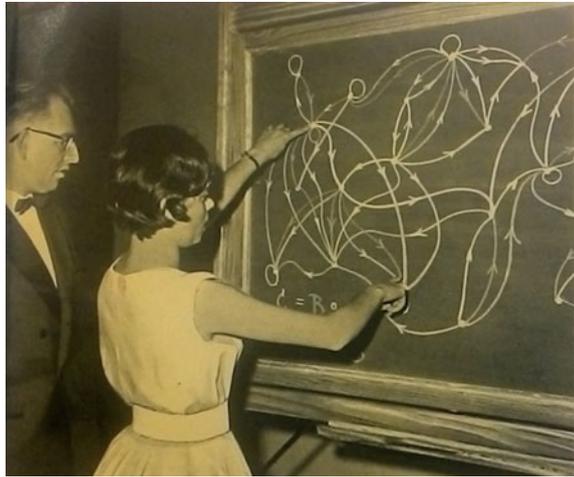
about 20 years New Math was the dominant paradigm for the teaching and learning
of mathematics in Belgium.

Also in the Netherlands, some mathematics educators fell for the charms of
New Math, among them the logician and Royaumont participant Piet Vredenduin
(1909–1996), who positively reported about Papy's experiments in *Euclides*, the jour-
nal of the Dutch associations of mathematics teachers (Vredenduin, 1967). But from
the outset, New Math was also strongly criticised. Freudenthal summarised his cri-
tique in two words: "anti-didactic inversion" (Freudenthal, 1973, p. 103), expressing
that an end product of mathematical activity, the most recently composed structure
of mathematics, is taken as a starting point for mathematics teaching (Fig. 11.2).
Although Freudenthal could not prevent that also in the Netherlands, a New Math
inspired curriculum for the secondary level was introduced in 1968, the implementa-
tion was less radical and New Math only lasted for a few years. New Math was never
introduced in Dutch primary schools. In 1971, the IOWO (Institute for the Develop-
ment of Mathematics Education) was founded with Freudenthal as its first director.
With a staff of 37 people, Freudenthal put into practice his ideas about mathemat-
ics education which resulted in Realistic Mathematics Education (RME), the Dutch
answer to New Math and to traditional, mechanistic approaches, both for the primary
and secondary level (La Bastide-van Gemert, 2015). The features of RME include
the use of rich contexts and realistic situations, that is, problem situations which
students can imagine, in order to develop mathematics, the use of students' own pro-
ductions as well as researched activities encouraging students to move from informal
to formal representations, less emphasis on algorithms and more on sense-making
and the use of guided reinvention.

Fig. 11.2 Papy and Freudenthal (caricatures by Léon Jesmanowicz, 1971)

11.2 Critique on New Math in Belgium and Search for Alternatives

In 1974, the SBPM/BVW was restructured on a linguistic basis into the SBPMef (Société Belge des Professeurs de Mathématiques d'expression française) and the VVWL (Vlaamse Vereniging Wiskundeleraars), with respective journals *Mathématique et Pédagogie* and *Wiskunde en Onderwijs*. The SBPMef did not continue some form of cooperation with the community of Dutch mathematics teachers, but instead started some networking with the French and later also with French speaking mathematics teachers in Switzerland. Hence, at least at the level of professional organisations, cooperation and even exchange of information between the French speaking Belgian community and the Dutch stopped. For the purpose of this chapter, we will further focus on the evolutions in Flanders.

Although at that time and until the late 1980s, the VVWL had become a fortress of New Math proponents (including, e.g., De Bruyn, De Munter, Holvoet, Laforce, Verhulst, Vermandel, Warrinnier), cooperation with the Dutch continued formally and Vredenduin became the main contact (Holvoet, 1996). Except for a number of articles by Vredenduin himself, mostly about logical issues, *Wiskunde en Onderwijs* at that time rarely published articles by Dutch authors or contributions informing its readership about ongoing evolutions in the Netherlands. The work of Freudenthal and his team was ignored by the official association of mathematics teachers in Flanders.

The stable position of New Math in Belgium during the late 1960s and 1970s and the absence of critique in official fora, does not imply that the whole mathematics education community in Belgium was unconditionally in favour of Papy's approach. Instead, Papy's method and its implementation at school had divided this community

into two camps, the so-called 'Papy'ists' and 'anti-Papy'ists' (Colot, 1969), but that latter group was not in power at that time and remained relatively silent in public fora. This silence was broken on March 11, 1980, when Albert Pirard and Paul Godfrind, professors at respectively the Université de Liège and the École Royale Militaire, published an opinion article in *La Libre Belgique*, a main newspaper of the French speaking Belgian community. The title of the article, "The Disasters of Modern Mathematics", set the tone. Among other things, the authors stated: "We now see in the entrance exam [for future students of engineering, *added by the authors*] candidates who have almost never practiced geometry" (Pirard & Godfrind, 1980, p. 15). They heckled the tough and abstract approach to geometry in the third year of secondary school in which, for example, the length of a segment was defined as the class of all congruent segments and an angle was briefly described as "a rotation that has lost its centre" (ibid.). They further argued that modern mathematics is absolutely useless and that a total aversion prevailed among students and science teachers. They advised "urgently to leave the abstract language and aberrations of modern mathematics and to return to a realistic, concrete and basic teaching of mathematics" (ibid.). However, as far as we know, the impact of Pirard and Godfrind's article in Flanders was limited.

In Flanders, the public debate opened only a few years later, in 1982, when the pedagogue and teacher educator Raf Feys published a virulent pamphlet "Moderne Wiskunde: Een Vlag op een Modderschuit" (Feys, 1982), in which he firmly criticised the fundamental principles of New Math and the way it was introduced at the primary level. In his close contacts with schools, Feys did not see the appearance of the promised fascinating world, but "artificial results in a fake reality" (ibid., p. 3) and also little enthusiasm in children, but "more disgust, disorientation and desperation" (ibid., p. 3). Feys not only criticised New Math, he also suggested how mathematics education at the primary level should evolve and the model he had in mind was the Dutch RME. Instead of taking the structure of mathematics as a starting point, mathematics education should start from and gradually develop, the intuitive, informal and real-world knowledge and skills of the children. Feys' pamphlet focused on primary education and had most impact at that level, but thanks to media attention, it also echoed at the secondary level.

A follow-up event and important step towards a broad and open societal debate in Flanders was initiated by the Foundation Lodewijk de Raet, which on April 30, 1983 organised a colloquium in Brussels titled What Kind of Mathematics for 5–15 Year Olds? (Stichting-Lodewijk de Raet, 1983). Nearly 150 people participated, including representatives of mathematics education from primary to university level, teacher educators, members of the Inspectorate of Education and of mathematics curriculum committees. In a lecture by Freudenthal, participants were confronted with developments abroad (including the decision of some German states to prohibit by law set theory at school because "it made children mentally ill" (Stichting-Lodewijk de Raet, 1983, p. 4) and they learned about the Dutch alternative, illustrated by plenty of IOWO materials. On the opposite side, Roger Holvoet (1938–1998), professor of mathematics at the University of Leuven and fervent Papy-ist, minimised elementary school students' difficulties with typical New Math elements and confirmed

his confidence in the current update and innovation of mathematics education for the primary level. Another speaker at the colloquium, Jan Vermeylen, mathematics teacher at a Flemish secondary school and board member of the VVWL, showed himself more critical of the ongoing modernisation: "Little has been realised from the beautiful dream, finding an easy way to learn mathematics" (Stichting-Lodewijk de Raet, 1983, p. 12). He argued that, in most study streams, we should teach mathematics being as useful as possible, based on students' experiences and interests. He further stated that, for this purpose, set theory is not strictly prohibited, but in most cases, unnecessary and even harmful. Clearly, the colloquium revealed diverging points of view. But more importantly, it definitely made it clear that an adjustment of New Math had to take place and that "the learning materials of the IOWO and the new Dutch textbooks inspired by the RME approach, could no longer be neglected" (Stichting-Lodewijk de Raet, 1983, p. 2).

At the secondary level in Flanders, another development took place. In 1983, Dirk Janssens, recently appointed as professor in mathematics education at the University of Leuven, launched the idea to start a new journal for mathematics teachers. A group of his former students and newly started mathematics teachers (Deprez, Eggermont, Gyssels, Kesselaers, Remels, Roelens and Roels) responded positively and started the journal entitled *Uitwiskeling* (Fig. 11.3), with the first issue being published in 1984. *Uitwiskeling* is an untranslatable neologism connecting 'wiskunde' (the Dutch word for mathematics) with 'uitwisselen' (the Dutch word for 'to exchange'). The name of the journal refers to the idea of creating a forum where mathematics teachers can exchange ideas and discuss questions related to the practice of mathematics education. So, the action of *Uitwiskeling* was basically a constructive one: not directly criticising New Math, but searching for and sharing teaching resources that motivate pupils and stimulate their active participation in the learning process. *Uitwiskeling* had some fixed columns, the most important ones being "Cobweb", intended for questions and answers, hints, ideas, suggestions, reports of lessons and other short contributions from the readership; "Under the Magnifying Glass", a larger article in which members of the editorial board scrutinised and elaborated a part of the curriculum or an aspect of mathematics education; and "Guide to the Library", in which articles and books (in most cases from abroad) that were considered useful for classroom practice were identified, summarised and discussed. Soon, *Uitwiskeling* reached a large audience of Flemish mathematics teachers and became a channel through which these teachers learned about the new developments in mathematics education in other countries. Special attention was paid to developments in the Netherlands, Germany, and the French speaking part of Belgium, in which at that time Nicolas Rouche (1925–2008), professor of mathematics education at the Université catholique de Louvain and his Groupe d'Enseignement Mathématique (GEM), founded in 1978, became very influential. The journal *Uitwiskeling* still exists (www.uitwiskeling.be)—currently, that is, 2019, the 35th volume is running—and the angle of incidence remained unchanged: the practice of mathematics education and the confrontation of that practice with new ideas from didactics of mathematics.

Fig. 11.3 Cover
Uitwiskeling, November
1987 (design: Abdon Van
Bogaert)

11.3   How During the Middle 1980s and 1990s New Developments in Neighbouring Countries Reached the Community of Flemish Mathematics Teachers

11.3.1   Rounding off the Rough Edges of New Math

During the middle 1980s, the call for change became ever louder in Flanders and an official response was therefore inevitable. A first modification of the curricula for the secondary level started in 1983 and lasted until 1988. It was a modest reform that mainly rounded off the rough edges of New Math (Roels, 1995). We briefly describe some new accents in the programmes for the catholic schools (which is the largest network of schools in Flanders). Changes in the other networks ran more or less in parallel. For the first two years, a more intuitive approach to arithmetic was proposed: the different types of numbers and number operations were no longer defined in a set-theoretic environment and many proofs were eliminated. So, it became permitted again to introduce negative numbers with reference to temperatures below zero or to profit and loss and, for the rational numbers, teachers were allowed again to refer to the fractions that pupils had learned at the primary level. The time saved on theoretical issues was spent on practicing the operations and on solving equations and word problems. The plane geometry of the first two years, which was affine, remained unchanged, but the curriculum committee chose a radically different approach for metric plane geometry, which is part of the programme from the third year on. The length of a segment and the measure of an angle were accepted as primitive concepts (such as, e.g., points or straight lines) in combination with a few intuitively

Fig. 11.4 Papy's (1967) version of the Pythagorean theorem ('Hoofdstelling' = Main theorem; 'Als' = If; 'dan' = then)

accessible axioms, and thus the need to define them fell away. Partly because of that, the extensive study of isometries (reflections, translations, rotations, …) was shortened so that more time and attention could be spent on the classical plane figures (triangles, quadrilaterals, circles, …) and their properties. The Pythagorean theorem, reduced to a special case of the formula for the norm of a sum of vectors during the New Math period (Papy, 1967) (Fig. 11.4), regained its central position of the past. Finally, in analytic geometry, the equation of a straight line could again be introduced without relying on vector spaces. The introduction of this new geometry programme for the third year was accompanied by a large-scale action of in-service teacher education (Janssens & Roels, 1985).

At the upper secondary level, the curriculum change was more radical. In solid geometry, in Flanders a topic traditionally reserved for study streams with a strong mathematics package (6 to 8 h per week), the synthetic perspective was revalued. The curriculum stated that solid geometry should be seen as an extension of plane geometry and no longer as a part or application of linear algebra. The synthetic approach, meant for developing students' spatial skills, had to precede the analytic treatment. Special attention had to go to the sketching and understanding of planar representations of spatial situations. It was further recommended to also include problems about area and volume of solids. But a new curriculum cannot succeed without new and appropriate teaching materials. An attempt to provide such materials was undertaken by *Uitwiskeling* (Deprez, Roelens, & Roels, 1987b) and developed in more detail in an in-service teacher education course (Deprez, Eggermont, Janssens, & Roelens, 1987a). One of the sources of inspiration was the work of the Dutch HEWET team (led by De Lange and Kindt) who had developed teaching materials, in line with the RME philosophy, on various mathematical topics for pre-university education (for an overview, see, e.g., De Lange, 1987). This work was realised under the umbrella of OW&OC (Mathematics Education Research and Educational Computer

Centre), the successor of the IOWO since 1981. *Uitwiskeling* was also influenced by publications of the GEM (Groupe d'Enseignement Mathématique) and of the French IREMs (Instituts de Recherche sur l'Enseignement des Mathématiques). A pragmatic-eclectic approach to solid geometry was presented, starting intuitively with a phase of exploration and investigation. In that phase, it was suggested, students had to work in groups on problems that are challenging, but easy to understand and imagine, such as, for example, "What types of plane figures can occur when one intersects a cube and a plane?" This phase should then result in a series of statements about possible mutual positions of straight lines and planes, of which students have to check the correctness. Next, the correct statements could be accepted as starting points—axioms—for the further development of solid geometry. Later on, based on the results of that synthetic phase and on what students have learned in the fourth year in their lessons on plane geometry (e.g., about the scalar product), space coordinates and vectors could be introduced and analytic descriptions of straight lines and planes could be deduced. Finally, a number of richer problems about solids should be investigated, using both synthetic and analytic tools.

Another important change at the upper secondary level related to the approach of analysis (calculus) in study streams with 2–4 hours of mathematics per week. This new approach was not a mere weakening of the corresponding curriculum part for study streams with 6–8 hours of mathematics per week, but tried to meet the specific needs of students who had, for some reason, chosen a limited package of mathematics in their final years of secondary school. The idea was to opt for a less formal approach by skipping the topological foundation and by introducing the concepts of continuity and limit in an intuitive-graphical way. The time saved had to be spent to derivatives and integrals and to applications of these basic concepts of calculus. Because the meaning(s) of these latter concepts was central, calculation techniques were limited to polynomial and rational functions. Another novelty was a change in the order of integral calculus for these study streams: to allow a more insightful and motivating approach, the definite integral was introduced first, as the (oriented) area under the graph of a function, before the concept of primitive function (or indefinite integral). Initially, for this important curriculum change, little or no didactical support was provided for the teachers involved, but quite soon, *Uitwiskeling* spent an issue on this new approach (Deprez, Gyssels, & Roels, 1985) and later, this was further developed in an in-service educational course for teachers (De Bock et al., 1986). This course presented a quite radical interpretation of the new curriculum: the authors immediately started with derivatives, continuity was omitted and limits were, to some extent, integrated into the section on derivatives. That section on derivatives was largely inspired by two Dutch HEWET cahiers (Kindt & De Lange, 1984, 1985) in which the derivative was distilled from different real-world contexts in which (rate of) change had to be measured. The idea of 'conceptual mathematisation', that is, mathematisation as a way to introduce mathematical concepts (De Lange, 1987), was quite innovative in Flanders at that time.

11.3.2   A Second Wave of Changes

The curriculum changes of the middle 1980s were positively welcomed by most of the Flemish mathematics teachers. The conviction grew that priority in school mathematics should be shifted from mathematics as a static, rigorous deductive system to a meaningful and useful activity related to the broader world and society. Of course, parts of the curriculum that had remained unchanged in the first curricular modification were now assessed more critically by the teachers. Therefore, a second adjustment became inevitable. The introduction of a new structure for Flemish secondary education in 1989 provided the opportunity for this second wave of curriculum changes. In terms of content, changes were in line with the foregoing: typical New Math elements, such as sets and relations, were further reduced, mathematical structures were no longer explicitly addressed and the ambition to set up a global deductive system was abandoned. The treatment of geometry in the early years of secondary school became metric from the start and the notion of area and corresponding calculations were no longer neglected. Another important innovation was the extension of combinatorics and probability to statistics, now also including descriptive statistics (data analysis) and the testing of hypotheses as a preview of the application of statistics in practice (Carbonez & Veraverbeke, 1994; Kesselaers & Roelens, 1992). Finally, the modernisation of analysis that had started with the first curricular modification in study streams with a limited package of mathematics was now added to the study streams with a strong mathematical component. The idea was to start with sequences as a basis for a mathematically rigorous, but at the same time more intuitive and dynamic approach to the concepts of continuity and limit (De Bock et al., 1992). In these study streams the time spent on continuity and limits was also diminished: derivatives and integrals were considered the core concepts of secondary school analysis and had to receive maximum attention.

This second wave of programme changes was not limited to the content of secondary school mathematics, but also brought a number of didactical innovations. First, and perhaps most importantly, there was the role given to modelling and applications. In contrast with the previous period in which mathematics was basically taught as an autonomous discipline, the applicability of mathematics in other domains was now strongly emphasised. These domains did not only include the traditional areas of application (such as physics), but also biology, economics and other social sciences. Once again, inspiration was found in the Dutch RME materials, in particular in the HEWET cahiers. In contrast to the classical role of applications in mathematics education (i.e., applying a pre-designed mathematical method in another domain), also the idea of conceptual mathematisation (as explained above) enjoyed increasing attention in Flanders: mathematical ideas that are developed from diverse contexts, get a richer meaning and may subsequently be applied more easily in new domains (De Lange, 1987). Emphasising the applied side of mathematics also fitted with the belief that it is motivating for students to realise that mathematics is closely related to their own living environment. Second, more attention was given to (guided) self-discovery and active learning processes in the teaching and learning of mathematics. Contemporary research in educational psychology had shown that

effective learning is based on constructive processes, mediated and guided by adequate and supportive intervention strategies (De Corte, 1996). Applied to mathematics education, it means that students should not only be confronted with 'end products' of mathematical activity (i.e., 'finished' mathematical texts), but should also have the opportunity to go through the process of mathematical discovery and to authentically (re-)build pieces of mathematics themselves. Third, the growing importance of graphing calculators and computers was recognised, not only as powerful calculation tools, but also as means for authentic mathematical exploration, discovery and simulation (Cleve, De Bock, & Roelens, 1993). These new technological tools made a more graphical approach to mathematics—and even an approach based on multiple representations—achievable in mathematics classrooms at the secondary level.

The above-mentioned curricular innovations were again supported by the large-scale action of in-service teacher education Mathematics Taught By Applications (De Bock, Janssens, Roelens, & Roels, 1994; Janssens, 1993; Roels et al., 1990). The idea was to integrate the modelling perspective into the study of elementary functions, matrices, derivatives and integrals. In line with the objective to activate pupils during the mathematics classes, an active involvement of the participants was promoted and therefore, the sessions were perceived as real working sessions. During these sessions, groups of upper secondary mathematics teachers were confronted with real, and hence rather complex, modelling problems. In order to find appropriate solutions, they had to go through the whole modelling cycle (see, e.g., Verschaffel, Greer, & De Corte, 2000). An interesting case concerned the journey of the drilling rig called 'Yatzy' on the river Scheldt near Antwerp. The passage of the Yatzy under a high-voltage cable, taking into account the tides of the river, created considerable tension—both on board the platform, and among the teachers who participated! This authentic and large-scale modelling exercise, in which several mathematical models were integrated, also caught the attention of Dutch colleagues: the case was published in *Nieuwe Wiskrant* (De Bock & Roelens, 1990) (Fig. 11.5), the mathematics education journal published by OW&OC, which in 1991 was renamed as Freudenthal Institute after its founder. By that time, Flemish mathematics educators had become very familiar with the design principles of Dutch realistic mathematics education, and therefore exchange was no longer one-way traffic!

11.3.3 Consolidation

History continued and soon a new development took place. In 1989, the Flemish Government became responsible for educational matters. To promote and control the quality of education, it was decided to develop attainment targets for mainstream education at the primary and secondary level. The targets were designed as minimal objectives, which the government considered necessary and attainable for school children at these levels. These objectives referred either to knowledge, to skills or to attitudes. The attainment targets were approved by the Flemish Parliament and

Fig. 11.5 Cover *Nieuwe Wiskrant* with the *Yatzy*, 1990
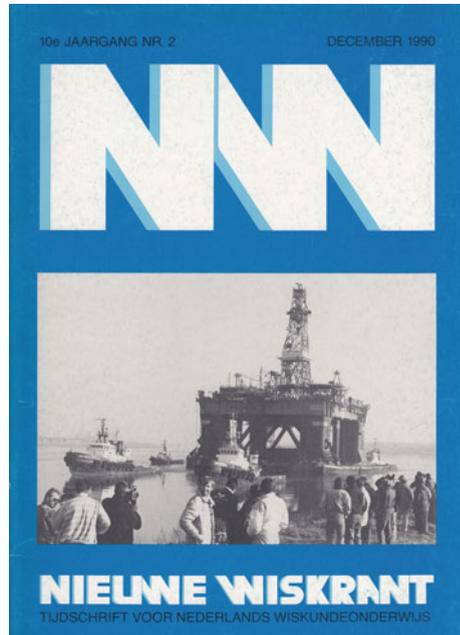
gradually implemented as parts of the curricula of the different educational networks (of Catholic and publicly run schools) from 1997 on; and they are still valid now. In this third wave of curriculum changes, the innovations of the previous phases, such as, e.g., the role of modelling and applications, the importance of a constructivist vision on learning and a meaningful implementation of ICT tools, were consolidated and continued. Remarkably, modern mathematics was no longer an issue and even elementary set theory was definitively removed from the new curricula: none of the attainment targets still referred to these icons of New Math! More importantly, a number of vertical learning trajectories were identified in the attainment targets. We briefly exemplify three such trajectories throughout students' secondary school careers.

A first vertical learning trajectory refers to statistics (Deprez, Roels, & Roelens, 1992). Nowadays, statistics starts in the first two years of secondary school with the analysis, representation and interpretation of real data and with some basic elements of probability related to fractions. Hence, a bridge with what students had learned at the primary level is made. In the middle two years of secondary education, the attainment targets state that students have to be able to select by themselves representations that are most appropriate in a given situation and should learn about different measures for the central tendency and the spread of a set of data. In addition, students learn to use probability trees to solve more complex probability problems. In the final two years, all students study the normal distribution, and students in a study stream with a strong mathematical component are introduced to confidence intervals

or the testing of hypotheses, with a strong emphasis on conceptual understanding over technical fluency (see above). This development is in line with those in many countries, especially in the Netherlands, which played a pioneering role in statistics education since the 1950s and which has served as a model for Flanders (Bunt, 1959; Garst, 1990; Zwaneveld, 2000).

A second vertical learning trajectory refers to functions (Eggermont & Roels, 1997). While in the New Math period, this topic immediately started with an abstract and technically advanced definition ('a special type of subset of the Cartesian product of two sets') in the first year in secondary school, now students first encounter tables, graphs and formulas as representations of various types of meaningful relationships (e.g., proportional and inverse proportional relationships). Gradually, more functional skills are developed (e.g., transforming the graph of a function), the level of abstraction is raised and several classes of functional relationships are studied, leading to the notion of a 'real function of a real variable'. A more general and abstract definition of the concept of function, treated as an independent mathematical object, only occurs in the final years of secondary education. That way, students of different ages learn about and work with different aspects of functions that are adapted to their situation. This approach was closely related to the Dutch 'TGF analysis' (tables, graphs, formulas), promoted from the late 1980s on by the HAWEX team (led by Kindt, Roodhardt, Van der Kooij and Van Reeuwijk), but also to contemporary developments in the United States (National Council of Teachers of Mathematics, 1989).

A third vertical learning trajectory refers to solid geometry (Deprez & Roels, 2000; Op de Beeck, Deprez, & Roels, 1997; Thaels, Eggermont, & Janssens, 2001). The emphasis came to lie on gaining insight into spatial objects and their planar representations. During the New Math period, this component of geometrical thinking was completely absent in the first four years of secondary education because it did not fit with an axiomatic approach at these levels. Now in primary school, pupils learn about solids by seeing and doing, most often in realistic contexts. They gain insight in such objects on the basis of three-dimensional models or on the basis of planar representations. This learning trajectory is continued in the first years of secondary school. At that level, the (re)construction of situations in space starting from a planar representation is developed further and some attention is already paid to argumentation related to properties of solids. Area and volume of elementary solids (cubes, cuboids and cylinders) are also part of the curriculum. In the middle years of secondary school, students learn to build more precise arguments about straight lines and planes in space, but this argumentation is always embedded in concrete problem situations, for example, about planar sections of solids. In the final two years of secondary school, for those students following a study stream with a strong mathematical component, a more structured—analytical—approach to solid geometry is developed. The development has a sound mathematical basis using a modern axiom system based on points and vectors. But the system works only at the background and can eventually be discussed at the end of the course. For some students, this gives the opportunity to sketch the way to 4-dimensional geometry (Deprez et al., 1987a, b). In the new view on solid geometry for the first four years of

secondary school, the influence from scholars of the Freudenthal Institute (see, e.g., the HAWEX and HEWET materials) and from Nicolas Rouche and his team is also very prominent.

11.4 Some Topics that Underwent a True Metamorphosis

In this section, we describe in some detail a number of topics that, compared to New Math, have been drastically changed and in which the influence of the RME approach is particularly clear. The first topic relates to exponential and logarithmic functions. The treatment of exponential and logarithmic functions during the 1960s and 1970s in Belgium was a textbook example of Freudenthal's (1973) notion of 'anti-didactical inversion'. In the sixth year of secondary education, first, the natural logarithmic function was defined as an integral function of $1/x$. Next, logarithmic functions with other bases were introduced as multiples of the natural logarithmic function and finally, the exponential functions were defined as the inverses of these logarithmic functions with arbitrary base. At the end, an aha experience was evoked: for rational exponents, the 'new' exponents coincide with the ones students had previously met in the fourth year. Needless to argue that this approach, although mathematically logical, had many educational disadvantages. The RME alternative is to start with exponential functions as models for exponential (or cumulative) growth, a context that gives a concrete meaning to the exponent (time) and proved to be very useful to understand various problems situations related to this class of functions (De Lange & Kindt, 1984a, 1986). The functions $x \rightarrow 2^x$ and $x \rightarrow 0.5^x$ (a model for negative growth) serve as prototypes of, respectively, increasing and decreasing exponential functions, and logarithmic functions are introduced as their inverses (and thus no longer vice versa). The transition from rational to real exponents is handled intuitively. The context of growth also proves to be very helpful for reasoning about logarithms and their properties (e.g., the fundamental theorem of logarithms was clarified as "time needed for doubling + time needed for tripling = time needed for multiplication by six"). Furthermore, it is 'proved' that the slope function of an exponential function is proportional to itself and the natural exponential function (and the number e) is introduced as the exponential function for which the proportionality factor equals 1 (hence, as the function that is equal to its derivative). A lot of additional applications are given (logarithmic scales, drawing log-log graphs, the Carbon 14 dating method, …).

A second topic that underwent a thorough metamorphosis was trigonometry. In fact, classical trigonometry was not so much influenced by New Math—those reformers generally showed little interest for this type of 'applied mathematics'—and was still taught in a rather mechanistic (pre-New Math) way, focusing on trigonometric formulas and the (technical) solution of trigonometric equations and inequalities. The new curricula separate the geometric part about angles and the solution of triangles, taught in the third and fourth year, from the functional part—in which the arguments of sine and cosine are real numbers—that is part of the fifth and sixth year curricu-

lum. In the functional part, trigonometric functions are seen as models for periodic phenomena, such as, for example, tides (De Lange & Kindt, 1984c, 1985). The 'sine model' and its main characteristics (amplitude, period, and horizontal and vertical translation) receive ample attention. These model characteristics are systematically explained and graphs of generalised sine functions are drawn using a grid frame. This system not only allows to approach some more complex modelling problems, but also provide a framework for the graphical solution of trigonometric equations and equalities.

Matrices are a third example. In the New Math period, this topic was part of a linear algebra course. After some practicing of computational techniques, matrices were studied in a very abstract way, with much emphasis on aspects related to properties of operations leading to the identification of underlying algebraic structures. Also, here, the HEWET materials provided an alternative for a more concrete point of view, connecting matrices to graphs and different types of contexts, such as distance and connectivity, population dynamics, consumer behaviour … (De Lange & Kindt, 1984b). In this vision, much attention is paid to the contextual interpretation of a matrix, its square and its product with another matrix. The widespread availability of ICT tools makes it also possible to perform calculations with big matrices that arose from real contexts. That way, matrices become a powerful tool for modelling various application problems in which blocks of numbers are involved. These problems originate from different disciplines (biology, economics, other social sciences, …) and proved to be a better introduction to abstract algebra than an immediate start with abstract structures.

We conclude this section with the fourth example: the new approach to differential and integral calculus. In this domain, the versatility in meanings of the concept of derivative and (definite) integral, that one cannot possibly understand based only on a definition, was given a central place. Meaning depends on the context in which these concepts occur, or, as Freudenthal (1973, p. 513) wrote: "What the differential quotient and the integral of a function mean depends on what the function means, and this can be many different things." Basically, the derivative is meant for measuring change (Kindt & De Lange, 1984, 1985). To arrive at a solid understanding of the concept of derivative, conceptual mathematisation was promoted. On the basis of different contexts in which change occurs and has slightly different meanings (speed of an animal, population growth, marginal cost, …), the notions of average and instantaneous change are explored and interpreted graphically as, respectively an average slope (slope of a secant) and the slope in a point (slope of the tangent). The transition from average to instantaneous change can be clarified on the basis of an intuitive concept of limit. This pre-formal phase is meant to equip the derivative with a rich and flexible meaning, and only at a later moment, a formal-analytical quantification is presented (as, respectively, a difference quotient and a differential quotient or derivative). In line with Poincaré's (1908, p. 83) words that "mathematics is the art of giving the same name to different things", the derivative appears as a common name for rate of change. The didactical track continues with the differentiation rules, first deduced for polynomial functions, but gradually expanded when other classes of functions are involved, and ending with various new applications,

e.g., problems related to optimisation, in which, thanks to the rich pre-formal phase, the derivative is now recognised and applied more easily.

Unfortunately, directly usable HEWET or other RME materials for the teaching of integral calculus were not available, but it proved possible to design a course in the same spirit, that is, giving a central place to the versatility of meaning of the concept of (definite) integral (De Bock, 1990; De Bock et al., 1994; Roelens, Roels, & Deprez, 1990). Inspiration was, among other sources, found in the work of German mathematics educators (see, e.g., Kirsch, 1976). Although integrals are defined geometrically—directly as the (oriented) area under the graph of a function or by means of lower and/or upper sums—their meaning in different contexts and, hence, the type of problems that can be solved using integrals, is emphasised. Three types of such problems are identified. A first type is labelled 'reconstruction problems': starting from the rate of change of a variable, the variable itself can be reconstructed. So, for example, the area under a graph of flow rate of a river, velocity or marginal costs enables to reconstruct, respectively, water volume passing, distance travelled or total costs. A second type relates to summation. Integrals are not only approximated by sums (the idea of 'numerical integration'), but also vice versa: sums of a large number of terms can be approximated or idealised by integrals. Sometimes, it is also helpful to perceive a magnitude (e.g., a volume, a surface area or the length of an arc) as a sum to discover how that magnitude can be calculated as the integral of some function. Third, integral problems are also related to averaging continuously changing magnitudes. Definite integrals with variable upper limit lead to the concepts of integral function and anti-derivative or indefinite integral. Finally, the link between the concepts of (anti-)derivative and (definite) integral is established (leading to the so-called 'fundamental theorem of calculus').

11.5   Conclusion

The Dutch RME model enriched Flemish mathematics education in secondary schools during the 1980s and 1990s and its positive influence is generally recognised. However, this does not mean that mathematics education in Flanders nowadays is a copy of that Dutch RME model (as embodied in, e.g., the HEWET and HAWEX materials). Elements of the more traditional approach, focusing on calculation drill and algebraic techniques, as well as more structural elements, focusing on a logical organisation of content and on proof and argumentation, are still essential parts of Flemish mathematics curricula and of classroom practice, although their importance has decreased. In our view, this has resulted in a more or less balanced approach to mathematics education in Flanders. This specificity of Flemish mathematics education, which was the result of multiple influences, is probably one of the reasons why an orthodox version of the RME model could never be implemented. For example, a project in the late 1980s in which some of the HEWET cahiers were 'translated' to the Flemish context, proved to be unsuccessful, likely because Flemish teachers missed a clear structure, the provided modelling problems were sometimes too open and

computational skills received insufficient attention. It may also be one of the reasons why the Math Wars, that originated in the 1990s in the United States, have had very limited impact in Flanders (while they have been an important educational-political issue in the Netherlands).

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