Chapter 5
Approaches to Investigating Complex Dynamical Systems

France Caron

Abstract This chapter reports on a short experiment conducted in Canada to explore the potential, and feasibility, of introducing complex dynamical systems into mathematics curricula, both in schools and in university undergraduate programs. In particular it aimed to identify those mathematical habits of mind that are (or could be) developed in schools, universities and outside these traditional learning environments through exploring complex systems when approaching real life situations. The use of game design to engage members of the Canadian mathematics education community in the modelling of a real-life ecosystem brought to light different mathematical habits of mind and provided a snapshot of where we are with respect to modelling in mathematics education. Put in perspective with current lines of inquiry in the modelling of complex dynamical systems, in integrating modelling in mathematics education, and in developing computational thinking, the experiment opened a reflection on the possibilities and feasibility of helping tackle the complexity of our world's most pressing challenges through mathematics education.

Keywords Dynamical systems · Complex systems · Curriculum · Habits of mind · Structuring

5.1 Introduction

More and more, and despite its relatively modest presence in today's mathematics curricula (Stillman and Kaiser 2017), modelling is being recognised as an important element of the learning of mathematics. Not only is it a means for students to understand and integrate the mathematics they learn, but also it is, more fundamentally, a goal in itself of mathematics education. As a goal of a mathematics curriculum its purpose is for students to develop competencies that enable them to tackle situations

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from a mathematical perspective, whether these situations come from everyday life, science, the workplace or the complex challenges our world currently faces (Niss et al. 2007). Many studies (e.g. Kaiser et al. 2011) suggest that modelling as a learning goal is best served with authentic situations and open-ended problems, where students are free to invoke whatever mathematical knowledge they feel serves their purpose. Under such conditions, students may experience the various phases of the modelling cycle; including simplifying and structuring a real world situation into a specific problem with its goal, components and relationships; mathematising the problem (i.e. translating it into a mathematical model); working mathematically to reach a solution; interpreting and validating the results (and the model or the data) with respect to the original situation; and repeating the whole process if need be (Blum et al. 2002).

To assist students in developing powerful models, one could think of revisiting curriculum content so as to provide a richer mathematical toolbox for addressing real life needs of today and tomorrow. Yet, on that front, the following warning was expressed more than twenty years ago:

> Given the uncertain needs of the next generation of high school graduates, how do we decide what mathematics to teach? Should it be graph theory or solid geometry? Analytic geometry or fractal geometry? Modeling with algebra or modeling with spreadsheets? These are the wrong questions, and designing the new curriculum around answers to them is a bad idea. (Cuoco et al. 1996, p. 375)

Instead of merely "replacing one set of established results by another one (perhaps newer or more fashionable)", Cuoco et al. suggested organising curriculum design around *habits of mind*, to allow students "to become comfortable with ill-posed and fuzzy problems, […] to look for and develop new ways of describing situations" (p. 373) and "to close the gap between what the users and makers of mathematics do and what they say" (p. 376).

Some general habits of mind described as useful for doing mathematics include pattern-sniffing, experimenting, formulating, tinkering, inventing, visualizing, and conjecturing. More specific mathematical habits of mind would include approaches that either rely on a particular mathematical idea or reflect "the way mathematicians approach things" (Cuoco et al. p. 384). Contrary to the notion of *mathematical thinking styles* (Borromeo Ferri 2010) or the more general notions of *thinking styles* (Sternberg 1997) and *learning styles* (Felder and Brent 2005), the consideration of mathematical habits of minds was not brought forward to capture individual differences in preferred ways of thinking, understanding or doing things, nor to examine their influence in teaching and learning. Rather, it was meant to serve as a framework to organise a mathematics curriculum, where each of these habits of mind as a research method becomes a learning goal of mathematics education, and the content to be learned is identified on the basis of its potential contribution to developing these goals. Wu et al. (2015) take a similar viewpoint when they use the development of the thinking underpinning mathematical modelling as the teaching and the learning goals in a tertiary mathematical modelling course. Rather than focusing on methods to be learned, they promote activities that build on intuition, help make connections and innovation through critical thinking, cultivate inductive thinking, etcetera.

Yet, as proposed by Hunt (2007), there may be the need for citizens in general, and policy makers in particular, to develop a better appreciation of some key ideas of the mathematical sciences that can help us understand and deal with society's major challenges. Addressing issues of greatest public concern often depends on the "reliability of prediction of multi-component complex systems" and the inclusion in the analysis of "possible sudden changes and isolated events" (Hunt 2007, p. 3).

Nature, human organisations, as well as the multiple layers where the two meet are indeed often best described as complex systems.

> Complex systems are systems that comprise many interacting parts with the ability to generate a new quality of collective behavior through self-organization, e.g. the spontaneous formation of temporal, spatial or functional structures. [...] This recognition, that the collective behavior of the whole system cannot be simply inferred from the understanding of the behavior of the individual components, has led to many new concepts and sophisticated mathematical and modeling tools for application to many scientific, engineering, and societal issues that can be adequately described only in terms of complexity and complex systems. (Meyers 2011, p. v)

The behaviour of a complex system is intrinsically hard to predict, as the effect can be very distant from the cause in both space and time; what may look like a radical new rule may have little long-term effect while relatively small perturbation can ultimately result in a major impact. The consequent conflict between long-term and short-term goals (Forrester 1996) may partly explain the challenges our world is currently facing.

If developing a better understanding of complex systems should be regarded as one of the goals of education, and if the study of complex systems has to call upon "new concepts and sophisticated mathematical and modelling tools" (Meyers 2011, p. v), this may bring us back to the apparent dilemma of having to choose between adding content to the general mathematical toolbox and developing the mathematical habits of mind "to close the gap between what the users and makers of mathematics do and what they say" (Cuoco et al. 1996, p. 376).

In an attempt to examine how mathematics education could better equip students to approach ill-defined problems of increasing complexity, colleagues and I explored the opportunity for, and feasibility of, introducing complex dynamical systems in the mathematics curricula, both in schools and university undergraduate programs. This was done through activities and discussions within the context of a working group at the Canadian Mathematics Education Study Group (CMESG) annual conference (Caron et al. 2015). The CMESG meetings are a place where people from mathematics and mathematics education across Canadian universities share and explore issues and ideas on the teaching and learning of mathematics. In the various activities and discussions that took place in the group, one modelling experiment proved particularly interesting, as it generated sustained engagement among the participants, allowed a variety of models to emerge and revealed different and complementary mathematical habits of mind. Although some connections could be made with individual thinking, or learning styles, the observed differences appeared to have more to do with the different formal training and experiences our participants brought to the table.

While providing an interesting snapshot of where we are in Canada with respect to modelling in mathematics education, this experiment opened a reflection on the possibilities and feasibility of tackling the complexity of our world's most pressing challenges through mathematics education. This reflection led to examining the different lines of inquiry for exploring complex systems and the way these lines have been reflected in curriculum and resource development, school experiments and research on modelling in mathematics education. In particular the aim here is to identify those mathematical habits of mind that are (or could be) developed in schools, universities and outside these traditional learning environments through exploring complex systems when approaching real life situations.

5.2 The Experiment

In planning the working group, the use of games quickly imposed itself as a potentially rich and accessible approach to engage students and teachers at different levels of mathematics in experimenting with the dynamics of complex systems. The game context later revealed its value for revisiting our conception of mathematical habits of mind, with the current state and ubiquity of technology.

The working group consisted of 14 individuals who had elected to explore over three half-day sessions the topic of complex dynamical systems. The group showed a rich diversity of profiles, with school mathematics teachers (elementary and secondary), along with university professors and graduate students in either mathematics or mathematics education.

The first of the three sessions was dedicated to playing with simulators and games (built on cellular automata or agent-based models) in order to bring forward some of the characteristics of complex dynamical systems (with their different patterns for long-term behaviour) and their typical applications (social contexts and environmental situations).

In the second session, participants regrouped in three teams (A, B and C) to design a game to replicate the recent change in the dynamics of the Yellowstone ecological system, after wolves had been reintroduced. The situation, for which they could find additional information on the internet, was described as follows:

The Yellowstone Game

In the 1990s, wolves were re-introduced into Yellowstone National Park. A couple of decades later, the grizzly bear population in the park increased significantly. Why did that happen? Researchers hypothesised that a chain of interactions was at play. As wolves prey on elk, they decrease their population. All kinds of berries, previously consumed by elk (which also destroyed berry shrubs), are now able to recover, thus providing an abundant source of food for bears, especially in fall, when they prepare for hibernation. Build a game that will allow you to verify researchers' claims about the chain of events that led to the increase in the bear population. Then use your game to predict what will happen with the wolf, elk, bear and berries populations in the long run, in particular if the elk population declines to very small numbers.

To anchor the activity in a game environment, participants were provided with concrete materials (i.e. many small coloured square tiles, spinners and dice, biased and unbiased) that they were free to use.

In Session 3, an alternative model of the same situation was presented to the participants who were invited to explore further the modelling software (Stella) with which it had been built. A discussion followed where the participants shared their view on the value and feasibility of introducing dynamical complex systems in school mathematics, and on the resources that could help moving in that direction. The remarkable variety and evolution in the games and models that emerged with the Yellowstone situation gave a unique perspective into how to reconcile, within a modelling exercise approachable by all, complementary or even conflicting habits of mind. This will be the focus of the next session.

5.3 Habits of Mind at Play

All teams engaged with great enthusiasm in the game design activity, well beyond the time that had been planned for. Within each of the three teams, the diversity of the participants' background resulted in an interesting negotiation of the game structure, format and rules.

Teams A and B, included mathematicians who rapidly led their team onto the road of compartmental models or differential equations *à la* Lotka-Volterra. Very little time was initially spent on structuring the situation, as models seemed readily available, only requiring adaptation. With such entry into the task, the mathematicians in the group displayed their habit of *thinking of change analytically, in terms of functions and differential equations* to capture continuous variations (especially over time) of variables and to look for dependences that might describe causal phenomena. The proposal of a compartmental model, in Team A came from a mathematician working with biologists, and reflected a particular way of modelling often used in biology: *thinking of systems in terms of flows*. From their non-verbal behaviour, we could perceive some frustration among other team participants who did not feel at ease with the proposed equations or could not envision how the compartmental description could be used. They disengaged from the conversation until they found a way of regaining access, either by reaffirming the game design objective or by proposing an alternative approach.

In Team A, the diagram associated with the proposed compartmental model appeared to have facilitated a shift of representation. The compartments associ-

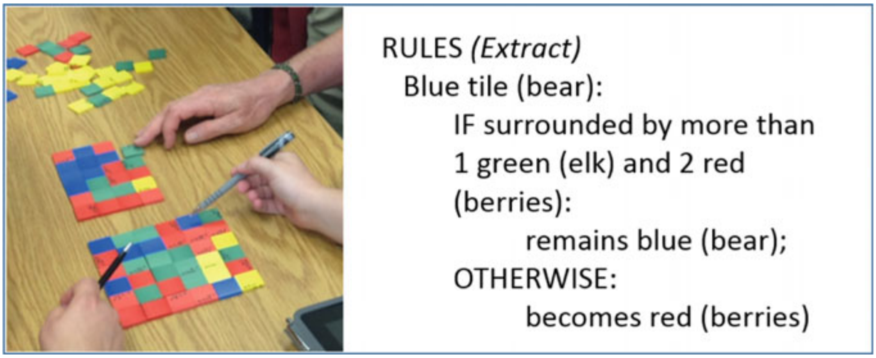

Fig. 5.1 Integrating spatial considerations with a cellular automaton

ated with the variables representing the species populations became distinct piles of tiles, enabling some physical enactment of compartmental models to take place as tiles were sequentially added or removed from piles, according to *rules that were being defined*, making the members *think iteratively* of the system. Perceiving a difficulty in capturing the various interactions (predation and competition) that characterise the Yellowstone situation with the addition and removal of tiles, Team A later moved to a *spatial representation* of the problem. The *domain was made discrete* as a 6×6 grid with coloured square tiles as cells where the colour indicated the dominating species in each small part of the territory (Fig. 5.1). The game would unfold like a cellular automaton similar to one they had experienced the day before, thereby maintaining an *iterative* view of the system.

In designing their cellular automaton for modelling the Yellowstone situation, the participants had to *define rules* that would reflect interactions between species. That was done by specifying when and how, with the *use of conditional statements*, the dominating species in one area (represented by a cell) would change in relation with the dominating species in the surrounding eight cells; one of these rules made use of randomness, to break a tie in competing species of surrounding cells. Generating the configuration for each iteration, one cell at a time, made the team appreciate the implications of the rules they had created. Despite the repetitive nature of the task, which seemed to call for a computer implementation, the team shared great enthusiasm in observing their game unfold and assessing the value of their model in capturing the evolution of the ecosystem. It turned out, to their great amusement, that within only 6 iterations, the berries ended up dominating the whole park, which could only *raise questions regarding the adequacy of the model*, its granularity and its rules. The team thus had gone through the whole modelling cycle and was in a position to reiterate.

A similar approach was eventually adopted by Team B, with a substantial increase in motivation from where they were when differential equations were being discussed. Their game had the extra feature of *keeping a record* of the changes in the dominating species for each cell, by superimposing tiles (i.e., stacking) for implementing change of colour. Their larger grid and high consumption of tiles prevented them from reaching a point in the simulation where they could assess the value of their model.

Structuring with cellular automata had brought forward this *algorithmic and iterative way of thinking* and made it a workable compromise for two of our three teams. While most team members could recognise mathematical structures, all could make sense of the model they were building and using for *simulating the dynamics* of the ecosystem. The fact that all participants were in the driving seat for *describing the interactions*, *inventing the rules*, and *experimenting* with them made them engage actively with these very useful habits of mind in mathematics. Their engagement may also have been favoured by the fact that a grid made out of colour tiles appealed to both the *analytical* and the *visual* thinkers in the teams.

Team C, where no member was a mathematician, went in a completely different direction. They elected to go for the most realistic representation of the situation they could think of, by *modelling at the level of the individual*, animal or plant, by *integrating movement* (a key feature of animals!) and by *addressing interactions as encounter events*. They *defined rules* such as the following:

> If a bear has not eaten in 10 turns, it disappears.
>
> After a tenth encounter between a bear and a wolf, a bear is replaced by a wolf. Otherwise, they bounce off.
>
> If a bear encounters berries, the bear sticks beside the berries for one step and then moves on. The bear gets power points. Every 10 cells, the bear drops a berry seed that will grow the next season.

This direction was inspired both by the exploration the previous day of an agent-based simulator (Caron et al. 2015), aimed at reproducing the movement and behaviour of two species (in a predator-prey relationship) within a given bounded region, and their experience of today's highly realistic and sophisticated computer games. The level of detail at which they positioned their modelling (the individual) prevented them from playing their game without a computer implementation. On this issue, Team C shared a very interesting viewpoint; they saw their role was to specify the components and rules of their game, and that they could turn to a programmer to implement them in a simulator. By 'outsourcing' the implementation of their model they had no means of validating yet, they were implicitly agreeing to their dependence on a potentially large black box.

After having presented their own games, the participants were shown a Stella model of the Yellowstone situation, equivalent to the one shown in Fig. 5.2, and later invited to explore the features of this modelling software. The working group came to appreciate how *modelling interactions as inflows and outflows* of reservoirs could support the development of skills for capturing the dynamics of a system. The ease of *building, defining and combining the relations* between system components, through graphical description and limited use of symbols, made it more appealing and accessible than the differential equations to participants with either less experience of such mathematical tools or an admittedly more visual learning style. The possibility to simulate directly from that description provided new meaning to the

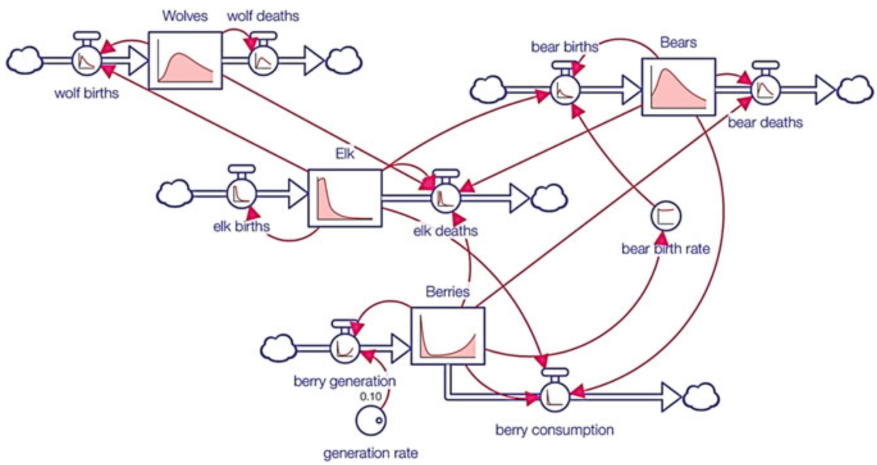

Fig. 5.2 Modelling and simulating with Stella

compartmental model that had been proposed in Team A. However, the perceived difficulty of *integrating space and movement* in the model made it less interesting than some of the other models that had been explored.

As the complexity of the situation and its perceived approachability within a game environment gave rise to radically different approaches, the experiment provided some insight into some mathematical habits of mind that are (or could be) developed in schools, universities and outside these traditional learning environments when approaching real life situations. This observation led to taking a closer look at each approach (where they emerged, where they are currently applied) as well as the initiatives for transposing them into mathematics or science education. Putting the activity in greater perspective allowed the author to connect some of the reflections that emerged in the working group discussion with affordances and obstacles that had been identified in those initiatives.

5.4 Modelling Complex Systems

As the study of complex systems is still quite recent and the tools and methods for doing so have evolved over the last decades, it is worth taking a look at what has been done, both inside and outside educational settings. Bosch et al. (2005) have stressed the importance of knowing the mathematical activities outside the school environment that motivate and justify the teaching and learning of mathematics with a given task.

5.4.1 Functions and Differential Equations

Differential equations constitute the classical tool for describing change in modelling nature and mechanisms. Dating back to Newton and Leibnitz, they are still at the core of the study of physics and its various applications, and have been used in many other disciplines: chemistry, biology, economics, etcetera. In a first course on differential equations, students are often presented with classical situations such as spring-mass systems, electric circuits, cooling of objects, where such equations are derived. With these fairly simple situations, the differential equations that describe the system can be solved analytically to produce a solution that can be expressed as a closed-form function that very closely approximates reality. In many ways, these are not complex systems.

A system does not have to be very complex to defeat classical *analytical methods* for finding the solution. In order to solve the typical prey-predator problems, even with only two species, one has to turn to techniques other than traditional calculus algebraic manipulations to capture the behaviour of the solution of the associated system of differential equations: *phase plane analysis*, *discretisation*, *numerical methods* and *computer simulation*. These ecology problems, along with those that come from a wide variety of domains (fluid dynamics, pharmacology, etc.) and that do not lend themselves to analytical solutions are typically not addressed before the third year of undergraduate mathematics, in mathematical modelling or numerical analysis courses. But they have contributed to making mathematicians, computer scientists, engineers, physicists, biologists and other users of mathematics also *think of functions algorithmically*, with recurrence and iteration. Solving the associated equations to simulate the evolution of such systems typically requires translating the mathematical model into a computer model (Greefrath et al. 2011). Depending on the complexity and peculiarities of the problem, and on the resources accessible to the user, this may entail possibly creative *use of a specialised software*, some *programming* and even new *algorithm design*. The actual mathematical work on the model, which is often reduced to a little box or a thin arrow in the different cyclical representations of the modelling process, can actually become a major endeavour on its own, requiring many steps, iterations and internal verification.

Even if a solution cannot be expressed in terms of known functions, this certainly does not mean that reasoning using functions does not contribute to the modelling of a complex situation and analysis of a model. But functions are used as building blocks to structure the system, to describe relations between variables and/or their rates of change, using known principles or formulating new assumptions. This *use of functions as building blocks* for modelling is quite different from the one that is typically promoted in the different Canadian school curricula, where, by the end of secondary school, learning to model with functions is often associated to "graphing data and determining the [specific] function that best approximates the data" (e.g. Western and Northern Canadian Protocol (WNCP) 2008). While curve fitting and regression may help approach unknown relationships between two variables and develop initial intuitions, they can only bring limited contribution to understanding of the situation, as no profound structuring is involved (Doerr et al. 2017; Galbraith 2007).

The learning of differential equations can help *structuring a situation using rate of change.* But as these equations are usually introduced after one or two courses of calculus, a relatively small portion of the population gets to develop and maintain such habit of mind.

5.4.2 System Dynamics Software

Enabling more people to *think in terms of structural relationships, flows, accumulations* and *feedback mechanisms* to explore the evolution of complex systems has been the driving force behind the development of system dynamics software. These ideas come from Jay W. Forrester, the founder of system dynamics, who transferred his engineering knowledge to study the dynamics behind a variety of industrial, urban and global situations (Forrester 2007). He was active in promoting system dynamics for K–12 education, encouraged by the emergence in the 1980s of user-friendly simulation software with advanced graphical user interfaces, such as *Stella, Powersim* and *Vensim.*

These tools, and the more recent web-based *Insight Maker*, provide an icon-based modelling environment where key aggregate variables (stocks represented by rectangular reservoirs) are defined by the user and made subject to inflows and/or outflows that will define their rate of change. These flows can be made to depend on the value of various variables and parameters. Running a simulation shows the evolution of the different variables over time; the resulting functions are constructed dynamically by the software, through numerical integration, for every simulation the user decides to run, and with the time step specified, allowing relatively smooth passage between discrete and continuous models.

With the possibility of combining multiple interactions, randomness and the systematic use of numerical integration, these environments lend themselves naturally to the modelling and simulation of progressively more complex systems. The ease of modifying or refining structural relationships allows for some *experimenting* and *tinkering*. Such modelling environments have been around for more than thirty years, and have been used in research in a variety of fields to model the behaviour of different phenomena or processes (e.g. spread of a virus and immunisation, carbon cycle and global warming, fisheries management, effect of taxes or social policies).

Doerr (1996) provided a rich retrospective of the first ten years of use of *Stella* in various mathematics or physics classes. She reported that early experiments in the USA in the late 1980s had led to recognition of the importance of computer modelling and system dynamics for secondary education. Modelling with diagrams was considered to favour a conceptual representation of the system that allowed students to both communicate their understanding of a system and experiment with it. With the system taking charge of the calculation, it was expected that students would engage in a more *qualitative analysis of problems*, focusing on principles and feedback mechanisms. Although the hydraulic metaphor was revealed to be generally more difficult for students to grasp than anticipated, there was some evidence of improvements in students' qualitative reasoning about problem situations, and in identifying structural similarities between them.

Despite the development of some resources (e.g., Fisher 2001; Kreith and Chakerian 1999), a perceived lack of alignment to school curriculum resulted in little implementation of system dynamics modelling in North American school mathematics. Apart from enthusiastic pioneers, mathematics teachers have not felt as comfortable with the approach as their science colleagues (Doerr 1996; Fisher 2011). It may be that this modelling approach cannot be associated with the mathematical habits of mind that they have developed. Indeed, some of the topics brought by Fisher (2011) in the lists she proposes to develop progressively systems thinking in K–12 (flows, feedback loops, equilibrium, tipping points, etc.) appear more solidly anchored in the science curriculum than in the current mathematics curriculum. It can also be that the teachers of mathematics are hesitant in having their students rely on a black box for time integration. Using a spreadsheet variant with direct access to the discretised integration formula, for example, was much better received by mathematics teachers than the use of stock-flow diagrams (Tinker 1993 as cited in Doerr 1996).

Resistance against the use of a black box also appeared in our working group discussion. Participants commented that the proper use of a tool like Stella would have to rely on the knowledge of calculus and the numerical methods responsible for allowing the simulation to unfold. But then, one participant suggested that we might have held initially similar views about dynamic geometry software (DGS) and the necessary geometry knowledge to use them; this was until experimentation with younger students showed that well designed exploration activities with DGS could contribute to developing new intuition and motivate the learning of concepts, properties and proofs. Similarly, he said, we could envision developing intuition with respect to functions and calculus through modelling with system dynamics software.

5.4.3 Cellular Automata

The distance between school mathematics and cellular automata may appear even greater than with system dynamics. Conceived in the 1940s by physicist, mathematician and computer scientist John Von Neumann, a cellular automaton is an array of "cells" (arranged in one or more dimensions) in which the state or behaviour of a cell in some generation is determined by rules that involve the states of neighbouring cells in the previous generation. Simulations consist of iterative application of the rules at each time step. Cellular automata are thus discretised models of dynamical systems, both in time and space. Instead of working with aggregate variables, they propose a distributed representation of a system, where one or many state variables are assigned to each of the cells of the domain. Changes to these variables are made locally, at the cell level, based on the exclusive knowledge of the states of immediate neighbours and on rules, which act as conditional statements.

As the focus on immediate neighbours in generating the future impedes us from envisioning and appreciating the system as a whole, it is mainly through running the system (typically via a computer implementation) that one can gain insight into its collective behaviour. This emphasises the notion that the "collective behavior of the whole system cannot be simply inferred from the understanding of the behavior of the individual components" (Meyers 2011, p. v), which is a defining feature of a complex system.

Cellular automata were popularised in 1970 through John Conway's "Game of Life", aimed at emulating the evolution of a community of living organisms (Gardner 1970). Since then, they have been used as models in a large array of applications: wildfire propagation, voting dynamics, land use and urbanisation, dynamics of infection, etcetera. In each area of application, experimentation and research have led to more sophisticated versions of cellular automata. In urban simulation, for instance, the notion of neighbourhood has been extended to allow action-at-a-distance; transition rules have been adapted to accommodate hierarchy, self-modification and stochasticity (Torrens and O'Sullivan 2001). As simple models of complex systems, cellular automata also have given rise to radically new approaches for solving problems of fluid dynamics in porous media and complex geometries: movement of the fluid has been modelled as pseudo-particles moving on a grid, from a node to one of its neighbours, while the rules for handling collisions are defined so as to *preserve conservation principles* (Raabe 2004).

Some cellular automata simulators were developed (but not necessarily maintained) as outreach material on the web. Spreadsheet versions have been documented (Catterall and Lewis 1985; Hand 2005), with the claimed advantages that the workings of the model are made explicit to students and that they can be implemented easily without the need for programming skills.

Participants of the working group noted that using tiles, as was done with the Yellowstone cellular automata, could help introduce grade 8 students to recursion. They valued recursion as a "big idea" that captures the notion of transformation, much better than the function does in the way it is usually taught. The possible progression from playing a game with predefined rules to changing the rules, and then to designing completely new rules was perceived to offer a realistic development perspective for engaging students in modelling complex systems.

5.4.4 Agent-Based Models

An agent-based model of a system is a collection of autonomous and adaptive entities called agents. Each agent individually assesses its situation and makes decisions for itself on the basis of a set of rules. *Algorithmic thinking* still applies, but it explicitly, and more effectively, addresses learning and adaptation that characterise individual behaviour. Well suited to capture the complexity of biological systems, agent-based modelling has also been used, amongst other things, to approach human behaviour in

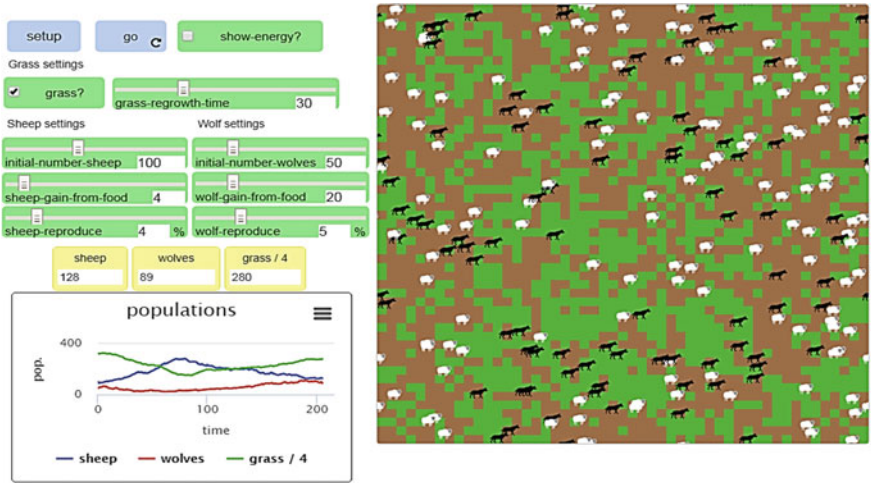

Fig. 5.3 A prey-predator simulator with NetLogo. http://www.netlogoweb.org/

collective settings (e.g. emergency evacuation) or human systems and organisations (e.g. finance markets).

As agent-based models aim to describe the behaviour of the system's constituent units (the agents) and their interactions, their coding and simulation can capture emergent phenomena, which may be hard to predict otherwise (Bonabeau 2001; Meyers 2011). Such emergent phenomena will be perceptible at a *level* higher than that of the agents, as in a flock of birds. The emerging phenomenon may be counterintuitive. For instance, students may be surprised to observe from a simulation that when too many cars try to move forward on the same highway, the resulting traffic jam will go backwards (Wilensky and Resnick 1999).

In order to favour the development of *multilevel thinking* and appreciation of how order can emerge from randomness, Wilensky and Resnick (1999) have each developed educational agent-based modelling environments for exploring complex dynamical systems. Named after the original Logo environment created by Seymour Papert, *StarLogo* and *NetLogo* were conceived as extensions of this pioneering microworld. Instead of instructing a single turtle to move, the users program the rules of movement for the agents, when and where they are generated or deleted, as well as their actions when they meet or collide. The ground where the agents move can also be organised as a cellular automaton (Fig. 5.3).

While *StarLogo* is aimed at K–12 students with a programming environment similar to the one of *Scratch*, *NetLogo* with its relatively simple programming language has been designed also for another type of user: the expert in his field with limited or no programming experience. It seems to have met a need, as the use of *NetLogo* in research papers has grown to a point where an extension has been developed to interface with R statistical software for more sophisticated interactive analysis of simulation results (Thiele and Grimm 2010).

From various experiments where he analysed student reasoning in exploring complex system simulations, Wilensky has come to the conclusion that in order to make sense of these systems, students need to use two complementary forms of reasoning: the "*agent-based*" *reasoning* to think in terms of properties and behaviour of the individual entities of the system, and the "*aggregate*" *reasoning* to think about the properties and rates of change of populations and higher level structure (Jacobson and Wilensky 2006).

In the working group, the team that went for an agent-based model of the Yellowstone situation mainly stayed at the agent level, and did not address the aggregate level explicitly. As a result, this team's game gave the impression that the potential for increased realism came with reduced opportunity for mathematical analysis.

5.5 Discussion

As can be gathered from this brief overview, research has developed and used a variety of approaches for investigating complex systems in nature and society. Although quite different, these approaches share some common features. For one thing, they all make extensive use of *computer simulation* to follow the evolution of such systems, through the *iterative application of rules or numerical schemes that describe change or interactions*. These rules or schemes are designed so as to *reflect the invariants in situations of change,* with possible adaptation. As another fundamental element, the passage to a computer treatment involves one or several kinds of *discretisation*: of time, of space, of matter.

Frejd (2017) argued that modelling as a professional activity is different from modelling as a school activity, as it is often "based on knowledge and experiences reaching far beyond what can be found in a secondary mathematics classroom" (p. 377), with computers, programming and specialised software playing a major role in the development of models and their use. That being said, there have been attempts at reducing the knowledge gap. Some of the models, techniques and underlying ideas involved in investigating complex dynamical systems appear within reach of secondary students and the development at school of such practices would expand substantially the class of problems that citizens could explore and potentially solve. Consequently, there has been sustained effort from dedicated individuals and teams in allowing students to develop these skills and notions, through the careful design and classroom use of modelling and simulation tools and learning activities.

Despite this long lasting commitment and promising beginnings in secondary education in the 1990s, the presence in school of these approaches has remained relatively marginal, at least in Canada. The same can be said within the literature on modelling in mathematics education; papers that report on experiments with students investigating dynamical complex systems in school mathematics are relatively few, and often have been written by the people that have developed resources for such investigation. Trying to explain this limited transfer of modelling practices raises questions and issues that align along different lines of inquiry regarding mathematical modelling integration in the teaching of mathematics. These will be addressed in turn.

5.5.1 Epistemological Issues

In applied mathematics, for numerical simulations based on mathematical techniques (i.e. discretisation, numerical methods), which by their very nature can only approximate the solution, a case must be made, when solving numerically a system of equations, that the error has been controlled and that the equations have been solved correctly; this is what is known as *verification* (Roache 1998). When these equations, or any other type of mathematical model, are meant to represent a situation, there is also the need for external consistency, that is, ensuring the model and its associated solution adequately represent the situation. This is the *validation* part of the modelling process, which is shared with science and engineering (Roache 1998), and for which simulations, as a new form of scientific production, have blurred the line between theorising and experimentation (Greca et al. 2014).

Verification and validation techniques for models, simulations and scientific theories are not part of the typical repertoire of teachers of mathematics, who are more familiar with proof. For them to engage with their students in simulation-based modelling activities and exploration of complex systems may require letting go of their attachment to rigour (with the risk of sending the unfortunate message that anything goes when it comes to modelling, simulation or science) or to define heuristics for the class that might help control the quality of the models and simulations. Alternatively, it could be an opportunity for interdisciplinary work with their colleagues of science.

5.5.2 Interdisciplinary Collaborations

Judging from the literature, there have been more classroom experiments with the investigation of dynamical complex systems in the teaching of science than in the teaching of mathematics. Many factors seem indeed to favour science education over mathematics education for adopting such a paradigm: the very visible presence of simulation in science, the central notion of model, the tradition with experimentation and validation, etcetera. In addition, familiar scientific principles, that are the objects of study in a science course, can be used when structuring a real-life situation. Using such principles (including those from geometry), pre-existing models, and deductive reasoning, one can build a theoretical model that not only *describes* the situation but also contributes to *explaining* it (Doerr et al. 2017).

Yet, as “a rather small number of relatively simple structures appear repeatedly in different businesses, professions, and real-life settings” (Forrester 1993, p. 189), there is an opportunity, and maybe even a need, for mathematics education to become involved. Some elements of these structures can already be mapped with mathemat-

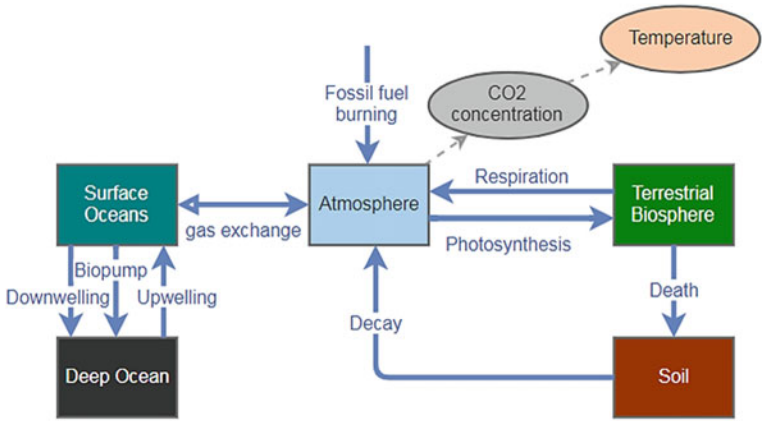

Fig. 5.4 Structure of a simple model of the carbon cycle with Insight Maker. https://insightmaker.com/insight/79473/Global-Carbon-Cycle

ical content (e.g., flow with rate of change or derivative). While simple generic structures such as feedback loops may not (yet) be part of the mathematics curriculum, they can still be used as a learning context to introduce prescribed content (e.g. exponential functions) and to provide opportunities for additional complexity, possibly in collaboration with science teachers.

Such collaboration can bring to the fore the role of mathematics in assessing the validity of assertions. For example, as some like to reduce climate change to a matter of belief, we could move the discussion down to simple models (Fig. 5.4) that can be used to both explain the phenomenon and predict its evolution. This would have students look into the assumptions that are made, the principles that are used, their translation into mathematical equations and their logical implications.

Another potential contribution of mathematics education to the study of complex systems would be to develop an understanding of the mathematical work involved in the simulations. If we are to teach modelling of complex systems to raise critical awareness in future citizens on important issues, then they must be equipped with the knowledge of how such models can produce results. This would mean crossing into algorithms and computational thinking.

5.5.3 Technology and Computational Thinking

In her seminal paper, Wing (2006) pleaded for the importance, in this 21st century, of developing *computational thinking* as a fundamental skill for everyone. Several elements she described can be associated with the modelling of complex dynamical systems or with the habits of mind that were unveiled within our experiment (Sect. 5.2):

> reformulating a seemingly difficult problem into one we know how to solve, perhaps by reduction, embedding, transformation, or simulation
>
> thinking recursively
>
> choosing an appropriate representation for a problem
>
> modeling the relevant aspects of a problem to make it tractable
>
> using invariants to describe a system's behavior succinctly and declaratively. (p. 33)

As summarised by Wing (2006, p. 34), "computational thinking is more than being able to program a computer. It requires *thinking at multiple levels of abstraction.*" It encompasses "the thought processes involved in formulating a problem and expressing its solution(s) in such a way that a computer—human or machine—can effectively carry out" (Wing 2014, p. 8). As the development of such a wide-ranging habit of mind may very well be influenced by the tools we use, and as the "practice of choosing effective computational tools" can also be considered part of computational thinking (Weintrop et al. 2016, p. 139), there may be a need to take a closer look at the tools and tasks with which K–12 students learn to model. The objective should be for students to grow with these tools, as they learn to explore, modify or build their own models of increasing complexity, and to develop through the process *conditional, recursive* and *multi-level thinking*. Some of the black boxes hidden in those tools should be opened progressively, as students learn the mathematics involved.

To support all of this, there was a general consensus within the working group that it would be beneficial to have students learn programming at school. But mathematics education could also help by building on concepts and approaches that are the most effective at opening the class of problems that can be solved. This brings us back to the mathematical content.

5.5.4 Curriculum and Mathematical Content

It may still be a question of perceived distance between the knowledge that appears useful to approach dynamical complex systems and the mathematical content to be taught, as prescribed by the curriculum. Yet, there are some elements in today's mathematics curriculum with which connections could be established or reinforced for the modelling and simulation of dynamical complex systems.

Prior learning of probability diagrams (grids, sets, trees,…) could be built upon to recognise and value such discrete structures as powerful models: not just for describing the situation and communicating the variables of interest (which already is an important asset for interdisciplinary collaborations), but also for suggesting the algorithm to do the mathematical work required to generate the solution.

Recurrence relations may be present in school mathematics, but they are sometimes limited to an auxiliary status to introduce or to characterise functions (e.g. Ministry of Education, Ontario (MEO) 2005). In the ICTMA book series, chapters addressing the use of discrete models are much fewer than those addressing the use of continuous functions. While there is a recognition that discrete models are rarely taught in school, and that students, when modelling a situation, go directly to well-known functions without much reflection (Kaiser et al. 2011), we see at the same time other researchers (e.g. Amit and Neria 2010) consider the recursive approaches that students naturally tend to adopt as a weaker form of generalisation than a functional approach. Without denying the importance of the concept of function, we may have to acknowledge that the attention it receives tends to eradicate other valid alternative modelling approaches, which could prove more effective in producing a solution. Encouraging students to describe patterns in a recursive fashion opens the door to computation on a spreadsheet and simulation. It may lead later to a better appreciation of how, when applied on a progressively smaller time scale, such change descriptions develop into differential equations, and how these can be integrated numerically, using the same recursive approach. With the ease of understanding Euler's method, the box of numerical integration of system dynamics software does not have to be black.

We could also value from a mathematical perspective explorations done with cellular automata; not only can they be seen as a generalisation in more dimensions of recursive formulas, where stability and periodicity can be simultaneously observed, but also they can be considered a valid initiation to the learning of multi-variable functions, where the space domain is discretised and where the table of values is stored in a matrix. Computing the state of a new cell based on the state of its neighbours is actually similar to what is done in numerical schemes that are used to solve partial differential equations. For instance, the partial differential equation that models 2D heat equilibrium can be discretised and transformed into an iterative numerical scheme in which the temperature at a point on a grid is repeatedly adjusted to the average of the temperature of its four closest neighbours on that grid. As numerical methods can sometimes be intuitive and disconcertingly easy, with only a few arithmetic operations involved, it may seem regrettable that these simple iterative schemes are not made part of the students' toolbox before they have managed to master all advanced calculus techniques.

Agent-based modelling may be harder to connect to specific mathematical content, but it could well be an interesting entry into programming, which constitutes, in itself, a very rich toolbox to tackle complexity and a limitless playground for using mathematics. If programming ever gets to be thought of as a mathematical activity, its learning and its use for integration of modelling in mathematics education could build on some mathematically productive habits of mind students may have developed in playing computer games.

5.6 Conclusion

Organising a mathematics curriculum around *habits of mind*, so that students "become comfortable with ill-posed and fuzzy problems" (Cuoco et al. 1996) appears more than relevant in our increasingly complex and vulnerable world. There is indeed a need both for future professionals to deal with our most pressing environmental and societal challenges, and for citizens in general to have a better understanding of what is at play when predictions, assessments or decisions are made, as "models have a huge impact on our world" (Doerr et al. 2017). Yet, some of the key mathematical habits of mind that are required for approaching complex systems are anchored in concepts, techniques and tools that, though accessible and sometimes even natural to secondary students, have received little attention in general mathematics education. If we consider that "mathematics is about the study of pattern and structure, and the logical analysis and calculation with patterns and structures" (Brown and Porter 1995), then we should be looking for a set of structures and approaches that, while allowing appreciation of the internal coherence of mathematics, could increase our capacity to understand our world and tackle its key challenges.

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