Erich Lehmann's Contributions to Orderings of Probability Distributions

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The initial contributions to the theory of comparison of experiments appeared in the late forties/early fifties (see, e.g., Bohnenblust, Shapley, and Sherman, 1949; Stein, 1951; Blackwell, 1951, 1953).

At around the same time, motivated by some problems in hypothesis testing, Erich Lehmann introduced some basic concepts in the theory of comparison of distributions. As later developments show, the two theories are strictly connected and they often examine conditions on two probability measures P and Q under which $\int \phi \, dP \leq \int \phi \, dQ$ for all functions ϕ in some class.

The success of the concept has been enormous both in probability theory and in applications in statistics, decision theory, economics, operations research, and reliability theory. We refer to Müller and Stoyan (2002); Shaked and Shanthikumar (2007) for an extensive analysis of stochastic orders and their applications.

Testing multiparameter hypotheses

Lehmann (1952) is the first paper where a monotonicity condition is considered, in the framework of hypothesis testing. The hypothesis studied in this paper concerns different parameters. It could be, for instance, of the type $H : \theta_1 \leq \theta_1^*, \theta_2 \leq \theta_2^*$. For this type of hypotheses an unbiased test does not exist in general.

Therefore a restriction of monotonicity is imposed, that is, the rejection region for H is assumed to be an increasing set. This makes sense when for $\theta_1 \leq \theta_1'$ and $\theta_2 \leq \theta_2'$ we have that F_{θ_1, θ_2} is stochastically dominated by $F_{\theta_1', \theta_2'}$. The uniformly most powerful test among all monotonic regression regions is found and it has a simple natural form.

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Ordered families of distributions

Lehmann (1955) introduces what is now known as the *usual stochastic order* and the *monotone likelihood ratio order*.

Using the same notation as in Lehmann's paper, two distributions P_θ and $P_{\theta'}$ can be compared for instance according to the following four criteria:

- (i) There exists a random vector Z and two functions g, g' such that $g(Z)$ has distribution P_θ , $g'(Z)$ has distribution $P_{\theta'}$ and $g(z) \leq g'(z)$ for all z .
- (ii) For every increasing function ϕ ,

$$\mathbb{E}_\theta \phi(X) \leq \mathbb{E}_{\theta'} \phi(X).$$

- (iii) For every increasing set S ,

$$P_\theta(S) \leq P_{\theta'}(S).$$

- (iv) Let P_θ and $P_{\theta'}$ have densities p_θ and $p_{\theta'}$ with respect to some common measure μ . Then the ratio $p_{\theta'}(x)/p_\theta(x)$ is increasing.

The three conditions (i), (ii), (iii) are equivalent and this has been proved later in great generality by Kamae, Krengel, and O'Brien (1977) using results by Strassen (1965). Lehmann (1955) shows the equivalence of (ii) and (iii) and provides two counterexamples (Examples 2.2 and 2.3 in the paper) to the equivalence of (i) and (ii). Unfortunately there is a mistake, since the two distributions used in the counterexample are actually not ordered.

Condition (iv) was studied extensively by Karlin and Rubin (1956a,b). For univariate distributions this condition is stronger than (i), but for multivariate distributions this is not the case, as Example 2.4 rightly shows. The sufficient multivariate condition that implies (i) is the following

$$p_{\theta'}(x)p_\theta(y) \leq p_\theta(x \wedge y)p_{\theta'}(x \vee y),$$

where $\wedge$ and $\vee$ indicate the componentwise minimum and maximum, respectively. This condition was studied by Karlin and Rinott (1980) and Whitt (1982).

Comparing location experiments

Lehmann (1988) starts with a review of the classical theory of comparison of experiments developed in the late forties by several scholars and culminated with Blackwell (1951, 1953). We refer to the book by Torgersen (1991) and to the nice survey by Le Cam (1996).

An experiment $\mathcal{E}$ is defined as $\{X, \mathcal{P}\}$, where X is a random quantity and $\mathcal{P} = \{P_\theta, \theta \in \Omega\}$ is a parametric family of possible distributions of X .

Call $\mathcal{F} = \{Y, \mathcal{Q} = \{Q_\theta, \theta \in \Omega\}\}$ another experiment where P_θ and Q_θ are indexed by the same state of nature. The experiment $\mathcal{F}$ is more informative than the experiment $\mathcal{E}$ if there exists a random quantity Z with known distribution and a function h such that for all $\theta \in \Omega$

Y is distributed as Q_θ implies that $h(Y, Z)$ is distributed as P_θ .

Blackwell proved that $\mathcal{F}$ is more informative than $\mathcal{E}$ if and only if for any decision procedure based on X and any loss function there exists a (possibly randomized) decision procedure based on Y that guarantees a weakly smaller risk.

Some examples show that the above criterion is too demanding and does not allow to compare experiments that should be intuitively comparable. Lehmann (1988) therefore suggests a weaker criterion that involves considering only a sub-class of decision problems, in particular the class $\mathcal{M}$ of monotone procedures introduced by Karlin and Rubin (1956b). In these procedures the correct action is a nondecreasing function $\alpha(\theta)$ and the loss function $L(\theta, d)$ is nondecreasing in $d - \alpha(\theta)$ for $d - \alpha(\theta) > 0$ and nonincreasing in $d - \alpha(\theta)$ for $d - \alpha(\theta) < 0$, with $L(\theta, \alpha(\theta)) = 0$.

Define $F_\theta(t) = P_\theta(X \leq t)$ and $G_\theta(t) = Q_\theta(Y \leq t)$, and let F_θ and G_θ have densities with monotone likelihood ratio. The main theorem states that Y is more effective than X with respect to the class $\mathcal{M}$ if and only if $h_\theta(t) = G_\theta^{-1}(F_\theta(t))$ is nondecreasing in θ for each t . This condition corresponds to the comparison of the two P-P plots $(F_{\theta_1}, F_{\theta_2})$ and $(G_{\theta_1}, G_{\theta_2})$, for $\theta_1 < \theta_2$, and, unlike Blackwell's condition, is easy to verify.

For location families with strongly unimodal densities the condition of the theorem reduces to domination in term of the "more spread out" order studied by Bickel and Lehmann (1979).

This idea has found extensive application in the economic literature (see, for instance, Persico, 2000; Levin, 2001; Athey and Levin, 2001; Bergemann and Välimäki, 2002; Quah and Strulovici, 2009, and references therein). In particular Jewitt (2007) examines conditions under which the Lehmann criterion is applicable to various economic problems.

Invariant directional orderings

Several stochastic orderings have been studied and used since Lehmann (1955). Some are stronger than others. For instance, the likelihood ratio order is stronger than the hazard order, which in turn is stronger than the usual stochastic order. These three orders are defined in totally different ways and the above chain of implications is not immediate. Lehmann and Rojo (1992) provide a unified way to define these three orders and many others.

The paper considers a class $\mathcal{F}$ of continuous strictly increasing distribution functions and a preorder $S \subset \mathcal{F} \times \mathcal{F}$.

A preorder S is invariant under monotone transformations if $(F, G) \in S$ implies $(F\psi^{-1}, G\psi^{-1}) \in S$ for all $\psi \in \Psi$, the class of strictly increasing continuous functions. If S is invariant under monotone transformations and $(F, G) \in S$, then $\mathcal{O}(F, G) \subset S$, where

$$\mathcal{O}(F, G) = \{(F\psi^{-1}, G\psi^{-1}) : \psi \in \Psi\}$$

is the orbit of (F, G) under the group of transformations Ψ .

Each orbit $\mathcal{O}(F, G)$ can be labeled by the function $k : [0, 1] \rightarrow [0, 1]$ defined as $k(u) = GF^{-1}(u)$. Different orderings can be characterized through properties of the function k . So a preorder S can be defined as

$$S = \{(F, G) : k(F, G) \in \mathcal{K}\}$$

for some class $\mathcal{K}$ that contain the identity function and is closed under composition. Different choices of $\mathcal{K}$ produce the likelihood ratio order, the hazard rate order, and the usual stochastic order.

This approach allows even to order ordered pairs. Let $(F_1, G_1), (F_2, G_2) \in S$. Then G_2 is said to be further to the right of F_2 than G_1 is of F_1 if $(k_1, k_2) \in S$, where $k_i = G_i F_i^{-1}$.

The paper also defines metrics that are coherent with the various orders, that is, metrics d_S that are invariant with respect to monotone transformations and such that $d_S(F_1, G_1) \leq d_S(F_2, G_2)$ whenever $(k_1, k_2) \in S$.

The technique discovered in this paper has been used by Bartoszewicz and Benduch (2009) to define orders based on a generalization of the total time on test transform.

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