The Asymptotic Distribution of Point Charges on a Conducting Sphere

Willem R. van Zwet

University of Leiden and University of North Carolina

Abstract. Consider n point charges, each with charge $\frac{1}{n}$, in electrostatic equilibrium on the surface $\mathcal{S}$ of a conducting sphere. It is shown that as n tends to infinity, the distribution of the total charge 1 on $\mathcal{S}$ tends to the uniform distribution on $\mathcal{S}$. Though this is an entirely deterministic result, the proof is probabilistic in nature.

1 Introduction

Consider n point charges (electrons), each with charge $\frac{1}{n}$, in equilibrium position on the surface $\mathcal{S}$ of a conducting unit sphere in $\mathbf{R}^3$. If d denotes Euclidean distance in $\mathbf{R}^3$, these charges will be located at points $\xi_{n1}, \xi_{n2}, \ldots, \xi_{nn}$ on $\mathcal{S}$ for which the potential energy

$$\sum_{i \neq j} \frac{1}{d(\xi_{ni}, \xi_{nj})}$$

is an absolute minimum. Let P_n denote the measure that assigns measure $\frac{1}{n}$ to each of the points $\xi_{n1}, \ldots, \xi_{nn}$ so that for any $E \subset \mathcal{S}$, $P_n(E)$ denotes the charge situated in E. Let λ be Lebesgue measure on $\mathcal{S}$ and $\Pi = \lambda/(4\pi)$ the uniform probability measure on $\mathcal{S}$. Is it true that P_n converges weakly to Π as $n \to \infty$? In other words, is the macroscopic model where the electrical charge is viewed as a continuous phenomenon compatible with the microscopic description in terms of point charges?

The problem of providing a rigorous proof of this intuitively obvious conjecture was raised by Korevaar (1972) and Robbins (1975). Two different proofs were given independently by Korevaar (1976) and van Zwet (1976), the former preceding the latter by some months. At the time, however, neither proof was published. Sixteen years later, matters of priority don't seem terribly relevant any more and since the present author's probabilistic proof is simple and perhaps somewhat amusing, it is presented here. We shall prove

Theorem 1 *P_n converges weakly to Π, so $\lim_{n \to \infty} P_n(B) = \Pi(B)$ for every Borel set $B \subset \mathcal{S}$ whose boundary relative to $\mathcal{S}$ has Π-measure zero.*

In fact we prove more. It will be shown that the result remains valid if $\mathcal{S}$ is replaced by an arbitrary compact set $K \subset \mathbf{R}^3$ with positive capacity and Π by the so-called minimizing measure P_0 on K. For a review of relevant literature see Korevaar (1976).

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2 Charges on a Compact Set

We begin by dealing with the general case. Let K be an infinite compact set in $\mathbf{R}^3$ and let $\mathcal{P}_n$ denote the class of probability measures that assign measure $\frac{1}{n}$ to each of n distinct points of K. Consider n identical point charges situated at points $x_1, x_2, \ldots, x_n$ in K. For convenience we take these charges to be $\frac{1}{n}$ so that the total charge is equal to 1. If the potential energy of this configuration is finite, the points $x_1, \ldots, x_n$ must be distinct and the charge distribution may be described by a measure $P \in \mathcal{P}_n$ assigning measure $\frac{1}{n}$ to $x_1, \ldots, x_n$. In this case the energy may be written as

$$\tilde{\psi}(P) = \frac{1}{n^2} \sum_{i \neq j} \frac{1}{d(x_i, x_j)} = \int_K \int_K \frac{1}{d(x,y)} 1_{\{x \neq y\}} \, dP(x) \, dP(y) \,. \tag{2.1}$$

Now let $\xi_{n1}, \ldots, \xi_{nn}$ be a configuration of the n point charges for which the energy is an absolute minimum. Because K is infinite and compact such configurations exist and have finite energy, and the points $\xi_{n1}, \ldots, \xi_{nn}$ are distinct. Let $P_n \in \mathcal{P}_n$ be the corresponding probability measure which puts mass $\frac{1}{n}$ at $\xi_{n1}, \ldots, \xi_{nn}$. Then clearly

$$\tilde{\psi}(P_n) \;=\; \int_K \int_K \frac{1}{d(x,y)} 1_{\{x \neq y\}} \, dP_n(x) \, dP_n(y) \;=\; \min_{P \in \mathcal{P}_n} \tilde{\psi}(P) \;<\; \infty \,. \tag{2.2}$$

Instead of the above discrete model with indivisible point charges which are not subject to internal forces that would make them explode, one can also consider a model where charge is viewed as a "continuous" phenomenon. In this model the distribution of a total charge 1 on K is given by a measure P in the class $\mathcal{P}$ of all probability measures on the Borel sets in K, where $P(B)$ denotes the charge in the Borel set B. For $P \in \mathcal{P}$ and $x \in \mathbf{R}^3$ one defines the potential of P by

$$U(P, x) \;=\; \int_K \frac{1}{d(x,y)} dP(y) \,, \tag{2.3}$$

the energy of P by

$$\psi(P) \;=\; \int_K \int_K \frac{1}{d(x,y)} dP(x) \, dP(y) \;=\; \int_K U(P,x) \, dP(x) \tag{2.4}$$

and the capacity of the set K by

$$C(K) \;=\; \left[\inf_{P \in \mathcal{P}} \psi(P) \right]^{-1} . \tag{2.5}$$

Note that under this model the presence of a point charge implies infinite energy.

If $C(K) = 0$, then clearly $\psi(P) = \infty$ for every $P \in \mathcal{P}$. On the other hand, if $C(K) > 0$, then there exists a unique measure $P_0 \in \mathcal{P}$ for which ψ assumes its absolute minimum on $\mathcal{P}$ (c.f. Landkof (1972) p.131–133). P_0 is called the minimizing measure on K and represents the equilibrium (i.e. minimum energy) distribution of a charge 1 on K under the continuous model. Note that since

$$\psi(P_0) = \min_{P \in \mathcal{P}} \psi(P) = \frac{1}{C(K)}, \tag{2.6}$$

the assumption $C(K) > 0$ ensures that P_0 assigns measure zero to single points and that the compact set K must therefore be non–denumerable.

Theorem 2 *Let K be a compact set in $\mathbf{R}^3$ with positive capacity. Then P_n converges weakly to P_0, so $\lim_{n\to\infty} P_n(B) = P_0(B)$ for every Borel set $B \subset K$ whose boundary relative to K has P_0–measure zero.*

Proof. Let $X_1, \ldots, X_n$ be independent random points in K that are identically distributed according to the probability measure P_0. Since P_0 assigns probability zero to single points, the points $X_1, \ldots, X_n$ are distinct with probability 1. Hence (2.1) and (2.2) imply that with probability 1

$$\tilde{\psi}(P_n) \le \frac{1}{n^2} \sum_{i \ne j} \frac{1}{d(X_i, X_j)}.$$

Taking the expectation on the right we find

$$\tilde{\psi}(P_n) \le \frac{n-1}{n} \int_K \int_K \frac{1}{d(x,y)} dP_0(x)\, dP_0(y) = \frac{n-1}{n} \psi(P_0). \tag{2.7}$$

Now let $Q_n = P_n \times P_n$ denote the product measure on $K \times K$. Because $K \times K$ is compact, the set $\{Q_n : n = 1, 2, \ldots\}$ is relatively compact in the topology of weak convergence $\left(\int f dQ_n \to \int f dQ \right.$ for bounded continuous $\left. f \right)$. To show that Q_n converges weakly to a probability measure Q on $K \times K$ it is therefore sufficient to show that every weakly convergent subsequence has limit Q. Let Q_{n_k}, $k = 1, 2, \ldots$, denote such a weakly converging subsequence with limit Q_0 and define a bounded and continuous function f_c on $K \times K$ by $f_c(x,y) = \min(c, 1/d(x,y))$ for $c > 0$. Noting that Q_n assigns probability $\frac{1}{n}$ to the set $\{(x,y) : x = y\}$ we see that for every $c > 0$,

$$\begin{aligned}
\liminf_{k\to\infty} \tilde{\psi}(P_{n_k}) &= \liminf_{k\to\infty} \int_{K\times K} \frac{1}{d(x,y)} 1_{\{x \ne y\}} dQ_{n_k}(x,y) \\
&\ge \liminf_{k\to\infty} \int_{K\times K} f_c(x,y) 1_{\{x\ne y\}} dQ_{n_k}(x,y) = \liminf_{k\to\infty} \Big[\int_{K\times K} f_c dQ_{n_k} - \frac{c}{n_k} \Big] \\
&= \liminf_{k\to\infty} \int_{K\times K} f_c dQ_{n_k} = \int_{K\times K} f_c dQ_0,
\end{aligned}$$

so that the monotone convergence theorem implies that

$$\liminf_{k\to\infty} \tilde{\psi}(P_{n_k}) \ge \int_{K\times K} \frac{1}{d(x,y)} dQ_0(x,y). \tag{2.8}$$

For every n, Q_n is the product of two identical probability measures on K and Q_0 must therefore have the same structure, say $Q_0 = P_0' \times P_0'$ with $P_0' \in \mathcal{P}$. Hence (2.7) and (2.8) yield

$$\psi(P_0') \leq \liminf_{k\to\infty} \tilde{\psi}(P_{n_k}) \leq \liminf_{k\to\infty} \frac{n_k - 1}{n_k} \psi(P_0) = \psi(P_0)$$

and since P_0 is the unique measure in $\mathcal{P}$ minimizing ψ, we find that $P_0' = P_0$ so that $Q_0 = P_0 \times P_0$. Since the limit Q_0 is independent of the weakly convergent subsequence Q_{n_k} we have chosen, it follows that Q_n converges weakly to $P_0 \times P_0$ and hence that P_n converges weakly to P_0.

3 Charges on the Surface of a Sphere

It remains to consider the special case where $K = \mathcal{S}$. Clearly $\mathcal{S}$ is compact and one easily verifies that the uniform probability measure $\Pi = \lambda/(4\pi)$ on $\mathcal{S}$ has finite energy $\psi(\Pi)$ and a constant potential $U(\Pi, x)$ for $x \in \mathcal{S}$. But this implies that $\mathcal{S}$ has positive capacity and that Π is the unique minimizing measure on $\mathcal{S}$ (see Landkof (1972, p. 137). Theorem 1 is therefore an immediate consequence of Theorem 2.

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