High-Order Boltzmann Machines for MAX-SAT and SAT

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We propose a mapping of the MAX-SAT problem, in the propositional calculus setting, into High-order Boltzmann Machines (HOBM). The approximate solution of MAX-SAT can be used as an approximate answer to the SAT question. An extensive experimental study of the behavior of HOBM for this problem has been conducted.

A HOBM can be defined as a pair (U, Λ) , where U is the set of units, $U = \{u_1, \dots, u_N\}$, Λ is the set of connections and $\lambda \in \Lambda$ denotes a connection. The configuration or state of the BM with N units is defined by $\mathbf{x} \in \{0, 1\}^N$ and $x_i \in \{0, 1\}$ is the state of the unit in the configuration $\mathbf{x}$, a connection is a set of units $\lambda \in \Lambda, \lambda \subseteq U$ and $\lambda \neq \emptyset$. The set of connections which contain the unit u_j , is denoted by $\Lambda^j = \{\lambda / u_j \in \lambda\}$. The connection strengths are given by a map $w: \Lambda \rightarrow \mathbb{R}$. We denote w_λ the weight of the connection λ . The consensus or energy function associates a real value to every configuration $C: \{0, 1\}^N \rightarrow \mathbb{R}$, is defined as:

$$C(\mathbf{x}) = \sum_{\lambda \in \Lambda} w_\lambda \prod_{i \in \lambda} x_i$$

We will state MAX-SAT as 0-1 nonlinear programming problem. Let $\{x_i \in L\}$ be a set of $\{0, 1\}$ variables. That is, we have a variable x_i for each literal, x_i takes value 0 when the literal is false, and 1 when true. MAX-SAT can be stated as the following maximization problem:

$$\max \left(\sum_{j=1}^{NC} (1 - \bar{c}_j) \right) \quad \text{where } \bar{c}_j = \prod_{i \in c_j} (1 - x_i) \quad \text{subject to the constraint } x_p = 1 - x_{\neg p} \quad \forall p \in P$$

Note that each term in the addition will be either 0 (when the clause is false) or 1 (when the clause is true). The products, in fact, compute the truth value of the clause negation. Finally, the constraint implies that each literal and its negation must have opposite truth values. In other words, the values of the variables x_i can be interpreted as a truth assignment to the propositions in P .

The formulation of MAX-SAT as a 0-1 programming problem leads to an immediate mapping of the problem into HOBM. Given a propositional expression E defined over the set of propositions P , an HOBM $= (U, \Lambda)$ for the approximate solution of MAX-SAT is built up as follows: The set of units is defined assigning an unit to each literal in the set of literals L , $U = \{u_i | i \in L\}$. The state x_i of each unit corresponds to the 0-1 variables. The set of connections is partitioned into two disjoint subsets $\Lambda = \Lambda^c \cup \Lambda^*$. The clauses are modeled by the connections in Λ^c . Those connections are constructed, following the formulation suggested above, as the clauses negations:

$$\Lambda^c = \{\lambda_j \text{ s.t. } u_{\neg i} \in \lambda_j \Leftrightarrow i \in c_j \quad j = 1..NC\}$$

Connections modelling clauses have a negative identical weight $w_\lambda = -\omega_c \lambda \in \Lambda^c$. The connections in Λ^* model the constraint that variables corresponding to opposite literals must have opposite 0-1 values. In its turn Λ^* is partitioned into bias and inhibitory connections:

$$\Lambda^* = \Lambda^i \cup \Lambda^b$$

$$\Lambda^b = \{\lambda_i = \{u_i\} \quad \forall u_i \in U\} \quad \Lambda^i = \{\lambda_p = \{u_p, u_{\neg p}\} \quad \forall p \in P\}$$

Inhibitory connections have a negative value $-\omega_i$ to force opposite units to take opposite states, and the bias connections have a positive value ω_b to avoid the null configuration (all units in state 0) to be an optimum of the consensus function. This last condition may appear because all the weights defined so far have negative values, our units are 0-1 variables and the assumed dynamic of the HOBM is the maximization of the consensus function.

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