ILLUMINATING SETS OF CONSTANT WIDTH

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Abstract. The problem of illuminating the boundary of sets having constant width is considered and a bound for the number of directions needed is given. As a corollary, an estimate for Borsuk's partition problem is inferred. Also, the illumination number of sufficiently symmetric strictly convex bodies is determined.

§1. Introduction. Let $\mathbf{x}$ be a point on the boundary ∂K of a convex body* K in Euclidean space, $\mathbf{R}^n$. A direction $\mathbf{u} \in S^{n-1}$ is said to *illuminate* K at $\mathbf{x}$ if the line $\{\mathbf{x} + t\mathbf{u} \mid t \in \mathbf{R}\}$ "enters" K in $\mathbf{x}$. More precisely, $\mathbf{u}$ illuminates K at $\mathbf{x}$ if $\mathbf{x} + t\mathbf{u}$ is an interior point of K , for some positive t . Instead of saying " $\mathbf{u}$ illuminates K at $\mathbf{x}$ " we will just say " $\mathbf{u}$ illuminates $\mathbf{x}$ ". This will cause no confusion, because the convex body K should be clear from the context.

Directions $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m \in S^{n-1}$ are said to illuminate K if every point on the boundary of K is illuminated by at least one of these directions. We denote by $I(K)$ the minimal number of directions sufficient to illuminate K and call it the *illumination number* of K . This concept of illumination was introduced by V. G. Boltjansky in [2]. There he proved that for convex bodies K , $I(K)$ is equal to $H(K)$, the number of smaller, positively homothetic copies of K required to cover K (see [3]).

The maximum values of $I(K)$ for convex bodies K in $\mathbf{R}^n$, are unknown when $n > 2$. By the above result of Boltjansky, Hadwiger's conjecture, about the covering of a set by homothetic copies of it, is equivalent to

$$I(\text{convex body in } \mathbf{R}^n) \leq I(n\text{-dimensional parallelogram}) = 2^n.$$

See [3] for a discussion of this conjecture.

A set of *constant width* d is a convex body such that the distance between any two distinct parallel supporting hyperplanes of it is d (see [5, pp. 122–131], [4]). In this note we prove

THEOREM 1. *If W is a set of constant width in $\mathbf{R}^n$ then*

$$I(W) < 5n\sqrt{n}(4 + \log n) \left(\frac{3}{2}\right)^{n/2}.$$

The research exposed in this note was done while I was at the Hebrew University in Jerusalem, as a student of Professor Gil Kalai. I am grateful to Kalai for his interest, encouragement and advice, and for valuable improvements of the original manuscript.

* A convex body is a compact convex set that has interior points.

Using this theorem, and the equivalence $I(W) = H(W)$, we get

COROLLARY 2. *Every set of constant width $W \subset \mathbf{R}^n$ can be covered by less than $5n\sqrt{n}(4 + \log n)(3/2)^{n/2}$ homothetic copies of itself, having some homothety coefficient α , $0 < \alpha < 1$.*

Since every bounded set of positive diameter is contained in a set of constant width having the same diameter (see [5, p. 126]), we have the following estimate for Borsuk's partition problem.

COROLLARY 3. *Every set of diameter d ($0 < d < \infty$) in $\mathbf{R}^n$ can be covered by less than $5n\sqrt{n}(4 + \log n)(3/2)^{n/2}$ sets having smaller diameters.*

To the best of our knowledge, this is, asymptotically, the best bound known for Borsuk's partition problem (see [7] for references and results concerning this problem). The approach to Borsuk's problem through the illumination problem is suggested in [3] and [8, p. 420].

Theorem 1 is proved using a probabilistic argument: The probability that a "small" region in ∂W will be illuminated by a random, uniformly distributed, direction is estimated from below. A straightforward computation then shows that if "enough" directions are chosen randomly, uniformly, and independently, the probability that they will completely illuminate W is nonzero. An essential part of the proof relies on finding lower bounds for volumes of spherical sets of a certain type. In [11] analogous results for sets in $\mathbf{R}^n$ give lower bounds for the volumes of sets having constant width.

The only known result in the direction opposite to Theorem 1 is that $I(K) \geq n + 1$ for every convex body $K \subset \mathbf{R}^n$ ([3]). It is not known if there is a set of constant width $W \subset \mathbf{R}^n$, that cannot be illuminated by $n + 1$ directions.

Rogers [10] has shown that every $K \subset \mathbf{R}^n$, having diameter $0 < d < \infty$ and invariant under the group of congruences that leave invariant an n -dimensional regular simplex, can be partitioned into $n + 1$ subsets, each with diameter $< d$. Inspired by this, we prove in Section 4 that if K is a strictly convex body† invariant under a group of orthogonal transformations that is generated by reflections through hyperplanes and acts irreducibly‡ on $\mathbf{R}^n$, then $I(K) = n + 1$. This can give an alternate proof of Rogers' result.

§2. First we will introduce some notation and give a condition for the illumination of a single boundary point. Throughout this note, K will denote an arbitrary convex body, and W will denote a set of constant width in $\mathbf{R}^n$. For a set $A \subset S^{n-1}$ we define

$$A^+ = \{\mathbf{u} \in S^{n-1} \mid \mathbf{u} \cdot \mathbf{v} > 0 \text{ for all } \mathbf{v} \in A\}.$$

† A strictly convex body is a convex body whose boundary contains no line segment.

‡ "G acts irreducibly on $\mathbf{R}^n$ " means that there is no subspace of $\mathbf{R}^n$, other than 0 and $\mathbf{R}^n$, which is invariant under all elements of G.

When $\mathbf{x}$ is a boundary point of a convex body K , we use $N_K(\mathbf{x})$ to denote the set of inward normal unit vectors of K at $\mathbf{x}$:

$$N_K(\mathbf{x}) = \{\mathbf{u} \in S^{n-1} \mid \mathbf{u} \cdot \mathbf{p} \geq \mathbf{u} \cdot \mathbf{x} \text{ for all } \mathbf{p} \in K\}.$$

Notice that $N_K(\mathbf{x})$ is nonempty for $\mathbf{x} \in \partial K$.

LEMMA 4. *Let K be a convex body in $\mathbf{R}^n$, let $\mathbf{x}$ be a boundary point of K , and let $\mathbf{u} \in S^{n-1}$. Then $\mathbf{x}$ is illuminated by the direction $\mathbf{u}$, if, and only if, $\mathbf{u} \in N_K(\mathbf{x})^+$.*

Proof. Suppose $\mathbf{u} \in N_K(\mathbf{x})^+$. If the line $L = \{\mathbf{x} + t\mathbf{u} \mid t \in \mathbf{R}\}$ contains no interior points of K , then it is a supporting line of K . Therefore there is a hyperplane, $H = \{\mathbf{p} \in \mathbf{R}^n \mid \mathbf{p} \cdot \mathbf{w} = \mathbf{x} \cdot \mathbf{w}\}$, supporting K and containing L . Thus one of the vectors $\pm \mathbf{w}/\|\mathbf{w}\|$ is in $N_K(\mathbf{x})$. That is a contradiction to $\mathbf{u} \in N_K(\mathbf{x})^+$, because $H \supset L$ implies that $\mathbf{w} \cdot \mathbf{u} = 0$. Therefore L contains interior points of K . Pick any $\mathbf{v} \in N_K(\mathbf{x})$. K is contained in the half space $\{\mathbf{p} \in \mathbf{R}^n \mid \mathbf{p} \cdot \mathbf{v} \geq \mathbf{x} \cdot \mathbf{v}\}$. Since $\mathbf{u} \cdot \mathbf{v} > 0$, this means that the points $\mathbf{x} + t\mathbf{u}$ with $t < 0$ are not in K . Thus the interior points of L correspond to positive values of t , and $\mathbf{u}$ illuminates $\mathbf{x}$.

Now suppose that $\mathbf{u}$ illuminates $\mathbf{x}$. Let t be a positive number such that $\mathbf{p} = \mathbf{x} + t\mathbf{u}$ is an interior point of K . Because $\mathbf{p}$ is an interior point, for every $\mathbf{v} \in N_K(\mathbf{x})$ we have $\mathbf{p} \cdot \mathbf{v} > \mathbf{x} \cdot \mathbf{v}$, which implies $\mathbf{u} \cdot \mathbf{v} > 0$, so $\mathbf{u} \in N_K(\mathbf{x})^+$.

From now on, we work with an arbitrary, but fixed, set of constant width $W \subset \mathbf{R}^n$. The lemma above shows that if E is a subsets of the boundary of W , then one direction can illuminate E , if, and only if,

$$\bigcap_{\mathbf{x} \in E} N_W(\mathbf{x})^+ = \left(\bigcup_{\mathbf{x} \in E} N_W(\mathbf{x}) \right)^+$$

is nonempty. The following proposition will help us find subsets of ∂W that are "easily" illuminated. For a subset $A \subset S^{n-1}$ define $U_W(A)$ to be the union of the sets $N_W(\mathbf{x})$, $\mathbf{x} \in \partial W$, that intersect A :

$$U_W(A) = \bigcup_{N_W(\mathbf{x}) \cap A \neq \emptyset} N_W(\mathbf{x}).$$

A direction in $U_W(A)^+$ illuminates every point $\mathbf{x} \in W$ that satisfies

$$N_W(\mathbf{x}) \cap A \neq \emptyset.$$

In order to show that when A is chosen properly these points are "easily" illuminated, we want to prove that $U_W(A)^+$ is "large". Our means of doing so is by estimating the diameter of $U_W(A)$. (We view S^{n-1} with the metric induced by the Euclidean metric in $\mathbf{R}^n$. The diameters of subsets of S^{n-1} refer to this metric.)

PROPOSITION 5. *Let A be a nonempty subset of S^{n-1} . Then*

$$\text{diameter } U_W(A) \leq 1 + \text{diameter } A.$$

Proof. Since $U_W(A)$ does not change when we replace W with a positively homothetic copy of itself, we may, and will, assume that W has constant width

1, and therefore also diameter 1. Let $\mathbf{v}_1, \mathbf{v}_2$ be unit vectors in $U_W(A)$. By the definition of $U_W(A)$, there are points $\mathbf{x}_1, \mathbf{x}_2 \in \partial W$ such that $\mathbf{v}_i \in N_W(\mathbf{x}_i)$ and $N_W(\mathbf{x}_i) \cap A \neq \emptyset$ for $i = 1, 2$. Suppose $\mathbf{u}_i$ is in $N_W(\mathbf{x}_i) \cap A$, $i = 1, 2$. Since $\mathbf{u}_i$ is an inward normal of W at $\mathbf{x}_i$, and since W has constant width 1, the hyperplane $\{\mathbf{p} \in \mathbb{R}^n \mid \mathbf{p} \cdot \mathbf{u}_i = \mathbf{x}_i \cdot \mathbf{u}_i + 1\}$ is a supporting hyperplane of W . The only point on this hyperplane whose distance from $\mathbf{x}_i$ is not greater than 1 is $\mathbf{x}_i + \mathbf{u}_i$. Since diameter $W = 1$, we conclude that $\mathbf{x}_i + \mathbf{u}_i \in \partial W$ for $i = 1, 2$. Therefore

$$1 = (\text{diameter } W)^2 \geq \|(\mathbf{x}_1 + \mathbf{u}_1) - \mathbf{x}_2\|^2 = \|\mathbf{x}_1 - \mathbf{x}_2\|^2 + 2\mathbf{u}_1 \cdot (\mathbf{x}_1 - \mathbf{x}_2) + 1$$

and

$$1 = (\text{diameter } W)^2 \geq \|(\mathbf{x}_2 + \mathbf{u}_2) - \mathbf{x}_1\|^2 = \|\mathbf{x}_2 - \mathbf{x}_1\|^2 + 2\mathbf{u}_2 \cdot (\mathbf{x}_2 - \mathbf{x}_1) + 1.$$

Summing these inequalities and rearranging we get

$$(\mathbf{u}_1 - \mathbf{u}_2) \cdot (\mathbf{x}_2 - \mathbf{x}_1) \geq \|\mathbf{x}_2 - \mathbf{x}_1\|^2.$$

Because $(\mathbf{u}_1 - \mathbf{u}_2) \cdot (\mathbf{x}_2 - \mathbf{x}_1) \leq \|\mathbf{u}_1 - \mathbf{u}_2\| \|\mathbf{x}_2 - \mathbf{x}_1\|$, this implies $\|\mathbf{u}_1 - \mathbf{u}_2\| \geq \|\mathbf{x}_2 - \mathbf{x}_1\|$. So

$$\|\mathbf{x}_2 - \mathbf{x}_1\| \leq \text{diameter } A. \quad (2.1)$$

As with $\mathbf{x}_i + \mathbf{u}_i$, the points $\mathbf{x}_i + \mathbf{v}_i$ lie in ∂W , and therefore

$$1 \geq \|(\mathbf{x}_1 + \mathbf{v}_1) - (\mathbf{x}_2 + \mathbf{v}_2)\| \geq \|\mathbf{v}_1 - \mathbf{v}_2\| - \|\mathbf{x}_2 - \mathbf{x}_1\|.$$

Using (2.1), this implies

$$1 + \text{diameter } A \geq \|\mathbf{v}_1 - \mathbf{v}_2\|.$$

We use μ to denote the standard probability measure on S^{n-1} . Define

$$g(n, d) = \inf \{ \mu(A^+) \mid A \subset S^{n-1}, \text{diameter } A \leq d \}.$$

Let $N(n, \varepsilon)$ be the number of sets having diameter ε that is required to cover S^{n-1} . The core of this note is:

PROPOSITION 6. For $0 < \varepsilon < \sqrt{2} - 1$ we have

$$I(W) \leq 1 + \frac{\log N(n, \varepsilon)}{-\log(1 - g(n, 1 + \varepsilon))}.$$

Proof. It is easily verified that $0 < g(n, 1 + \varepsilon) < 1$ (if $\emptyset \neq A \subset S^{n-1}$ then A^+ is contained in a hemisphere, so that $\mu(A^+) \leq \frac{1}{2}$. If also diameter $A = d < \sqrt{2}$ then A^+ contains a spherical cap of radius $\sqrt{2} - d$ around any point of A), so that the right-hand side is well defined. Let M be a natural number satisfying

$$M > \frac{\log N(n, \varepsilon)}{-\log(1 - g(n, 1 + \varepsilon))}.$$

It is sufficient to show that M directions can illuminate W . Set $N = N(n, \varepsilon)$, and let $A_1, \dots, A_N$ be a covering of S^{n-1} with sets of diameter ε . By Proposition 5, we have diameter $U_W(A_i) \leq 1 + \varepsilon$, and therefore

$$\mu(U_W(A_i)^+) \geq g(n, 1 + \varepsilon), \quad i = 1, 2, \dots, N.$$

Pick M directions $\mathbf{u}_1, \dots, \mathbf{u}_M$ at random, uniformly and independently distributed on S^{n-1} . Take any $i, j, 1 \leq i \leq N, 1 \leq j \leq M$. The probability that $\mathbf{u}_j$ will be in $U_W(A_i)^+$ is $\mu(U_W(A_i)^+)$, which is at least $g(n, 1 + \varepsilon)$. Therefore the probability that $U_W(A_i)^+$ will contain none of the points $\mathbf{u}_1, \dots, \mathbf{u}_M$ is at most $(1 - g(n, 1 + \varepsilon))^M$. Thus the probability p that at least one $U_W(A_l)^+, 1 \leq l \leq N$ will contain no points of $\mathbf{u}_1, \dots, \mathbf{u}_M$ satisfies

$$p \leq \sum_{l=1}^N (1 - g(n, 1 + \varepsilon))^M < N(1 - g(n, 1 + \varepsilon))^{\log N / -\log(1 - g(n, 1 + \varepsilon))} = 1.$$

This shows that one can choose M directions, so that each set $U_W(A_l)^+, l = 1, \dots, N$, contains at least one of them. Let $\mathbf{v}_1, \dots, \mathbf{v}_M$ be such directions, and let $\mathbf{x}$ be a point of ∂W . We claim that one of these directions illuminates $\mathbf{x}$. Since $N_W(\mathbf{x})$ is nonempty, and the sets $A_1, \dots, A_N$ cover S^{n-1} , one of them, say A_i , intersects $N_W(\mathbf{x})$. By the definition of $U_W(A_i)$, we have $N_W(\mathbf{x}) \subset U_W(A_i)$. So that

$$N_W(\mathbf{x})^+ \supset U_W(A_i)^+.$$

$U_W(A_i)^+$ contains at least one of $\mathbf{v}_1, \dots, \mathbf{v}_M$, say $\mathbf{v}_k$. We have

$$\mathbf{v}_k \in U_W(A_i)^+ \subset N_W(\mathbf{x})^+$$

and therefore, by Lemma 4, $\mathbf{v}_k$ illuminates $\mathbf{x}$. This shows that the directions $\mathbf{v}_1, \dots, \mathbf{v}_M$ illuminate W .

In order to deduce Theorem 1 from Proposition 6, we only have to estimate $g(n, 1 + \varepsilon)$ from below, and $N(n, \varepsilon)$ from above. The former will be done in the next section, and the latter is dealt with by the following well known fact.

LEMMA 7. $N(n, \varepsilon) \leq (1 + 4/\varepsilon)^n$.

Proof. Let E be a maximal subset of S^{n-1} having the property that $\|\mathbf{u} - \mathbf{v}\| > \frac{1}{2}\varepsilon$ for $\mathbf{u} \neq \mathbf{v}$ in E . The maximality of E shows that the balls with radius $\frac{1}{2}\varepsilon$ and centers in E cover S^{n-1} , therefore

$$|E| \geq N(n, \varepsilon).$$

All the balls $B(\mathbf{u}, \varepsilon/4)$, $\mathbf{u} \in E$, are disjoint and are contained in the ball $B(0, 1 + \varepsilon/4)$. Comparing volumes gives

$$|E|(\varepsilon/4)^n \leq (1 + \varepsilon/4)^n,$$

or

$$|E| \leq \left(\frac{4}{\varepsilon} + 1\right)^n.$$

Remark. Better estimates for $N(n, \varepsilon)$ are known (see [9]), but do not seem to contribute any significant improvement to Theorem 1.

§3. In this section we give a lower bound for $g(n, d)$, and prove Theorem 1.

PROPOSITION 8. Let $d > 0$ and let A be a nonempty subset of S^{n-1} having diameter $\leq d$. Suppose $\mathbf{u} \in S^{n-1}$, $a > 0$ and A is contained in the half-space $\{\mathbf{p} \in \mathbf{R}^n \mid \mathbf{p} \cdot \mathbf{u} \geq a\}$, then

$$A^+ \cup TA^+ \supset D_0(\mathbf{u}, \arctan(2a/d)),$$

where $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$ is the reflection through the line determined by $\mathbf{u}$, $-\mathbf{u}$:

$$T\mathbf{p} = 2(\mathbf{p} \cdot \mathbf{u})\mathbf{u} - \mathbf{p},$$

and $D_0(\mathbf{u}, \psi)$ is the open spherical cap consisting of all unit vectors having an angle with $\mathbf{u}$, which is smaller than ψ .

Proof. Suppose $\mathbf{x}$ is a point in S^{n-1} but not in $A^+ \cup TA^+$, and let θ be the angular distance between $\mathbf{x}$ and $\mathbf{u}$, $0 \leq \theta \leq \pi$. Write

$$\mathbf{x} = (\cos \theta)\mathbf{u} + (\sin \theta)\mathbf{v}, \quad (3.1)$$

where $\mathbf{v}$ is a unit vector orthogonal to $\mathbf{u}$ (we ignore the trivial case $n = 1$). Since $\mathbf{x} \notin A^+$, there is a point $\mathbf{y} \in A$ with

$$0 \geq \mathbf{y} \cdot \mathbf{x} = \mathbf{y} \cdot \mathbf{u} \cos \theta + \mathbf{y} \cdot \mathbf{v} \sin \theta. \quad (3.2)$$

Since $T^{-1} = T$ and $\mathbf{x} \notin TA^+$ we have $T\mathbf{x} \notin A^+$. Thus there is a point $\mathbf{z} \in A$ with

$$0 \geq \mathbf{z} \cdot T\mathbf{x} = \mathbf{z} \cdot \mathbf{u} \cos \theta - \mathbf{z} \cdot \mathbf{v} \sin \theta. \quad (3.3)$$

Summing (3.2) and (3.3), and using $\|\mathbf{y} - \mathbf{z}\| \leq d$, $\sin \theta \geq 0$, we have

$$0 \geq (\mathbf{y} \cdot \mathbf{u} + \mathbf{z} \cdot \mathbf{u}) \cos \theta + (\mathbf{y} - \mathbf{z}) \cdot \mathbf{v} \sin \theta \geq (\mathbf{y} \cdot \mathbf{u} + \mathbf{z} \cdot \mathbf{u}) \cos \theta - d \sin \theta. \quad (3.4)$$

Temporarily suppose that $\theta < \frac{1}{2}\pi$. Then $\cos \theta > 0$, so by (3.4)

$$\tan \theta \geq \frac{\mathbf{y} \cdot \mathbf{u} + \mathbf{z} \cdot \mathbf{u}}{d} \geq \frac{2a}{d}.$$

The last inequality is justified by the hypothesis that $A \subset \{\mathbf{p} \in \mathbf{R}^n \mid \mathbf{p} \cdot \mathbf{u} \geq a\}$. Whether or not $\theta < \frac{1}{2}\pi$, we have $\theta \geq \arctan(2a/d)$. This shows that $A^+ \cup TA^+ \supset D_0(\mathbf{u}, \arctan(2a/d))$.

PROPOSITION 9.

$$g(n, d) \geq \frac{1}{\sqrt{8\pi n}} \left(\frac{3}{2} + \frac{(2n+1)d^2 - (2n+2)}{4n+4-2d^2n} \right)^{-(n-1)/2} \quad \text{for } 0 < d \leq \sqrt{2}.$$

Proof. Let $0 < d \leq \sqrt{2}$ and let A be a nonempty subset of S^{n-1} having diameter $\leq d$. By Jung's theorem [5, p. 111], there is, in $\mathbf{R}^n$, a ball with radius $d\sqrt{n/(2n+2)}$ containing A . Let $\mathbf{q}$ be the center of this ball. Write $\mathbf{q} = t\mathbf{u}$ with $t \geq 0$ and $\mathbf{u} \in S^{n-1}$. For every $\mathbf{x} \in A$ we have

$$d^2 \frac{n}{2n+2} \geq \|\mathbf{x} - \mathbf{q}\|^2 = \|\mathbf{x} - t\mathbf{u}\|^2 = 1 - 2t\mathbf{x} \cdot \mathbf{u} + t^2. \quad (3.5)$$

Since $1 - 2t\mathbf{x} \cdot \mathbf{u} + t^2 = 1 - (\mathbf{x} \cdot \mathbf{u})^2 + (\mathbf{x} \cdot \mathbf{u} - t)^2 \geq 1 - (\mathbf{x} \cdot \mathbf{u})^2$, inequality (3.5)

implies

$$d^2 \frac{n}{2n+2} \geq 1 - (\mathbf{x} \cdot \mathbf{u})^2. \quad (3.6)$$

From (3.5), $t \geq 0$ and $d \leq \sqrt{2}$ it can be seen that $\mathbf{x} \cdot \mathbf{u} \geq 0$, so, using (3.6), we obtain

$$\mathbf{x} \cdot \mathbf{u} \geq \sqrt{1 - d^2 \frac{n}{2n+2}}.$$

Set

$$a = \sqrt{1 - d^2 \frac{n}{2n+2}}. \quad (3.7)$$

The above argument shows that A is contained in the half-space

$$\{\mathbf{p} \in \mathbf{R}^n \mid \mathbf{p} \cdot \mathbf{u} \geq a\}.$$

Proposition 8 can be applied, yielding

$$A^+ \cup TA^+ \supset D_0(\mathbf{u}, \arctan(2a/d)).$$

Now, since T is an orthogonal transformation, we have $\mu(A^+) = \mu(TA^+)$, and

$$\begin{aligned} \mu(A^+) &= \frac{1}{2}(\mu(A^+) + \mu(TA^+)) \geq \frac{1}{2}\mu(A^+ \cup TA^+) \\ &\geq \frac{1}{2}\mu\left(D_0\left(\mathbf{u}, \arctan \frac{2a}{d}\right)\right) = \frac{1}{2} \frac{\text{Vol}_{n-1} D_0(\mathbf{u}, \arctan 2a/d)}{\text{Vol}_{n-1} S^{n-1}} \\ &= \frac{\text{Vol}_{n-1} D_0(\mathbf{u}, \arctan 2a/d)}{2n\Omega_n}. \end{aligned} \quad (3.8)$$

Here Ω_n denotes the volume of the n -dimensional unit ball, and Vol_{n-1} is the $(n-1)$ -dimensional volume.

Let D' be the orthogonal projection of $D_0(\mathbf{u}, \arctan 2a/d)$ to the hyperplane $\{\mathbf{p} \in \mathbf{R}^n \mid \mathbf{p} \cdot \mathbf{u} = 0\}$. Obviously we have

$$\text{Vol}_{n-1} D_0\left(\mathbf{u}, \arctan \frac{2a}{d}\right) \geq \text{Vol}_{n-1} D'. \quad (3.9)$$

D' is an $(n-1)$ -dimensional ball having radius

$$\sin\left(\arctan \frac{2a}{d}\right) = \left(1 + \frac{d^2}{4a^2}\right)^{-1/2},$$

so

$$\text{Vol}_{n-1} D' = \Omega_{n-1} \left(1 + \frac{d^2}{4a^2}\right)^{-(n-1)/2}. \quad (3.10)$$

Using (3.8), (3.9), (3.10), we get

$$\mu(A^+) \geq \frac{\Omega_{n-1}}{2n\Omega_n} \left(1 + \frac{d^2}{4a^2}\right)^{-(n-1)/2}. \quad (3.11)$$

Now

$$\frac{\Omega_{n-1}}{\Omega_n} = \frac{\pi^{(n-1)/2}/\Gamma((1+n)/2)}{\pi^{n/2}/\Gamma(1+n/2)} = \frac{\Gamma(1+n/2)}{\sqrt{\pi}\Gamma((1+n)/2)} \quad (3.12)$$

where Γ is the Gamma function. Since $\log \Gamma$ is convex ([1, p. 12]), we have

$$\Gamma(1+n/2)\Gamma(n/2) \geq \Gamma((1+n)/2)^2,$$

and therefore

$$\begin{aligned} \frac{\Gamma(1+n/2)}{\Gamma((1+n)/2)} &\geq \frac{\Gamma(1+n/2)}{\Gamma((1+n)/2)} \sqrt{\frac{\Gamma((1+n)/2)^2}{\Gamma(1+n/2)\Gamma(n/2)}} \\ &= \sqrt{\frac{\Gamma(1+n/2)}{\Gamma(n/2)}} = \sqrt{\frac{n}{2}}. \end{aligned} \quad (3.13)$$

Using (3.11), (3.12), (3.13), we get

$$\mu(A^+) \geq \frac{1}{2n} \frac{1}{\sqrt{\pi}} \sqrt{\frac{n}{2}} \left(1 + \frac{d^2}{4a^2}\right)^{-(n-1)/2}.$$

And after substituting the value of a ,

$$\begin{aligned} \mu(A^+) &\geq \frac{1}{\sqrt{8\pi n}} \left(1 + \frac{d^2}{4-2d^2n/(n+1)}\right)^{-(n-1)/2} \\ &= \frac{1}{\sqrt{8\pi n}} \left(\frac{3}{2} + \frac{(2n+1)d^2-2n-2}{4n+4-2d^2n}\right)^{-(n-1)/2}. \end{aligned}$$

Now proving Theorem 1 is just a matter of putting the pieces together.

Proof of Theorem 1. Since $t < -\log(1-t)$ for $0 < t < 1$, Proposition 6 implies

$$I(W) < 1 + \frac{\log N(n, \varepsilon)}{g(n, 1+\varepsilon)}, \quad 0 < \varepsilon < \sqrt{2}-1.$$

Choose

$$\varepsilon = \sqrt{1 + \frac{1}{2n+1}} - 1.$$

From Lemma 7 and Proposition 9 we get

$$\begin{aligned} I(W) &< 1 + \frac{\log N(n, \varepsilon)}{g(n, 1+\varepsilon)} \leq 1 + \frac{\log \left(1 + \frac{4}{\varepsilon}\right)^n}{g\left(n, \sqrt{\frac{2n+2}{2n+1}}\right)} \\ &\leq 1 + \sqrt{8\pi n} \left(\frac{3}{2}\right)^{(n-1)/2} n \log \left(1 + \frac{4}{\varepsilon}\right) \\ &= 1 + 4n\sqrt{\pi n/3} \left(\frac{3}{2}\right)^{n/2} \log \left(1 + \frac{4}{\varepsilon}\right). \end{aligned}$$

Since one easily sees that $\varepsilon > 1/(4n+3)$, we have

$$I(W) < 1 + 4n\sqrt{\pi n/3} \log(13 + 16n) \left(\frac{3}{2}\right)^{n/2} \leq 5n\sqrt{n}(4 + \log n) \left(\frac{3}{2}\right)^{n/2}$$

Remarks. 1. In [11] we give a lower bound for the volumes of sets of constant width in $\mathbf{R}^n$, using results analogous to Propositions 8 and 9.

2. The factor $5n\sqrt{n}(4 + \log n)$ in Theorem 1 should not be taken seriously. It can be improved with some more careful estimates. However, any improvement of the exponential factor, $(3/2)^{n/2}$, would be interesting. A possible way to do this may be to try to get a better lower bound for $g(n, d)$. An advance in this direction may lead to better estimates for the minimal volume of a set having constant width 1 in $\mathbf{R}^n$.

§4.

THEOREM 10. *Let $K \subset \mathbf{R}^n$ be a strictly convex body, invariant under a group of orthogonal transformations G that is generated by reflections through hyperplanes and acts irreducibly on $\mathbf{R}^n$. Then $I(K) = n + 1$.*

We preface the proof with a few definitions and a lemma. A unit vector $\mathbf{r}$ is called a *root* of G if the orthogonal reflection through the subspace orthogonal to $\mathbf{r}$ is an element of G . We denote this reflection by $S_{\mathbf{r}}$:

$$S_{\mathbf{r}}\mathbf{x} = \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{r})\mathbf{r}, \quad \mathbf{x} \in \mathbf{R}^n.$$

If $\mathbf{v}, \mathbf{r}$ are roots of G then $S_{\mathbf{r}}\mathbf{v}$ is also a root of G , because $S_{S_{\mathbf{r}}\mathbf{v}} = S_{\mathbf{r}}S_{\mathbf{v}}S_{\mathbf{r}}$. In particular $-\mathbf{r} = S_{\mathbf{r}}\mathbf{r}$ is a root.

LEMMA 11. *Let G be as in Theorem 10. If $n > 1$ then there are $n + 1$ roots of G , $\mathbf{r}_0, \dots, \mathbf{r}_n$, such that every nonzero $\mathbf{x} \in \mathbf{R}^n$ has a negative inner product with at least one of them.*

At least when G is finite this follows from known results (see [6]).

Proof of the Lemma. We will say that a set of vectors $\{\mathbf{v}_0, \dots, \mathbf{v}_m\}$ is *almost independent* if every proper subset of it is linearly independent but $\{\mathbf{v}_0, \dots, \mathbf{v}_m\}$ is linearly dependent. It is easily checked that $\{\mathbf{v}_0, \dots, \mathbf{v}_m\}$ is almost independent, if, and only if, $\mathbf{v}_1, \dots, \mathbf{v}_m$ are linearly independent and $\mathbf{v}_0$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_m$ with *nonzero* coefficients.

Because of the hypotheses on G and because $n \geq 2$, G has at least one root, $\mathbf{r}$. $\{\mathbf{r}, -\mathbf{r}\}$ is an almost independent set. Suppose $\{\mathbf{r}_0, \dots, \mathbf{r}_m\}$ is the largest almost independent set of roots of G . Let U be the subspace generated by $\mathbf{r}_0, \dots, \mathbf{r}_m$. We claim that $U = \mathbf{R}^n$ and therefore $n = m$. Let $\mathbf{r}$ be a root of G and suppose that U is not invariant under $S_{\mathbf{r}}$. This implies $\mathbf{r} \notin U$ and $S_{\mathbf{r}}\mathbf{r}_i \neq \mathbf{r}_i$ for some $i = 0, 1, \dots, m$. Assume, without loss of generality, that $S_{\mathbf{r}}\mathbf{r}_0 \neq \mathbf{r}_0$. $\mathbf{r}_0$ is a linear combination of $\mathbf{r}_1, \dots, \mathbf{r}_m$ with nonzero coefficients. Since $S_{\mathbf{r}}\mathbf{r}_0 - \mathbf{r}_0$ is a nonzero multiple of $\mathbf{r}$, $S_{\mathbf{r}}\mathbf{r}_0$ is a linear combination of $\mathbf{r}, \mathbf{r}_1, \dots, \mathbf{r}_m$ with nonzero coefficients. Because $\mathbf{r}, \mathbf{r}_1, \dots, \mathbf{r}_m$ are linearly independent, this means

that the set $\{S_r r_0, r, r_1, \dots, r_m\}$ is almost independent, contradicting the choice of $\{r_0, \dots, r_m\}$.

This forces us to conclude that U is invariant under the reflections generating G , and therefore under every transformation in G . G acts irreducibly on $\mathbf{R}^n$ and $U \neq 0$. This implies $U = \mathbf{R}^n$ and $n = m$.

Because $\{r_0, \dots, r_n\}$ is almost independent, it is possible to write

$$0 = \sum_{i=0}^n a_i r_i$$

with all coefficients nonzero. We replace some of the roots r_i by their negatives, to have all the coefficients a_i positive. If x is any nonzero vector, we must have $x \cdot r_i \neq 0$ for some $i = 0, 1, \dots, n$, because the r_i span $\mathbf{R}^n$. Since $0 = x \cdot 0 = \sum_{i=0}^n a_i (x \cdot r_i)$ and $a_i > 0$, we must have $x \cdot r_i < 0$ for some i .

Proof of Theorem 10. As mentioned in Section 1, $I(K) \geq n+1$ holds for every convex body in $\mathbf{R}^n$, thus we only need to show that $I(K) \leq n+1$. The Theorem is obviously true when $n = 1$, so we assume $n > 1$.

Let $r_0, \dots, r_n$ be the roots of G guaranteed by the lemma and let $x \in \partial K$. First observe that the origin is necessarily an interior point of K so $x \neq 0$. For some r_i , $x \cdot r_i < 0$. We claim that this r_i illuminates x . $x - 2(x \cdot r_i)r_i = S_{r_i}x \in K$, since $S_{r_i} \in G$. For some positive t , $x + tr_i$ is an interior point of K , because $x, x - 2(x \cdot r_i)r_i \in K$, K is strictly convex and $-2(x \cdot r_i) > 0$. This verifies the claim and we see that $r_0, \dots, r_n$ illuminate K .

Remark. A set of constant width is strictly convex; therefore Theorem 10 applies to (sufficiently symmetric) sets of constant width.

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