Chapter 9
Lower Semicontinuous Convex Functions

The theory of convex functions is most powerful in the presence of lower semicontinuity. A key property of lower semicontinuous convex functions is the existence of a continuous affine minorant, which we establish in this chapter by projecting onto the epigraph of the function.

9.1 Lower Semicontinuous Convex Functions

We start by observing that various types of lower semicontinuity coincide for convex functions.

Theorem 9.1 *Let* $f\colon \mathcal{H} \to \left]-\infty, +\infty\right]$ *be convex. Then the following are equivalent:*

(i) f *is weakly sequentially lower semicontinuous.*
(ii) f *is sequentially lower semicontinuous.*
(iii) f *is lower semicontinuous.*
(iv) f *is weakly lower semicontinuous.*

Proof. The set $\operatorname{epi} f$ is convex by Definition 8.1. Hence, the equivalences follow from Lemma 1.24, Lemma 1.35, and Theorem 3.32. ☐

Definition 9.2 The set of lower semicontinuous convex functions from $\mathcal{H}$ to $[-\infty, +\infty]$ is denoted by $\Gamma(\mathcal{H})$.

The set $\Gamma(\mathcal{H})$ is closed under several important operations. For instance, it is straightforward to verify that $\Gamma(\mathcal{H})$ is closed under multiplication by strictly positive real numbers.

Proposition 9.3 *Let* $(f_i)_{i\in I}$ *be a family in* $\Gamma(\mathcal{H})$. *Then* $\sup_{i\in I} f_i \in \Gamma(\mathcal{H})$.

Proof. Combine Lemma 1.26 and Proposition 8.14. ☐

H.H. Bauschke and P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, CMS Books in Mathematics, DOI 10.1007/978-1-4419-9467-7_9,

Corollary 9.4 *Let $(f_i)_{i\in I}$ be a family in $\Gamma(\mathcal{H})$. Suppose that one of the following holds:*

(i) *I is finite and $-\infty \notin \bigcup_{i\in I} f_i(\mathcal{H})$.*
(ii) *$\inf_{i\in I} f_i \geq 0$.*

Then $\sum_{i\in I} f_i \in \Gamma(\mathcal{H})$.

Proof. (i): A consequence of Lemma 1.27 and Proposition 8.15.

(ii): Let $\mathcal{I}$ be the class of nonempty finite subsets of I and set $(\forall J \in \mathcal{I})$ $g_J = \sum_{i\in J} f_i$. Then it follows from (i) that $(\forall J \in \mathcal{I})$ $g_J \in \Gamma(\mathcal{H})$. However, (2.4) yields $\sum_{i\in I} f_i = \sup_{J\in\mathcal{I}} g_J$. In view of Proposition 9.3, the proof is complete. ☐

Proposition 9.5 *Let $\mathcal{K}$ be a real Hilbert space, let $L \in \mathcal{B}(\mathcal{H},\mathcal{K})$, and let $f \in \Gamma(\mathcal{K})$. Then $f \circ L \in \Gamma(\mathcal{H})$.*

Proof. This is a consequence of Proposition 8.18. ☐

Proposition 9.6 *Let $f \in \Gamma(\mathcal{H})$ and suppose that $-\infty \in f(\mathcal{H})$. Then f is nowhere real-valued, i.e., $f(\mathcal{H}) \subset \{-\infty, +\infty\}$.*

Proof. Let $x \in \mathcal{H}$ be such that $f(x) = -\infty$, let $y \in \mathcal{H}$, and let $\alpha \in \left]0,1\right[$. If $f(y) \neq +\infty$, then Proposition 8.4 yields $f(\alpha x + (1-\alpha)y) = -\infty$. In turn, since f is lower semicontinuous, $f(y) \leq \underline{\lim}_{\alpha\downarrow 0} f(\alpha x + (1-\alpha)y) = -\infty$, i.e., $f(y) = -\infty$. ☐

The function $x \mapsto -\infty$ belongs to $\Gamma(\mathcal{H})$, which makes the following notion well defined.

Definition 9.7 Let $f\colon \mathcal{H} \to [-\infty,+\infty]$. Then

$$\breve{f} = \sup\big\{g \in \Gamma(\mathcal{H}) \;\big|\; g \leq f\big\} \tag{9.1}$$

is the *lower semicontinuous convex envelope* of f.

Proposition 9.8 *Let $f\colon \mathcal{H} \to [-\infty,+\infty]$. Then the following hold:*

(i) *$\breve{f}$ is the largest lower semicontinuous convex function majorized by f.*
(ii) *$(\forall x \in \mathcal{H})$ $\breve{f}(x) = \underline{\lim}_{y\to x} \breve{f}(y)$.*
(iii) *$\operatorname{epi} \breve{f}$ is closed and convex.*
(iv) *$\operatorname{conv}\operatorname{dom} f \subset \operatorname{dom} \breve{f} \subset \overline{\operatorname{conv}}\operatorname{dom} f$.*

Proof. (i): This is a consequence of (9.1) and Proposition 9.3.

(ii): This follows from (i) and Lemma 1.31(iv).

(iii): Combine (i), Lemma 1.24, and Definition 8.1.

(iv): By (i), $\breve{f} \leq f$ and $\breve{f}$ is convex. Hence, Proposition 8.2 yields

$$\operatorname{conv}\operatorname{dom} f \subset \operatorname{conv}\operatorname{dom} \breve{f} = \operatorname{dom} \breve{f}. \tag{9.2}$$

Now set $C = \overline{\text{conv}}\, \text{dom}\, f$ and

$$g\colon \mathcal{H} \to [-\infty,+\infty] : x \mapsto \begin{cases} \breve{f}(x), & \text{if } x \in C; \\ +\infty, & \text{if } x \notin C. \end{cases} \tag{9.3}$$

Using (iii), we note that $\text{epi}\, g = (\text{epi}\, \breve{f}) \cap (C \times \mathbb{R})$ is closed and convex. It follows from Lemma 1.24 and Definition 8.1 that

$$g \in \Gamma(\mathcal{H}). \tag{9.4}$$

Now fix $x \in \mathcal{H}$. If $x \in C$, then $g(x) = \breve{f}(x) \leq f(x)$; otherwise, $x \notin \text{dom}\, f \subset C$ and therefore $g(x) = f(x) = +\infty$. Altogether, $g \leq f$ and, in view of (9.4), we obtain $g \leq \breve{f}$. Thus, $\text{dom}\, \breve{f} \subset \text{dom}\, g \subset C = \overline{\text{conv}}\, \text{dom}\, f$. ☐

Theorem 9.9 *Let* $f\colon \mathcal{H} \to [-\infty,+\infty]$. *Then* $\text{epi}\, \breve{f} = \overline{\text{conv}}\, \text{epi}\, f$.

Proof. Set $E = \overline{\text{conv}}\, \text{epi}\, f$. Since $\breve{f} \leq f$, we have $\text{epi}\, f \subset \text{epi}\, \breve{f}$. Hence $E \subset \overline{\text{conv}}\, \text{epi}\, \breve{f} = \text{epi}\, \breve{f}$ by Proposition 9.8(iii). It remains to show that $\text{epi}\, \breve{f} \subset E$. We assume that $f \not\equiv +\infty$, since otherwise $\breve{f} = f$ and the conclusion is clear. Let us proceed by contradiction and assume that there exists

$$(x,\xi) \in \text{epi}\, \breve{f} \smallsetminus E. \tag{9.5}$$

Since E is a nonempty closed convex subset of $\mathcal{H} \times \mathbb{R}$, Theorem 3.14 implies that the projection (p,π) of (x,ξ) onto E satisfies

$$\big(\forall (y,\eta) \in E\big) \quad \langle y - p \mid x - p\rangle + (\eta - \pi)(\xi - \pi) \leq 0. \tag{9.6}$$

Letting $\eta \uparrow +\infty$ in (9.6), we deduce that $\xi \leq \pi$. Let us first assume that $\xi = \pi$. Then (9.6) yields $(\forall y \in \overline{\text{conv}}\, \text{dom}\, f)\ \langle y - p \mid x - p\rangle \leq 0$. Consequently, since Proposition 9.8(iv) asserts that $x \in \text{dom}\, \breve{f} \subset \overline{\text{conv}}\, \text{dom}\, f$, we obtain $\|x - p\|^2 = 0$ and, in turn, $(p,\pi) = (x,\xi)$, which is impossible since $(x,\xi) \notin E$ by (9.5). Therefore, we must have

$$\xi < \pi. \tag{9.7}$$

Setting $u = (x-p)/(\pi - \xi)$ and letting $\eta = f(y)$ in (9.6), we get

$$(\forall y \in \text{dom}\, f) \quad \langle y - p \mid u\rangle + \pi \leq f(y). \tag{9.8}$$

Consequently, f is minorized by the lower semicontinuous convex function $g\colon y \mapsto \langle y - p \mid u\rangle + \pi$, and it follows that $g \leq \breve{f}$. In particular, since $(x,\xi) \in \text{epi}\, \breve{f}$, we have

$$\pi \leq \frac{\|x-p\|^2}{\pi - \xi} + \pi = g(x) \leq \breve{f}(x) \leq \xi, \tag{9.9}$$

which contradicts (9.7). We conclude that $\text{epi}\, \breve{f} \subset E$. ☐

Corollary 9.10 *Let* $f\colon \mathcal{H} \to [-\infty,+\infty]$ *be convex. Then* $\bar{f} = \breve{f}$.

Proof. Combine Lemma 1.31(vi) and Theorem 9.9. ☐

Corollary 9.11 *Let* $f\colon \mathcal{H} \to \left]-\infty,+\infty\right]$ *be convex and such that* $\operatorname{lev}_{<0} f \neq \varnothing$. *Then* $\operatorname{lev}_{<0} f \subset \operatorname{lev}_{\leq 0} f \subset \operatorname{lev}_{\leq 0} \breve{f}$ *and* $\overline{\operatorname{lev}_{<0} f} = \overline{\operatorname{lev}_{\leq 0} f} = \operatorname{lev}_{\leq 0} \breve{f}$.

Proof. Take $x \in \mathcal{H}$. Then $f(x) < 0 \Rightarrow f(x) \leq 0 \Rightarrow \breve{f}(x) \leq 0$, which shows the inclusions. Now assume that $x \in \operatorname{lev}_{\leq 0} \breve{f}$. Then, since f is convex, Theorem 9.9 yields $(x, \breve{f}(x)) \in \operatorname{epi} \breve{f} = \overline{\operatorname{epi}} f$. Hence there exists a sequence $(x_n, \xi_n)_{n\in\mathbb{N}}$ in $\operatorname{epi} f$ that converges to $(x, \breve{f}(x))$. Now fix $z \in \operatorname{lev}_{<0} f$ and define a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\left]0,1\right]$ by

$$(\forall n \in \mathbb{N}) \quad \alpha_n = \begin{cases} \dfrac{1}{n+1}, & \text{if } \xi_n \leq 0; \\ \min\left\{1, \dfrac{1}{n+1} + \dfrac{\xi_n}{\xi_n - f(z)}\right\}, & \text{otherwise.} \end{cases} \tag{9.10}$$

Then eventually

$$\begin{aligned} f(\alpha_n z + (1-\alpha_n)x_n) &\leq \alpha_n f(z) + (1-\alpha_n) f(x_n) \\ &\leq \alpha_n f(z) + (1-\alpha_n)\xi_n \\ &< 0. \end{aligned} \tag{9.11}$$

Therefore the sequence $(\alpha_n z + (1-\alpha_n)x_n)_{n\in\mathbb{N}}$, which converges to x, lies eventually in $\operatorname{lev}_{<0} f$. The result follows. ☐

9.2 Proper Lower Semicontinuous Convex Functions

As illustrated in Proposition 9.6, nonproper lower semicontinuous convex functions are of limited use. By contrast, proper lower semicontinuous convex functions will play a central role in this book.

Definition 9.12 The set of proper lower semicontinuous convex functions from $\mathcal{H}$ to $\left]-\infty,+\infty\right]$ is denoted by $\Gamma_0(\mathcal{H})$.

Example 9.13 Let $(e_i)_{i\in I}$ be a family in $\mathcal{H}$ and let $(\phi_i)_{i\in I}$ be a family in $\Gamma_0(\mathbb{R})$ such that $(\forall i \in I)\ \phi_i \geq \phi_i(0) = 0$. Set $f\colon \mathcal{H} \to \left]-\infty,+\infty\right] : x \mapsto \sum_{i\in I} \phi_i(\langle x \mid e_i\rangle)$. Then $f \in \Gamma_0(\mathcal{H})$.

Proof. Set $(\forall i \in I)\ f_i\colon \mathcal{H} \to \left]-\infty,+\infty\right] : x \mapsto \phi_i(\langle x \mid e_i\rangle)$. Then $f = \sum_{i\in I} f_i$ and $(\forall i \in I)\ 0 \leq f_i \in \Gamma_0(\mathcal{H})$. Thus, it follows from Corollary 9.4(ii) that $f \in \Gamma(\mathcal{H})$. Finally, since $f(0) = 0$, f is proper. ☐

Proposition 9.14 *Let* $f \in \Gamma_0(\mathcal{H})$, *let* $x \in \mathcal{H}$, *and let* $y \in \operatorname{dom} f$. *For every* $\alpha \in \left]0,1\right[$, *set* $x_\alpha = (1-\alpha)x + \alpha y$. *Then* $\lim_{\alpha\downarrow 0} f(x_\alpha) = f(x)$.

Proof. Using the lower semicontinuity and the convexity of f, we obtain $f(x) \leq \underline{\lim}_{\alpha\downarrow 0} f(x_\alpha) \leq \overline{\lim}_{\alpha\downarrow 0} f(x_\alpha) \leq \overline{\lim}_{\alpha\downarrow 0}(1-\alpha)f(x) + \alpha f(y) = f(x)$. Therefore, $\lim_{\alpha\downarrow 0} f(x_\alpha) = f(x)$. ☐

Corollary 9.15 *Let $f \in \Gamma_0(\mathbb{R})$. Then $f|_{\overline{\operatorname{dom} f}}$ is continuous.*

The conclusion of Corollary 9.15 fails in general Hilbert spaces and even in the Euclidean plane (see Example 9.27 below).

We conclude this section with an extension of Fact 6.13.

Fact 9.16 [233, Corollary 13.2] *Let f and g be in $\Gamma_0(\mathcal{H})$. Then*

$$\operatorname{int}(\operatorname{dom} f - \operatorname{dom} g) = \operatorname{core}(\operatorname{dom} f - \operatorname{dom} g). \tag{9.12}$$

9.3 Affine Minorization

A key property of functions in $\Gamma_0(\mathcal{H})$ is that they possess continuous affine minorants. To see this, we require the following two results.

Proposition 9.17 *Let $f \in \Gamma_0(\mathcal{H})$, let $(x,\xi) \in \mathcal{H}\times\mathbb{R}$, and let $(p,\pi) \in \mathcal{H}\times\mathbb{R}$. Then $(p,\pi) = P_{\operatorname{epi} f}(x,\xi)$ if and only if*

$$\max\{\xi, f(p)\} \leq \pi \tag{9.13}$$

and

$$(\forall y \in \operatorname{dom} f) \quad \langle y-p \mid x-p\rangle + \big(f(y)-\pi\big)\big(\xi-\pi\big) \leq 0. \tag{9.14}$$

Proof. Since $f \in \Gamma_0(\mathcal{H})$, the set $\operatorname{epi} f$ is nonempty, closed, and convex. Now set $(p,\pi) = P_{\operatorname{epi} f}(x,\xi)$. Then Theorem 3.14 implies that (p,π) is characterized by $(p,\pi) \in \operatorname{epi} f$ and $(\forall (y,\eta) \in \operatorname{epi} f)$ $\langle y-p \mid x-p\rangle + (\eta-\pi)(\xi-\pi) \leq 0$, which is equivalent to $f(p) \leq \pi$ and $(\forall y \in \operatorname{dom} f)(\forall \lambda \in \mathbb{R}_+)$ $\langle y-p \mid x-p\rangle +$ $(f(y)+\lambda-\pi)(\xi-\pi) \leq 0$. By letting $\lambda \uparrow +\infty$, we deduce that $\xi \leq \pi$. The characterization follows. ☐

Proposition 9.18 *Let $f \in \Gamma_0(\mathcal{H})$, let $x \in \operatorname{dom} f$, let $\xi \in \,]-\infty, f(x)[$, and let $(p,\pi) \in \mathcal{H}\times\mathbb{R}$. Then $(p,\pi) = P_{\operatorname{epi} f}(x,\xi)$ if and only if*

$$\xi < f(p) = \pi \tag{9.15}$$

and

$$(\forall y \in \operatorname{dom} f) \quad \langle y-p \mid x-p\rangle \leq \big(f(y)-f(p)\big)\big(f(p)-\xi\big). \tag{9.16}$$

Proof. Suppose first that $(p,\pi) = P_{\operatorname{epi} f}(x,\xi)$. Since $p \in \operatorname{dom} f$, (9.14) yields

$$\big(f(p)-\pi\big)\big(\xi-\pi\big) \leq 0. \tag{9.17}$$

To establish that $\xi < f(p)$, we argue by contradiction. Suppose that $f(p) \leq \xi$. Then $f(p) - \pi \leq \xi - \pi$ and hence, since $\xi - \pi \leq 0$ by (9.13), we obtain $(f(p) - \pi)(\xi - \pi) \geq (\xi - \pi)^2$. In view of (9.17), we deduce that $\xi = \pi$. In turn, since $x \in \operatorname{dom} f$, (9.14) implies that $\langle x - p \mid x - p \rangle \leq 0$. Thus $x = p$ and hence $(p, \pi) = (x, \xi)$. This is impossible, since $(p, \pi) \in \operatorname{epi} f$ and $(x, \xi) \notin \operatorname{epi} f$. Thus,

$$\xi < f(p), \tag{9.18}$$

and (9.13) implies that $\xi < \pi$ and $f(p) \leq \pi$. Hence, (9.17) yields $f(p) = \pi$ and (9.15) holds. Combining (9.15) and Proposition 9.17, we obtain (9.16).

Conversely, if (9.15) and (9.16) hold, then Proposition 9.17 implies directly that $(p, \pi) = P_{\operatorname{epi} f}(x, \xi)$. ☐

Theorem 9.19 *Let $f \in \Gamma_0(\mathcal{H})$. Then f possesses a continuous affine minorant.*

Proof. Fix $x \in \operatorname{dom} f$ and $\xi \in \left]-\infty, f(x)\right[$, and set $(p, \pi) = P_{\operatorname{epi} f}(x, \xi)$. Then, by (9.15), $f(p) > \xi$. Now set $u = (x - p)/(f(p) - \xi)$ and $g \colon \mathcal{H} \to \mathbb{R} \colon y \mapsto \langle y - p \mid u \rangle + f(p)$. Then (9.16) yields $g \leq f$. ☐

Corollary 9.20 *Let $f \in \Gamma_0(\mathcal{H})$. Then f is bounded below on every nonempty bounded subset of $\mathcal{H}$.*

Proof. Let C be a nonempty bounded subset of $\mathcal{H}$ and set $\beta = \sup_{x \in C} \|x\|$. Theorem 9.19 asserts that f has a continuous affine minorant, say $\langle \cdot \mid u \rangle + \eta$. Then, by Cauchy–Schwarz, $(\forall x \in C)\ f(x) \geq \langle x \mid u \rangle + \eta \geq -\|x\|\,\|u\| + \eta \geq -\beta\|u\| + \eta > -\infty$. ☐

Example 9.21 Suppose that $\mathcal{H}$ is infinite-dimensional and let $f \colon \mathcal{H} \to \mathbb{R}$ be a discontinuous linear functional (see Example 2.20 and Example 8.33). Then f has no continuous affine minorant.

Proof. Assume that the conclusion is false, i.e., that there exist $u \in \mathcal{H}$ and $\eta \in \mathbb{R}$ such that $(\forall x \in \mathcal{H})\ \langle x \mid u \rangle + \eta \leq f(x)$. Then, since f is odd, $(\forall x \in \mathcal{H})$ $f(x) \leq \langle x \mid u \rangle - \eta \leq \|x\|\,\|u\| - \eta$. Consequently, $\sup f(B(0;1)) \leq \|u\| - \eta$ and therefore f is bounded above on a neighborhood of 0. This contradicts Corollary 8.30(i) since f is nowhere continuous. ☐

Theorem 9.22 *Let $f \in \Gamma_0(\mathcal{H})$ and let $x \in \operatorname{int}\operatorname{dom} f$. Then there exists a continuous affine minorant a of f such that $a(x) = f(x)$. In other words, $(\exists\, u \in \mathcal{H})(\forall y \in \mathcal{H})\ \langle y - x \mid u \rangle + f(x) \leq f(y)$.*

Proof. In view of Corollary 8.30, $x \in \operatorname{cont} f$. Hence, it follows from Theorem 8.29 and Proposition 8.36 that $\operatorname{int}\operatorname{epi} f \neq \varnothing$. In turn, Proposition 7.5 implies that $(x, f(x)) \in \operatorname{spts}(\operatorname{epi} f)$, and we therefore derive from Theorem 7.4 that there exists $(z, \zeta) \in (\mathcal{H} \times \mathbb{R}) \smallsetminus (\operatorname{epi} f)$ such that $(x, f(x)) = P_{\operatorname{epi} f}(z, \zeta)$. In view of Proposition 3.19 and since $x \in \operatorname{int}\operatorname{dom} f$, we assume that $z \in \operatorname{int}\operatorname{dom} f$. Thus, by Proposition 9.17, $\max\{\zeta, f(x)\} \leq f(x)$, i.e.,

$$f(x) \geq \zeta \tag{9.19}$$

and

$$(\forall y \in \operatorname{dom} f) \quad \langle y - x \mid z - x \rangle + \big(f(y) - f(x)\big)\big(\zeta - f(x)\big) \leq 0. \tag{9.20}$$

If $f(x) = \zeta$, then the above inequality evaluated at $y = z$ yields $z = x$, which contradicts the fact that $(z, \zeta) \neq (x, f(x))$. Hence $f(x) > \zeta$. Now set $u = (z - x)/(f(x) - \zeta)$. Then (9.20) becomes $(\forall y \in \operatorname{dom} f)$ $\langle y - x \mid u \rangle + f(x) - f(y) \leq 0$, and the result follows. ☐

Proposition 9.23 (Jensen's inequality) *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space such that $\mu(\Omega) \in \mathbb{R}_{++}$, let $\phi \in \Gamma_0(\mathbb{R})$, and let $x \colon \Omega \to \mathbb{R}$ be a measurable function such that $\mu(\Omega)^{-1} \int_\Omega x(\omega)\mu(d\omega) \in \operatorname{int}\operatorname{dom} \phi$. Then*

$$\phi\left(\frac{1}{\mu(\Omega)} \int_\Omega x(\omega)\mu(d\omega)\right) \leq \frac{1}{\mu(\Omega)} \int_\Omega \phi(x(\omega))\mu(d\omega). \tag{9.21}$$

Proof. Since ϕ is lower semicontinuous, it is measurable, and so is therefore $\phi \circ x$. Now set $\xi = \mu(\Omega)^{-1} \int_\Omega x \, d\mu$. It follows from Theorem 9.22 that there exists $\alpha \in \mathbb{R}$ such that $(\forall \eta \in \mathbb{R})$ $\alpha(\eta - \xi) + \phi(\xi) \leq \phi(\eta)$. Thus, for μ-almost every $\omega \in \Omega$, $\alpha(x(\omega) - \xi) + \phi(\xi) \leq \phi(x(\omega))$. Integrating these inequalities over Ω with respect to μ yields $\phi(\xi)\mu(\Omega) \leq \int_\Omega \phi(x(\omega))\mu(d\omega)$. ☐

Example 9.24 Let $(\Omega, \mathcal{F}, \mu)$ be a measure space such that $\mu(\Omega) \in \mathbb{R}_{++}$, let $(\mathsf{H}, \|\cdot\|_\mathsf{H})$ be a real Hilbert space, and take p and q in $\mathbb{R}_{++}$ such that $p < q$. Then the following hold:

(i) Let $x \in L^p((\Omega, \mathcal{F}, \mu); \mathsf{H}))$. Then

$$\left(\int_\Omega \|x(\omega)\|_\mathsf{H}^p \mu(d\omega)\right)^{1/p} \leq \mu(\Omega)^{1/p - 1/q} \left(\int_\Omega \|x(\omega)\|_\mathsf{H}^q \mu(d\omega)\right)^{1/q}. \tag{9.22}$$

(ii) $L^q((\Omega, \mathcal{F}, \mu); \mathsf{H}) \subset L^p((\Omega, \mathcal{F}, \mu); \mathsf{H})$.

Proof. (i): Set $\phi = |\cdot|^{q/p}$. Then it follows from Example 8.21 that ϕ is convex. Now let $x \in L^p((\Omega, \mathcal{F}, \mu); \mathsf{H}))$ and set $y \colon \omega \mapsto \|x(\omega)\|_\mathsf{H}^p$. Since y is integrable, $\mu(\Omega)^{-1} \int_\Omega y \, d\mu \in \mathbb{R} = \operatorname{dom} \phi$, and Proposition 9.23 applied to y yields (9.22).

(ii): An immediate consequence of (i). ☐

Example 9.25 Let X be a random variable, and take p and q in $\mathbb{R}_{++}$ such that $p < q$ and $\mathsf{E}|X|^p < +\infty$. Then $\mathsf{E}^{1/p}|X|^p \leq \mathsf{E}^{1/q}|X|^q$.

Proof. Let μ be a probability measure and set $\mathsf{H} = \mathbb{R}$ in Example 9.24(i) (see Example 2.8). ☐

9.4 Construction of Functions in $\Gamma_0(\mathcal{H})$

We start with a basic tool for constructing functions in $\Gamma_0(\mathcal{H})$.

Proposition 9.26 *Let $g\colon \mathcal{H} \to \left]-\infty,+\infty\right]$ be a proper convex function such that* $\operatorname{dom} g$ *is open and g is continuous on* $\operatorname{dom} g$. *Set*

$$f\colon \mathcal{H} \to \left]-\infty,+\infty\right] : x \mapsto \begin{cases} g(x), & \text{if } x \in \operatorname{dom} g;\\ \varliminf\limits_{y\to x} g(y), & \text{if } x \in \operatorname{bdry}\operatorname{dom} g;\\ +\infty, & \text{if } x \in \mathcal{H} \smallsetminus \overline{\operatorname{dom}}\, g. \end{cases} \tag{9.23}$$

Then $f = \breve{g}$ and $f \in \Gamma_0(\mathcal{H})$.

Proof. Set $C = \operatorname{dom} g$. To show that $f = \breve{g}$ we shall repeatedly utilize Proposition 9.8. Note that, since $g \geq \breve{g}$, we have $C \subset \operatorname{dom}\breve{g} \subset \overline{C}$. Let $x \in \mathcal{H}$ and assume first that $x \in C$. Then $+\infty > g(x) \geq \breve{g}(x)$. By Theorem 9.9, there exists a sequence $(x_n, \xi_n)_{n\in\mathbb{N}}$ in $\operatorname{epi} g$ such that $(x_n,\xi_n) \to (x, \breve{g}(x))$. Hence $\breve{g}(x) = \lim \xi_n = \varliminf \xi_n \geq \varliminf g(x_n) \geq \varliminf \breve{g}(x_n) \geq \breve{g}(x)$ and so $f(x) = g(x) = \lim g(x_n) = \varliminf g(x_n) = \breve{g}(x)$. Consequently, $f = \breve{g}$ on C. If $x \in \mathcal{H} \smallsetminus \overline{C}$, then $f(x) = +\infty = \breve{g}(x)$ and thus $f = \breve{g}$ on $\mathcal{H} \smallsetminus \overline{C}$. If $x \in (\operatorname{bdry} C) \smallsetminus (\operatorname{dom}\breve{g})$, then $+\infty \geq f(x) = \varliminf_{y\to x} g(y) \geq \varliminf_{y\to x} \breve{g}(y) = \breve{g}(x) = +\infty$ and thus $f(x) = \breve{g}(x) = +\infty$. Finally, we assume that $x \in (\operatorname{bdry} C) \cap (\operatorname{dom}\breve{g})$. Using Theorem 9.9 again, we see that there exists a sequence $(x_n,\xi_n)_{n\in\mathbb{N}}$ in $\operatorname{epi} g$ such that $(x_n,\xi_n) \to (x,\breve{g}(x))$. Hence $f(x) = \varliminf_{y\to x} g(y) \geq \varliminf_{y\to x} \breve{g}(y) = \breve{g}(x) = \lim \xi_n = \varliminf \xi_n \geq \varliminf g(x_n) \geq \varliminf_{y\to x} g(y) = f(x)$ and therefore $f(x) = \breve{g}(x)$. We have verified that $f = \breve{g}$. It follows that f is lower semicontinuous and convex. Since f is real-valued on C, Proposition 9.6 implies that f is also proper. ☐

Example 9.27 The function

$$f\colon \mathbb{R}^2 \to \left]-\infty,+\infty\right] : (\xi,\eta) \mapsto \begin{cases} \eta^2/\xi, & \text{if } \xi > 0;\\ 0, & \text{if } (\xi,\eta) = (0,0);\\ +\infty, & \text{otherwise,} \end{cases} \tag{9.24}$$

belongs to $\Gamma_0(\mathbb{R}^2)$ and $f\big|_{\operatorname{dom} f}$ is not continuous at $(0,0)$.

Proof. Set

$$g\colon \mathbb{R}^2 \to \left]-\infty,+\infty\right] : (\xi,\eta) \mapsto \begin{cases} \eta^2/\xi, & \text{if } \xi > 0;\\ +\infty, & \text{otherwise.} \end{cases} \tag{9.25}$$

The convexity of $t \mapsto t^2$ and Proposition 8.23 imply that g is proper and convex. Moreover, Proposition 9.26 yields $\breve{g} = f \in \Gamma_0(\mathbb{R}^2)$. Now set $x = (0,0)$, fix a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_{++}$ such that $\alpha_n \downarrow 0$, and set $(\forall n \in \mathbb{N})$

$x_n = (\alpha_n^2, \alpha_n)$. Then $(x_n)_{n\in\mathbb{N}}$ lies in $\operatorname{dom} f$ and $x_n \to x$, but $\lim f(x_n) = 1 \neq 0 = f(x)$. ☐

The following result concerns the construction of strictly convex functions in $\Gamma_0(\mathbb{R})$.

Proposition 9.28 *Let* $g\colon \mathbb{R} \to \left]-\infty, +\infty\right]$ *be strictly convex and proper, and suppose that* $\operatorname{dom} g = \left]\alpha, \beta\right[$, *where* α *and* β *are in* $[-\infty, +\infty]$ *and* $\alpha < \beta$. *Set*

$$f\colon \mathbb{R} \to \left]-\infty, +\infty\right] : x \mapsto \begin{cases} g(x), & \text{if } x \in \left]\alpha, \beta\right[; \\ \lim\limits_{y\downarrow\alpha} g(y), & \text{if } x = \alpha; \\ \lim\limits_{y\uparrow\beta} g(y), & \text{if } x = \beta; \\ +\infty, & \text{otherwise.} \end{cases} \tag{9.26}$$

Then f *is strictly convex,* $f = \breve{g}$, *and* $f \in \Gamma_0(\mathbb{R})$.

Proof. Proposition 9.14, Corollary 8.30(iii), and Proposition 9.26 imply that f is convex and that $f = \breve{g} \in \Gamma_0(\mathbb{R})$. To verify strict convexity, suppose that x and y are distinct points in $\operatorname{dom} f$, take $\gamma \in \left]0, 1\right[$, and suppose that $f(\gamma x + (1-\gamma)y) = \gamma f(x) + (1-\gamma) f(y)$. By Exercise 8.1, $(\forall \lambda \in \left]0, 1\right[)\ f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda) f(y)$. Since $\left]x, y\right[\subset \left]\alpha, \beta\right[$ and $f = g$ on $\left]\alpha, \beta\right[$, this contradicts the strict convexity of g. ☐

The next two examples are consequences of Proposition 9.28 and Proposition 8.12(ii).

Example 9.29 (entropy) The negative *Boltzmann–Shannon entropy* function

$$\mathbb{R} \to \left]-\infty, +\infty\right] : x \mapsto \begin{cases} x\ln(x) - x, & \text{if } x > 0; \\ 0, & \text{if } x = 0; \\ +\infty, & \text{if } x < 0, \end{cases} \tag{9.27}$$

is strictly convex and belongs to $\Gamma_0(\mathbb{R})$.

Example 9.30 The following are strictly convex functions in $\Gamma_0(\mathbb{R})$:

(i) $x \mapsto \exp(x)$.

(ii) $x \mapsto |x|^p$, where $p \in \left]1, +\infty\right[$.

(iii) $x \mapsto \begin{cases} 1/x^p, & \text{if } x > 0; \\ +\infty, & \text{otherwise,} \end{cases}$ where $p \in \left[1, +\infty\right[$.

(iv) $x \mapsto \begin{cases} -x^p, & \text{if } x \geq 0; \\ +\infty, & \text{otherwise,} \end{cases}$ where $p \in \left]0, 1\right[$.

(v) $x \mapsto \begin{cases} 1/\sqrt{1-x^2}, & \text{if } |x| < 1; \\ +\infty, & \text{otherwise.} \end{cases}$

(vi) $x \mapsto \begin{cases} -\sqrt{1-x^2}, & \text{if } |x| \leq 1; \\ +\infty, & \text{otherwise.} \end{cases}$

(vii) $x \mapsto \begin{cases} x\ln(x) + (1-x)\ln(1-x), & \text{if } x \in \,]0,1[; \\ 0, & \text{if } x \in \{0,1\}; \\ +\infty, & \text{otherwise.} \end{cases}$

(viii) $x \mapsto \begin{cases} -\ln(x), & \text{if } x > 0; \\ +\infty, & \text{otherwise} \end{cases}$ (negative *Burg entropy* function).

Remark 9.31 By utilizing direct sum constructions (see Proposition 8.25 and Exercise 8.12), we can construct a (strictly) convex function in $\Gamma_0(\mathbb{R}^N)$ from (strictly) convex functions in $\Gamma_0(\mathbb{R})$.

We now turn our attention to the construction of proper lower semicontinuous convex integral functions (see Example 2.5 for notation).

Proposition 9.32 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, let $(\mathsf{H}, \langle \cdot \mid \cdot \rangle_{\mathsf{H}})$ be a real Hilbert space, and let $\varphi \in \Gamma_0(\mathsf{H})$. Suppose that $\mathcal{H} = L^2((\Omega, \mathcal{F}, \mu); \mathsf{H})$ and that one of the following holds:*

(i) $\mu(\Omega) < +\infty$.
(ii) $\varphi \geq \varphi(0) = 0$.

Set

$$f \colon \mathcal{H} \to \,]-\infty, +\infty] \\ x \mapsto \begin{cases} \displaystyle\int_\Omega \varphi\big(x(\omega)\big)\mu(d\omega), & \text{if } \varphi \circ x \in L^1\big((\Omega, \mathcal{F}, \mu); \mathbb{R}\big); \\ +\infty, & \text{otherwise.} \end{cases} \tag{9.28}$$

Then $f \in \Gamma_0(\mathcal{H})$.

Proof. We first observe that, since φ is lower semicontinuous, it is measurable, and so is therefore $\varphi \circ x$ for every $x \in \mathcal{H}$. Let us now show that $f \in \Gamma_0(\mathcal{H})$.

(i): By Theorem 9.19, there exists a continuous affine function $\psi \colon \mathsf{H} \to \mathbb{R}$ such that $\varphi \geq \psi$, say $\psi = \langle \cdot \mid \mathsf{u} \rangle_{\mathsf{H}} + \eta$ for some $\mathsf{u} \in \mathsf{H}$ and $\eta \in \mathbb{R}$. Let us set $u \colon \Omega \to \mathsf{H} \colon \omega \mapsto \mathsf{u}$. Then $u \in \mathcal{H}$ since $\int_\Omega \|u(\omega)\|_{\mathsf{H}}^2 \mu(d\omega) = \|\mathsf{u}\|_{\mathsf{H}}^2 \mu(\Omega) < +\infty$. Moreover, for every $x \in \mathcal{H}$, $\varphi \circ x \geq \psi \circ x$ and

$$\int_\Omega \psi\big(x(\omega)\big)\mu(d\omega) = \int_\Omega \langle x(\omega) \mid \mathsf{u} \rangle_{\mathsf{H}}\, \mu(d\omega) + \eta\mu(\omega) = \langle x \mid u \rangle + \eta\mu(\omega) \in \mathbb{R}. \tag{9.29}$$

Thus, Proposition 8.22 asserts that f is well defined and convex, with $\operatorname{dom} f = \big\{x \in \mathcal{H} \;\big|\; \varphi \circ x \in L^1((\Omega, \mathcal{F}, \mu); \mathbb{R})\big\}$. It also follows from (9.28) and (9.29) that

$$(\forall x \in \operatorname{dom} f) \quad f(x) = \int_\Omega (\varphi - \psi)\big(x(\omega)\big)\mu(d\omega) + \langle x \mid u \rangle + \eta\mu(\omega). \tag{9.30}$$

Now take $\mathsf{z} \in \operatorname{dom}\varphi$ and set $z\colon \Omega \to \mathsf{H}\colon \omega \mapsto \mathsf{z}$. Then $z \in \mathcal{H}$ and $\int_\Omega |\varphi\circ z| d\mu = |\varphi(\mathsf{z})|\mu(\Omega) < +\infty$. Hence, $\varphi \circ z \in L^1((\Omega, \mathcal{F}, \mu); \mathbb{R})$. This shows that f is proper. Next, to show that f is lower semicontinuous, let us fix $\xi \in \mathbb{R}$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in $\operatorname{lev}_{\leq\xi} f$ that converges to some $x \in \mathcal{H}$. In view of Lemma 1.24, it suffices to show that $f(x) \leq \xi$. Since $\|x_n(\cdot) - x(\cdot)\|_{\mathsf{H}} \to 0$ in $L^2((\Omega, \mathcal{F}, \mu); \mathbb{R})$, there exists a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ such that $x_{k_n}(\omega) \overset{\mathsf{H}}{\to} x(\omega)$ for μ-almost every $\omega \in \Omega$ [3, Theorem 2.5.1 & Theorem 2.5.3]. Now set $\phi = (\varphi - \psi) \circ x$ and $(\forall n \in \mathbb{N})$ $\phi_n = (\varphi - \psi) \circ x_{k_n}$. Since $\varphi - \psi$ is lower semicontinuous, we have

$$\phi(\omega) = (\varphi - \psi)\big(x(\omega)\big) \leq \underline{\lim}(\varphi - \psi)\big(x_{k_n}(\omega)\big) = \underline{\lim}\,\phi_n(\omega) \quad \mu\text{-a.e. on } \Omega. \tag{9.31}$$

On the other hand, since $\inf_{n\in\mathbb{N}} \phi_n \geq 0$, Fatou's lemma [3, Lemma 1.6.8] yields $\int_\Omega \underline{\lim}\,\phi_n d\mu \leq \underline{\lim} \int_\Omega \phi_n d\mu$. Hence, we derive from (9.30) and (9.31) that

$$\begin{aligned} f(x) &= \int_\Omega \phi d\mu + \langle x \mid u\rangle + \eta\mu(\Omega) \\ &\leq \int_\Omega \underline{\lim}\,\phi_n d\mu + \langle x \mid u\rangle + \eta\mu(\Omega) \\ &\leq \underline{\lim} \int_\Omega \phi_n d\mu + \lim \langle x_{k_n} \mid u\rangle + \eta\mu(\Omega) \\ &= \underline{\lim} \int_\Omega (\varphi \circ x_{k_n}) d\mu \\ &= \underline{\lim}\, f(x_{k_n}) \\ &\leq \xi. \end{aligned} \tag{9.32}$$

(ii): Since (8.16) holds with $\varrho = 0$, it follows from Proposition 8.22 that f is a well-defined convex function. In addition, since $\varphi(0) = 0$, (9.28) yields $f(0) = 0$. Thus, f is proper. Finally, to prove that f is lower semicontinuous, we follow the same procedure as above with $\psi = 0$. ☐

Example 9.33 (Boltzmann–Shannon entropy) Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and suppose that $\mathcal{H} = L^2(\Omega, \mathcal{F}, \mu)$ (see Example 2.6). Using the convention $0\ln(0) = 0$, set

$$\begin{aligned} f\colon\ &\mathcal{H} \to \left]-\infty, +\infty\right] \\ &x \mapsto \begin{cases} \displaystyle\int_\Omega \big(x(\omega)\ln(x(\omega)) - x(\omega)\big)\mu(d\omega), & \text{if } x \geq 0 \ \ \mu\text{-a.e. on } \Omega; \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned} \tag{9.33}$$

Then $f \in \Gamma_0(\mathcal{H})$. In particular, this is true in the following cases:

(i) Entropy of a random variable: $\mathcal{H} = L^2(\Omega, \mathcal{F}, \mathsf{P})$, where $(\Omega, \mathcal{F}, \mathsf{P})$ is a probability space (see Example 2.8), and

$$f\colon \mathcal{H} \to \left]-\infty,+\infty\right]$$
$$X \mapsto \begin{cases} \mathsf{E}\big(X\ln(X) - X\big), & \text{if } X \geq 0 \text{ a.s.;} \\ +\infty, & \text{otherwise.} \end{cases} \tag{9.34}$$

(ii) Discrete entropy: $\mathcal{H} = \mathbb{R}^N$ and

$$f\colon \mathcal{H} \to \left]-\infty,+\infty\right]$$
$$(\xi_k)_{1\leq k\leq N} \mapsto \begin{cases} \displaystyle\sum_{k=1}^{N} \xi_k \ln(\xi_k) - \xi_k, & \text{if } \displaystyle\min_{1\leq k\leq N} \xi_k \geq 0; \\ +\infty, & \text{otherwise.} \end{cases} \tag{9.35}$$

Proof. Denote by φ the function defined in (9.27). Then Example 9.29 asserts that $\varphi \in \Gamma_0(\mathbb{R})$. First, take $x \in \mathcal{H}$ such that $x \geq 0$ μ-a.e., and set $C = \big\{\omega \in \Omega \;\big|\; 0 \leq x(\omega) < 1\big\}$ and $D = \big\{\omega \in \Omega \;\big|\; x(\omega) \geq 1\big\}$. Since, for every $\xi \in \mathbb{R}_+$, $|\varphi(\xi)| = |\xi\ln(\xi) - \xi| \leq 1_{[0,1[}(\xi) + \xi^2 1_{[1,+\infty[}(\xi)$, we have

$$\begin{aligned}\int_\Omega \big|\varphi\big(x(\omega)\big)\big|\mu(d\omega) &= \int_C \big|\varphi\big(x(\omega)\big)\big|\mu(d\omega) + \int_D \big|\varphi\big(x(\omega)\big)\big|\mu(d\omega) \\ &\leq \mu(C) + \int_D |x(\omega)|^2\mu(d\omega) \\ &\leq \mu(\Omega) + \|x\|^2 \\ &< +\infty, \end{aligned} \tag{9.36}$$

and therefore $\varphi \circ x \in L^1\big((\Omega,\mathcal{F},\mu);\mathbb{R}\big)$. Now take $x \in \mathcal{H}$ and set $A = \big\{\omega \in \Omega \;\big|\; x(\omega) \geq 0\big\}$ and $B = \big\{\omega \in \Omega \;\big|\; x(\omega) < 0\big\}$. Then

$$\begin{aligned}\int_\Omega \varphi(x(\omega))\mu(d\omega) &= \int_A \varphi(x(\omega))\mu(d\omega) + \int_B \varphi(x(\omega))\mu(d\omega) \\ &= \begin{cases} \displaystyle\int_\Omega x(\omega)\big(\ln(x(\omega)) - 1\big)\mu(d\omega), & \text{if } x \geq 0 \;\; \mu\text{-a.e. on } \Omega; \\ +\infty, & \text{otherwise} \end{cases} \\ &= f(x). \end{aligned} \tag{9.37}$$

Altogether, it follows from Proposition 9.32(i) with $\mathsf{H} = \mathbb{R}$ that $f \in \Gamma_0(\mathcal{H})$.

(i): Special case when μ is a probability measure.

(ii): Special case when $\Omega = \{1,\ldots,N\}$, $\mathcal{F} = 2^\Omega$, and μ is the counting measure, i.e., for every $C \in 2^\Omega$, $\mu(C)$ is the cardinality of C. ☐

Exercises

Exercise 9.1 Let $f\colon \mathcal{H} \to \left]-\infty,+\infty\right]$ be lower semicontinuous and *midpoint convex* in the sense that

$$(\forall x \in \operatorname{dom} f)(\forall y \in \operatorname{dom} f) \quad f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}. \tag{9.38}$$

Show that f is convex.

Exercise 9.2 Let $f\colon \mathcal{H} \to \left]-\infty,+\infty\right]$ be midpoint convex. Show that f need not be convex.

Exercise 9.3 Provide a family of continuous linear functions the supremum of which is neither continuous nor linear.

Exercise 9.4 Let $f \in \Gamma_0(\mathcal{H})$. Show that $\mathbb{R} \cap \operatorname{ran} f$ is convex, and provide an example in which $\operatorname{ran} f$ is not convex.

Exercise 9.5 Provide an example of a convex function $f\colon \mathcal{H} \to [-\infty,+\infty]$ such that $\operatorname{ran} f = \{-\infty, 0, +\infty\}$. Compare with Proposition 9.6.

Exercise 9.6 Set $\mathcal{C} = \big\{C \subset \mathcal{H} \;\big|\; C \text{ is nonempty, closed, and convex}\big\}$ and set

$$(\forall C \in \mathcal{C}) \quad \Upsilon_C\colon \mathcal{H} \to [-\infty,+\infty] : x \mapsto \begin{cases} -\infty, & \text{if } x \in C;\\ +\infty, & \text{otherwise.}\end{cases} \tag{9.39}$$

Prove that $\mathcal{C} \to \big\{f \in \Gamma(\mathcal{H}) \;\big|\; -\infty \in f(\mathcal{H})\big\}\colon C \mapsto \Upsilon_C$ is a bijection.

Exercise 9.7 Let $f\colon \mathcal{H} \to \left]-\infty,+\infty\right]$ be convex. Show that f is continuous if and only if it is lower semicontinuous and $\operatorname{cont} f = \operatorname{dom} f$.

Exercise 9.8 Let $f\colon \mathcal{H} \to \left]-\infty,+\infty\right]$ be convex and set $\mu = \inf f(\mathcal{H})$. Prove the following statements:

(i) $f \in \Gamma(\mathcal{H}) \Leftrightarrow (\forall \xi \in \left]\mu,+\infty\right[)\ \operatorname{lev}_{\leq \xi} f = \overline{\operatorname{lev}_{< \xi} f}$.
(ii) $\operatorname{cont} f = \operatorname{dom} f \Leftrightarrow (\forall \xi \in \left]\mu,+\infty\right[)\ \operatorname{lev}_{< \xi} f = \operatorname{int} \operatorname{lev}_{\leq \xi} f$.
(iii) f is continuous $\Leftrightarrow (\forall \xi \in \left]\mu,+\infty\right[)\ \operatorname{lev}_{= \xi} f = \operatorname{bdry} \operatorname{lev}_{\leq \xi} f$.

Exercise 9.9 Let $(e_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$, let $(\omega_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}_+$, and let $(p_n)_{n\in\mathbb{N}}$ be a sequence in $\left[1,+\infty\right[$. Set $f\colon \mathcal{H} \to \left]-\infty,+\infty\right] : x \mapsto \sum_{n\in\mathbb{N}} \omega_n \,|\langle x \mid e_n\rangle|^{p_n}$. Show that $f \in \Gamma_0(\mathcal{H})$.

Exercise 9.10 Use Proposition 8.12(ii) and Proposition 9.28 to verify Example 9.29 and Example 9.30.