Deconvolution of non-Gaussian linear processes with vanishing spectral values

(non-minimum phase/cumulant spectra/filter/noise)

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ABSTRACT We consider the problem of estimating the filter generating a non-Gaussian linear process and the deconvolution of that process when the spectral density of the process has zeros. Without using a minimum phase assumption we show that often if there are only finitely many zeros there are procedures to effect such an estimation and deconvolution.

Linear processes X_t are generated by an independent identically distributed input sequence η_t convolved with a linear filter α_t

$$X_t = \sum_j \alpha_j \eta_{t-j}. \quad [1]$$

This model is of considerable interest in geoexploration (1–4). Given observations on X_t only, a main interest is to estimate the filter coefficients α_t and the input sequence η_t by deconvolving X_t .

Let $\alpha(z) = \sum \alpha_j z^j$ be the z -transform of the filter α_t . Then the spectral density of $\{X_t\}$ is $f(\lambda) = \sigma_\eta^2 |\alpha(e^{-i\lambda})|^2 / 2\pi$, which is completely determined by the second-order properties of $\{X_t\}$ and does not contain the phase information of $\alpha(z)$. If η_t has a Gaussian distribution then the process $\{X_t\}$ is completely determined by second-order properties and one can only determine $\alpha(z)$ up to $|\alpha(z)|$. There is the indeterminacy of the phase of $\alpha(z)$. Often this is resolved by making the ad hoc assumption that $\alpha(z)$ is minimum phase. If $\alpha(z)$ is polynomial this means the roots are outside the unit disc.

Often, η_t appears to have a non-Gaussian distribution and then one need not assume $\alpha(z)$ minimum phase. By using information provided by the higher-order (greater than 2) cumulants, the phase of $\alpha(z)$ can be estimated. Results using higher-order cumulant spectral techniques have been given by Rosenblatt (5) and by Lii and Rosenblatt (3, 6). A main result is summarized below.

LEMMA. Consider a non-Gaussian linear process X_t given in Eq. 1 with η_t having all their moments finite. Let $\sum |\alpha_j| < \infty$ and assume $\alpha(e^{-i\lambda}) \neq 0$ for all λ . The transfer function $\alpha(e^{-i\lambda})$ can then be identified in terms of observations only on $\{X_t\}$ up to an indeterminate integer n in a factor $e^{in\lambda}$ that corresponds to a shift of index of X_t and an indeterminate sign of $\alpha(1) = \sum \alpha_k$. For this result it is sufficient to have some finite moment of order $k > 2$ with cumulant $\gamma_k \neq 0$.

The basic idea is to obtain the modulus of $\alpha(e^{-i\lambda})$ using the spectral density $f(\lambda)$ of $\{X_t\}$ [since $|\alpha(e^{-i\lambda})| = [2\pi f(\lambda)]^{1/2}$] assuming $E\eta_t = 0$ and $E\eta_t^2 = 1$. There are a number of consistent estimates of $f(\lambda)$ discussed by Priestley (7). The phase $h(\lambda) = \arg\{\alpha(e^{-i\lambda})\alpha(1)/\alpha(1)\}$ of $\alpha(e^{-i\lambda})$ is obtained

by making use of the relation

$$h(\lambda_1) + \dots + h(\lambda_{k-1}) - h(\lambda_1 + \dots + \lambda_{k-1}) \\ = \arg \left[\frac{\alpha(1)}{|\alpha(1)|} \right]^k \gamma_k^{-1} b_k(\lambda_1, \dots, \lambda_{k-1}), \quad [2]$$

which is derived from the k th-order cumulant spectral density of the process $\{X_t\}$

$$b_k(\lambda_1, \dots, \lambda_{k-1}) \\ = \frac{\gamma_k}{(2\pi)^{k-1}} \alpha(e^{-i\lambda_1}) \dots \alpha(e^{-i\lambda_{k-1}}) \alpha(e^{i(\lambda_1 + \dots + \lambda_{k-1})}). \quad [3]$$

Given an estimate $\hat{b}_k$ of b_k one can obtain an estimate $\hat{h}(\lambda)$ of $h(\lambda)$. $\hat{\alpha}(e^{-i\lambda}) = |\hat{\alpha}(e^{-i\lambda})|e^{i\hat{h}(\lambda)}$ is an estimate of $\alpha(e^{-i\lambda})$. One can deconvolve X_t and obtain estimates of b_k by using

$$\hat{\eta}_t = \hat{\alpha}^{-1} * X_t. \quad [4]$$

THE RESULTS

In many contexts the spectral density $f(\lambda)$ does have zeros and it is of some interest to see whether one can carry out deconvolution and in the case of this model recover the original non-Gaussian white noise sequence η_n . We have already remarked that, under the assumption of the lemma, *Condition A* is valid.

Condition A. The function $\alpha(e^{-i\lambda})$ can be consistently estimated up to a factor $e^{in\lambda}$ with an integer and the sign ± 1 .

In the results we mention, the important details of how to estimate $f(\lambda)$ and $\alpha(e^{-i\lambda})$ will not be dealt with in any detail here. Some of the aspects of estimation of $\alpha(e^{-i\lambda})$ can be found in the papers by Lii and Rosenblatt (3, 6).

One should note that *Condition A* will be satisfied under the following assumption concerning $\alpha(e^{-i\lambda})$.

PROPOSITION. If $\alpha(e^{-i\lambda})$ is the restriction to the unit circle of a function $\alpha(z) \neq 0$ analytic in an annulus about the unit circle, *Condition A* is satisfied. Such a function can have at most a finite number of zeros on $|z| = 1$.

Under this assumption the procedure suggested via equations 2 and 3 of the paper for the estimation of $\alpha(e^{-i\lambda})$ will still be asymptotically valid (see additional remark at end of paper).

THEOREM 1. Under *Condition A* the process X_n can be deconvolved so as to obtain the process η_n up to sign and a time shift by making use of the filter corresponding to

$$\{f_n(\lambda) + a_n\}^{-1/2} \exp\{-i \arg \alpha_n(e^{-i\lambda})\}, \quad [5]$$

where $f_n(\lambda) (\geq 0)$, $\alpha_n(e^{i\lambda})$ are appropriate consistent estimates of $f(\lambda)$, $\alpha(e^{-i\lambda})$ based on a sample size n and a_n an appropriate sequence of positive numbers with limit zero.

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We shall also consider the case in which the process x_n has added to it Gaussian white noise ξ_t with small variance $a > 0$. We thus have the observed process

$$\begin{aligned}\xi_n = x_n + \xi_n &= \int_{-\pi}^{\pi} e^{in\lambda} \alpha(e^{-i\lambda}) dz_{\eta}(\lambda) \\ &+ \int_{-\pi}^{\pi} e^{in\lambda} dz_{\xi}(\lambda),\end{aligned}\quad [6]$$

with $z_{\eta}(\lambda)$, $z_{\xi}(\lambda)$ random spectral processes corresponding to the processes η and ξ .

THEOREM 2. Under Condition A and $\sum |a_i| < \infty$, the process X_n can be deconvolved so as to obtain the process η_n up to sign and a time shift by applying the filter corresponding to Eq. 5 to the process ξ_n and letting $n \rightarrow \infty$.

Details, particularly those concerned with estimation and derivation of results, will be provided later.

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