**SOME RENEWAL THEOREMS WITH APPLICATION TO A FIRST
PASSAGE PROBLEM**

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SOME RENEWAL THEOREMS WITH APPLICATION TO A FIRST PASSAGE PROBLEM¹

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1. Introduction. Let X_i , $i=1, 2, 3, \dots$ be a sequence of independent and identically distributed random variables with $E|X_i| < \infty$, $EX_i = \mu > 0$. Write $X_i^- = -\min(0, X_i)$, $S_n = \sum_{i=1}^n X_i$ and $M_n = \max_{1 \leq k \leq n} S_k$. In this paper we shall discuss the asymptotic behaviour as $x \rightarrow \infty$ of the sums $\sum_{n=1}^{\infty} a_n \Pr(S_n \leq x)$ and $\sum_{n=1}^{\infty} a_n \Pr(M_n \leq x)$ for certain classes of positive coefficient sequences $\{a_n\}$ and use the results on the latter sums to investigate the behaviour of the first passage time out of the interval $(-\infty, x]$ for the process S_n as $x \rightarrow \infty$.

The analysis that we shall use in obtaining the theorems on asymptotic behaviour follows closely on that of Smith [6] who discussed sums $\sum_{n=1}^{\infty} a_n \Pr(S_n \leq x)$ for a class of coefficient sequences that we shall also discuss and for non-identically distributed random variables. In fact, our Theorem 1 follows directly from a specialization of the analysis of Smith. One of the particularly interesting characteristics of this technique is that it enables us to study the asymptotic behaviour of the sums $\sum_{n=1}^{\infty} a_n \Pr(S_n \leq x)$ and $\sum_{n=1}^{\infty} a_n \Pr(M_n \leq x)$ in the one operation in spite of essential differences in their behaviour.

2. Renewal theorems. For the first set of positive term coefficient sequences $\{a_n\}$ that we consider we shall suppose (as in [6]) that there exist real numbers $\alpha > 0$, $\gamma \geq 0$ and some non-negative function of slow growth $L(x)$ such that

$$(1) \quad \sum_{n=1}^{\infty} a_n x^n \sim [\alpha/(1-x)^\gamma] L(1-x)^{-1}, \quad \text{as } x \rightarrow 1^-.$$

This is satisfied, for example, if

$$a_n \sim [\alpha/\Gamma(\gamma)] n^{\gamma-1} L(n) \quad \text{as } n \rightarrow \infty$$

using an Abelian theorem of Doetsch [3], 460.

In the subsequent work we shall need the following definition:

DEFINITION. The index k of the sequence $\{a_n\}$ is the least real k such that $a_n = O(n^k)$.

Consideration will be restricted to cases where $\sum a_n$ diverges.

THEOREM 1. Suppose $E|X| < \infty$, $EX = \mu > 0$. Let k be the index of the sequence $\{a_n\}$ and m be non-negative. In order that

$$\sum_{n=1}^{\infty} a_n \Pr(S_n \leq x) \sim [\alpha L(x)/\Gamma(1+\gamma)] (x/\mu)^\gamma \quad \text{as } x \rightarrow \infty$$

for each sequence $\{a_n\}$ such that $k \leq m$ it is necessary and sufficient that $E|X^-|^{m+2} < \infty$.

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THEOREM 2. Suppose $E|X| < \infty$, $EX = \mu > 0$. Let k be the index of the sequence $\{a_n\}$ and m be non-negative. In order that

$$\sum_{n=1}^{\infty} a_n \Pr(M_n \leq x) \sim [\alpha L(x)/\Gamma(1 + \gamma)](x/\mu)^\gamma \quad \text{as } x \rightarrow \infty$$

for each sequence $\{a_n\}$ such that $k \leq m$ it is necessary that $E|X^-|^{m+1} < \infty$ and sufficient that $E|X^-|^{m+2} < \infty$.

I conjecture that the condition $E|X^-|^{m+1} < \infty$ is both necessary and sufficient in Theorem 2. It is certainly known that in the particular case where $a_n = 1$ for all n so that $k = 0$ we only need $E|X^-| < \infty$ (see for example Chow and Robbins [2]).

The proof of the two theorems shall be deferred until we have given four lemmas. The first of these is given in a form which is more general than we shall need subsequently as it has some independent interest.

LEMMA 1. Suppose the random variables X_i , $i = 1, 2, 3, \dots$ are independent and identically distributed. Write $S_n = \sum_{i=1}^n X_i$, $M_n = \max_{1 \leq k \leq n} S_k$, and, in the case where $E|X_i| < \infty$, $EX_i = \mu$. If $E|X_i|^r < \infty$ with $1 \leq r < 2$ and $\mu \geq 0$, then

$$n^{-1/r}(M_n - n\mu) \rightarrow_{\text{a.s.}} 0.$$

If $E|X_i|^r < \infty$ with $0 < r < 1$ or $E|X_i|^r < \infty$ with $1 \leq r < 2$ and $\mu < 0$, then

$$n^{-1/r}M_n \rightarrow_{\text{a.s.}} 0.$$

("a.s." denotes almost sure convergence).

The corresponding almost sure convergence versions for the sums S_n have been given by Kolmogorov ($r = 1$) and Marcinkiewicz ($r \neq 1$) (see for example Loève [5] 242, 243).

PROOF. Suppose $E|X_i|^r < \infty$. Let $c_r = \mu$ if $1 \leq r < 2$; $c_r = 0$ if $0 < r < 1$. If $c_r \geq 0$, we have

$$\begin{aligned} M_n - nc_r &= \max_{1 \leq j \leq n} S_j - nc_r \\ &\leq \max_{1 \leq j \leq n} (S_j - jc_r) \\ &\leq \max_{1 \leq j \leq n} j^{1/r} a_j, \end{aligned}$$

where $a_j = j^{-1/r}|S_j - jc_r|$. Therefore, if $M_n \geq nc_r$, $0 \leq M_n - nc_r \leq \max_{1 \leq j \leq n} j^{1/r} a_j$, and if $M_n < nc_r$, $0 > M_n - nc_r \geq S_n - nc_r = -n^{1/r}a_n \geq -\max_{1 \leq j \leq n} j^{1/r} a_j$, so that $0 \leq n^{-1/r}|M_n - nc_r| \leq \max_{1 \leq j \leq n} j^{1/r} a_j$. In order to obtain the desired result in this case it suffices to show that for arbitrary $\epsilon > 0$,

$$\Pr(\bigcup_{k \geq n} \{\max_{1 \leq j \leq k} j^{1/r} k^{-1/r} a_j \geq \epsilon\}) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Now the strong laws of Kolmogorov and Marcinkiewicz ([5], 242, 243) imply that for arbitrary $\epsilon > 0$,

$$\Pr(\max_{j \geq n} a_j \geq \epsilon) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Therefore, given $\eta > 0$ arbitrarily small we can choose an N so large that

$$(2) \quad \Pr(\max_{j > N} a_j \geq \tfrac{1}{2}\epsilon) \leq \tfrac{1}{2}\eta$$

and then select an $n_0(>N)$ so large that for $n \geq n_0$,

$$(3) \quad \Pr(N^{1/r} n^{-1/r} \max_{1 \leq j \leq N} a_j \geq \tfrac{1}{2}\epsilon) \leq \tfrac{1}{2}\eta.$$

Then, for $n \geq n_0$,

$$\begin{aligned} \Pr(\bigcup_{k \geq n} \{\max_{1 \leq j \leq k} j^{1/r} k^{-1/r} a_j \geq \epsilon\}) \\ &= \Pr(\bigcup_{k \geq n} \{\max_{1 \leq j \leq N} j^{1/r} k^{-1/r} a_j + \max_{N < j \leq k} j^{1/r} k^{-1/r} a_j \geq \epsilon\}) \\ &\leq \Pr(\bigcup_{k \geq n} \{\max_{1 \leq j \leq N} j^{1/r} k^{-1/r} a_j + \max_{N < j \leq k} a_j \geq \epsilon\}) \\ &\leq \Pr(N^{1/r} n^{-1/r} \max_{1 \leq j \leq N} a_j + \max_{j > N} a_j \geq \epsilon) \\ &\leq \Pr(N^{1/r} n^{-1/r} \max_{1 \leq j \leq N} a_j \geq \tfrac{1}{2}\epsilon) + \Pr(\max_{j > N} a_j \geq \tfrac{1}{2}\epsilon) \\ &\leq \eta, \end{aligned}$$

using (2) and (3). This completes the proof of this part of the lemma. It remains to consider the case $c_r < 0$.

In the case $c_r < 0$, $M_n^+ = \max(0, M_n)$ actually has a proper limiting distribution (finite with probability one), $M^+ = \lim_{n \rightarrow \infty} M_n^+$.

We have

$$\begin{aligned} \Pr(\bigcup_{k \geq n} \{k^{-1/r} |M_k| \geq \epsilon\}) \\ &= \Pr(\bigcup_{k \geq n} \{k^{-1/r} M_k \geq \epsilon\} \cup \bigcup_{k \geq n} \{k^{-1/r} M_k \leq -\epsilon\}) \\ &\leq \Pr(\bigcup_{k \geq n} \{k^{-1/r} M_k \geq \epsilon\}) + \Pr(\bigcup_{k \geq n} \{k^{-1/r} M_k \leq -\epsilon\}) \\ &\leq \Pr(\bigcup_{k \geq n} \{M^+ \geq k^{1/r} \epsilon\}) + \Pr(\bigcup_{k \geq n} \{X_1 \leq -k^{1/r} \epsilon\}) \\ &= \Pr(M^+ \geq n^{1/r} \epsilon) + \Pr(X_1 \leq -n^{1/r} \epsilon) \end{aligned}$$

and both these terms approach zero as $n \rightarrow \infty$. Therefore, $n^{-1/r} M_n \rightarrow_{a.s.} 0$ as required. This completes the proof of the lemma.

In the subsequent work we shall write $F_n(x) = \Pr(S_n \leq x)$, $G_n(x) = \Pr(M_n \leq x)$ and $H_n(x)$ to mean either $F_n(x)$ or $G_n(x)$ (so that if a property holds for both $F_n(x)$ and $G_n(x)$ it holds for $H_n(x)$ and conversely).

LEMMA 2. $\int_{\mu}^{\infty} \{1 - H_n(nx)\} dx \rightarrow 0$ as $n \rightarrow \infty$.

PROOF. Introduce the new random variables $Y_i = \max(0, X_i)$, $i = 1, 2, 3, \dots$. Suppose $EY_i = \alpha$. If $M_n > x$ then $\sum_{i=1}^n Y_i > x$ so that $\Pr(\sum_{i=1}^n Y_i > x) \geq \Pr(M_n > x) \geq \Pr(S_n > x)$ and hence

$$(4) \quad 1 - F_n(x) \leq 1 - G_n(x) \leq 1 - K_n(x),$$

where we have written $K_n(x) = \Pr(\sum_{i=1}^n Y_i \leq x)$.

We show firstly that $\int_{\alpha}^{\infty} \{1 - K_n(nx)\} dx \rightarrow 0$ as $n \rightarrow \infty$. We have, by a simple integration by parts,

$$\begin{aligned} \alpha &= \int_0^{\infty} \{1 - K_n(nx)\} dx \\ &= \int_0^{\alpha} \{1 - K_n(nx)\} dx + \int_{\alpha}^{\infty} \{1 - K_n(nx)\} dx. \end{aligned}$$

Further, by the law of large numbers, $K_n(nx) \rightarrow 0$ as $n \rightarrow \infty$ for $x < \alpha$. It follows

from the mean value theorem that $\int_0^\alpha \{1 - K_n(nx)\} dx \rightarrow \alpha$ as $n \rightarrow \infty$ and hence that $\int_\alpha^\infty \{1 - K_n(nx)\} dx \rightarrow 0$ as $n \rightarrow \infty$.

Then, making use of (4),

$$0 \leq \int_\mu^\infty \{1 - F_n(nx)\} dx \leq \int_\mu^\infty \{1 - G_n(nx)\} dx \\ \leq \int_\mu^\alpha \{1 - G_n(nx)\} dx + \int_\alpha^\infty \{1 - K_n(nx)\} dx.$$

Now, in view of Lemma 1 (case $r = 1, \mu > 0$), $G_n(nx) \rightarrow 1$ as $n \rightarrow \infty$ for $x > \mu$ so that by the mean value theorem $\int_\mu^\alpha \{1 - G_n(nx)\} dx \rightarrow 0$ as $n \rightarrow \infty$. We have shown that $\int_\alpha^\infty \{1 - K_n(nx)\} dx \rightarrow 0$ as $n \rightarrow \infty$ so the proof is complete.

LEMMA 3. (Smith [6]). *If the non-negative constants $\{a_n\}$ satisfy (1) then as $s \rightarrow 0+$,*

$$\sum_{n=1}^\infty a_n e^{-\mu sn} \sim \alpha(\mu s)^{-\gamma} L(s^{-1}).$$

LEMMA 4. *If the non-negative constants $\{a_n\}$ satisfy (1) then as $s \rightarrow 0+$,*

$$\sum_{n=1}^\infty n a_n e^{-\mu sn} \sim \alpha(\mu s)^{-\gamma-1} L(s^{-1}) \quad \text{if } \gamma > 0 \\ \limsup_{s \rightarrow 0+} s [L(s^{-1})]^{-1} \sum_{n=1}^\infty n a_n e^{-\mu sn} < \alpha \mu^{-1} \quad \text{if } \gamma = 0.$$

Smith [6] has established the former result, ignoring the possibility of the case $\gamma = 0$. The result given above for $\gamma = 0$ can be readily extracted from Smith's proof and is adequate in the present context.

PROOF OF THEOREMS 1 AND 2. We follow the methods of [6] but work the proof in terms of $H_n(x)$.

Suppose firstly that $E|X|^{m+2} < \infty$. Take β arbitrary with $0 < \beta < \mu$. Consider

$$0 \leq K_n = \int_{n\beta}^{n\mu} e^{-sx} H_n(x) dx \quad (s \geq 0) \\ = n \int_\beta^\mu e^{-nsx} H_n(nx) dx \\ \leq n e^{-n\beta s} \int_\beta^\mu H_n(nx) dx.$$

Now using the law of large numbers and the inequality $H_n(y) \leq F_n(y)$ (or alternatively referring to Lemma 1 as well), we see that $H_n(nx) \rightarrow 0$ as $n \rightarrow \infty$ for all $x < \mu$. Hence, using the mean value theorem, we may write

$$(5) \quad K_n = n e^{-n\beta s} \delta_n',$$

where $\delta_n' \rightarrow 0$ as $n \rightarrow \infty$, uniformly in $s \geq 0$.

Now consider

$$0 \leq L_n = \int_{n\mu}^\infty e^{-sx} \{1 - H_n(x)\} dx \quad (s \geq 0) \\ = n \int_\mu^\infty e^{-nsx} \{1 - H_n(nx)\} dx \\ \leq n e^{-n\beta s} \int_\mu^\infty \{1 - H_n(nx)\} dx.$$

In view of Lemma 2, we may write

$$(6) \quad L_n = n e^{-n\beta s} \delta_n''$$

where $\delta_n'' \rightarrow 0$ as $n \rightarrow \infty$, uniformly in $s \geq 0$.

Combining (5) and (6) and putting $\delta_n = \delta_n' - \delta_n''$, we obtain

$$(7) \quad \sum_{n=1}^{\infty} a_n(L_n - K_n) = - \sum_{n=1}^{\infty} n a_n \delta_n e^{-n\beta s}.$$

Now given arbitrary $\epsilon > 0$ we can choose an integer $n_0(\epsilon)$ so large that $|\delta_n| < \epsilon$ for all $n > n_0$. Then,

$$|\sum_{n=1}^{\infty} n a_n \delta_n e^{-n\beta s}| < \sum_{n=1}^{n_0} n a_n |\delta_n| e^{-n\beta s} + \epsilon \sum_{n=1}^{\infty} n a_n e^{-n\beta s},$$

so that by Lemma 4,

$$\begin{aligned} \limsup_{s \rightarrow 0+} [(\mu s)^{\gamma+1}/L(s^{-1})] |\sum_{n=1}^{\infty} n a_n \delta_n e^{-n\beta s}| &\leq \epsilon \gamma \alpha \text{ if } \gamma > 0 \\ &\leq \epsilon \alpha \text{ if } \gamma = 0 \end{aligned}$$

and hence

$$(8) \quad [(\mu s)^{\gamma+1}/L(s^{-1})] \sum_{n=1}^{\infty} a_n(L_n - K_n) \rightarrow 0 \quad \text{as } s \rightarrow 0+.$$

Next write

$$\begin{aligned} (9) \quad \Phi_{\beta}(x) &= \sum_{n=1}^{\infty} a_n H_n(x) U(x - n\beta) \\ &= \sum_{n=1}^{\infty} a_n U(x - n\mu) - \sum_{n=1}^{\infty} a_n \{U(x - n\mu) - H_n(x)\} U(x - n\beta) \end{aligned}$$

where

$$\begin{aligned} U(x) &= 1, & x &\geq 0, \\ &= 0, & x &< 0. \end{aligned}$$

If we denote the Laplace transform of a function $A(x)$ by $A^0(s) = \int_0^{\infty} e^{-sx} A(x) dx$ we have

$$(10) \quad \Phi_{\beta}^0(s) = s^{-1} \sum_{n=1}^{\infty} a_n e^{-n\mu s} - \sum_{n=1}^{\infty} a_n(L_n - K_n);$$

the term by term integration being justified by monotone convergence.

From (8), (10) and Lemma 3 it follows that

$$(11) \quad [(\mu s)^{\gamma+1}/L(s^{-1})] \Phi_{\beta}^0(s) \rightarrow \alpha \mu \quad \text{as } s \rightarrow 0+.$$

Appealing to a Tauberian theorem of Doetsch [3], 511, we then obtain

$$(12) \quad [1/t^{\gamma+1}L(t)] \int_0^t \Phi_{\beta}(x) dx \rightarrow \alpha/\mu^{\gamma} \Gamma(\gamma + 2).$$

Now for $t > 0$ and $0 < \theta < 1$,

$$\Phi_{\beta}(\theta t)(t - \theta t) \leq \int_{\theta t}^t \Phi_{\beta}(x) dx \leq \Phi_{\beta}(t)(t - \theta t)$$

so that

$$\begin{aligned} [1/t^{\gamma}L(t)] \Phi_{\beta}(\theta t) &\leq [1/(1 - \theta)] \{ [1/t^{\gamma+1}L(t)] \int_0^t \Phi_{\beta}(x) dx \\ &\quad - [1/t^{\gamma+1}L(t)] \int_0^{\theta t} \Phi_{\beta}(x) dx \} \leq [1/t^{\gamma}L(t)] \Phi_{\beta}(t). \end{aligned}$$

Then, for fixed θ and $t \rightarrow \infty$, $[1/t^{\gamma+1}L(t)] \int_0^t \Phi_{\beta}(x) dx \rightarrow \alpha/\mu^{\gamma} \Gamma(\gamma + 2)$, $[1/(\theta t)^{\gamma+1}L(t)] \int_0^{\theta t} \Phi_{\beta}(x) dx \rightarrow [1/(\theta t)^{\gamma+1}L(\theta t)] \int_0^{\theta t} \Phi_{\beta}(x) dx \rightarrow \alpha/\mu^{\gamma} \Gamma(\gamma + 2)$ by

(12) so that

$$(13) \quad \limsup_{t \rightarrow \infty} [1/t^\gamma L(t)]\Phi_\beta(\theta t) \leq [\alpha/\mu^\gamma \Gamma(\gamma + 2)](1 - \theta^{\gamma+1})/(1 - \theta) \\ \leq \liminf_{t \rightarrow \infty} [1/t^\gamma L(t)]\Phi_\beta(t).$$

Taking $\theta \rightarrow 1$ in the right hand part of inequality (13) gives

$$(14) \quad \liminf_{t \rightarrow \infty} [1/t^\gamma L(t)]\Phi_\beta(t) \geq \alpha/\mu^\gamma \Gamma(\gamma + 1).$$

Further, the left hand part of the inequality (13) can be written

$$\limsup_{t \rightarrow \infty} [1/(\theta t)^\gamma L(\theta t)]\Phi_\beta(\theta t) \leq [-\theta\alpha/\mu^\gamma \Gamma(\gamma + 2)](1 - \theta^{-(\gamma+1)})/(1 - \theta)$$

and the left hand side of this is equal to $\limsup_{t \rightarrow \infty} [1/t^\gamma L(t)]\Phi_\beta(t)$ for θ fixed. Then, taking $\theta \rightarrow 1$ in the right hand side, we obtain

$$(15) \quad \limsup_{t \rightarrow \infty} [1/t^\gamma L(t)]\Phi_\beta(t) \leq \alpha/\mu^\gamma \Gamma(\gamma + 1),$$

so that combining (14) and (15),

$$(16) \quad \lim_{t \rightarrow \infty} [1/t^\gamma L(t)]\Phi_\beta(t) = \alpha/\mu^\gamma \Gamma(\gamma + 1).$$

Now under the condition $E|X^-|^{m+2} < \infty$ we certainly have for all x , $-\infty < x < \infty$, $\sum n^m H_n(x) < \infty$ in view of Theorem A of Heyde [4] and the inequality $H_n(x) \leq F_n(x)$. Hence, $\sum a_n H_n(x) < \infty$, since the sequence $\{a_n\}$ has index $k \leq m$. Write

$$(17) \quad \sum_{n=1}^{\infty} a_n H_n(x) = \Phi_\beta(x) + \Psi_\beta(x),$$

where

$$(18) \quad \Psi_\beta(x) = \sum_{n=1}^{\infty} a_n H_n(x) \{1 - U(x - n\beta)\}.$$

We shall go on to show that $\Psi_\beta(x)/x^\gamma L(x) \rightarrow 0$ as $x \rightarrow \infty$.

Define a new sequence of random variables Y_i , $i = 1, 2, 3, \dots$ by

$$Y_i = X_i - \beta.$$

Then, $EY > 0$ and $E|Y^-|^{m+2} < \infty$ since $E|X^-|^{m+2} < \infty$. It follows from Theorem A of [4] applied to the Y 's that for $k \leq m$, $\sum n^k F_n(n\beta) < \infty$, and since $H_n(n\beta) \leq F_n(n\beta)$,

$$(19) \quad \sum n^k H_n(n\beta) < \infty.$$

Also, it is clear from (18) that $\Psi_\beta(x) \leq \sum_{n=1}^{\infty} a_n H_n(n\beta)$ and (19) ensures that this upper bound is finite since k is the index of the sequence $\{a_n\}$. We therefore have $\Psi_\beta(x)/x^\gamma L(x) \rightarrow 0$ as $x \rightarrow \infty$ and hence, using (16) and (17),

$$\sum_{n=1}^{\infty} a_n H_n(x) \sim [\alpha L(x)/\Gamma(1 + \gamma)](x/\mu)^\gamma \quad \text{as} \quad x \rightarrow \infty.$$

This result is true for all sequences $\{a_n\}$ with index $k \leq m$ and hence establishes the sufficiency parts of both Theorems 1 and 2.

The necessity parts of both the theorems are easy to establish. It suffices to

note that in particular $\sum n^m H_n(x) < \infty$ for all x , $-\infty < x < \infty$, and hence by Theorem A of [4] in the case $H_n(x) = F_n(x)$ we obtain $E|X^-|^{m+2} < \infty$ and by Theorem A of [4] together with the well known inequality $G_n(0) = \Pr(M_n \leq 0) \geq n^{-1} \Pr(S_n \leq 0)$ in the case $H_n(x) = G_n(x)$ we obtain $E|X^-|^{m+1} < \infty$. This completes the proof of both Theorems 1 and 2.

Some remarks on the possibility that the condition $E|X^-|^{m+1} < \infty$ might be both necessary and sufficient in Theorem 2 would seem in order. To establish that this is the case, it would be adequate to show that if $EX = \mu > 0$ and $E|X^-|^{k+1} < \infty$, then for $0 < \beta < \mu$,

$$\sum n^k G_n(n\beta) = \sum n^k \Pr(M_n \leq n\beta) < \infty.$$

The nearest approach to this that I have obtained is summarized in the following theorem:

THEOREM 3. *Suppose $E|X| < \infty$, $EX > 0$, $\Pr(X < 0) > 0$, and let k be a non-negative integer. A necessary and sufficient condition for the convergence of the series*

$$\sum_{n=1}^{\infty} n^k \Pr(M_n \leq x), \quad -\infty < x < \infty,$$

is that $E|X^-|^{k+1} < \infty$.

PROOF. It follows from the work of Heyde [4] that a necessary and sufficient condition for the convergence of the series $\sum n^k \Pr(M_n \leq 0)$ is that $E|X^-|^{k+1} < \infty$. Therefore, in order to complete the proof it is only necessary to show that the convergence of $\sum n^k \Pr(M_n \leq 0)$ implies that of $\sum n^k \Pr(M_n \leq x)$, $0 < x < \infty$.

To accomplish this we define a new sequence of random variables:

$$N_0 = S_0 = 0,$$

$$N_1 = \max(S_0, S_1) = X_1^+,$$

$$N_2 = \max(S_0, S_1, S_2) = (X_1 + X_2^+)^+,$$

...

$$N_n = \max(S_0, S_1, S_2, \dots, S_n) = (X_1 + (X_2 + \dots + (X_{n-1} + X_n^+)^+ \dots)^+)^+ \\ \dots$$

Since the X_i are independent and identically distributed we may write $N_0 = 0$, $N_{n+1} \sim (X + N_n)^+$, $n \geq 1$, where for two random variables X and Y , we write $X \sim Y$ if they have the same distribution. Clearly, $N_n = M_n^+ = \max(0, M_n)$, so that

$$\Pr(N_n \leq x) = \Pr(M_n \leq x), \quad x > 0,$$

$$\Pr(N_n = 0) = \Pr(M_n \leq 0).$$

Now for $A > 0$, $n \geq 1$, we have

$$\Pr(M_{n+1} \leq A) = \int_{-\infty}^A \Pr(M_n \leq A - y) d\Pr(X \leq y)$$

so that for arbitrary $B > 0$,

$$\begin{aligned}\sum_{n=1}^N n^\alpha \Pr(M_{n+1} \leq A) &= \int_{-\infty}^A \left\{ \sum_{n=1}^N n^\alpha \Pr(M_n \leq A - y) \right\} d\Pr(X \leq y) \\ &\geq \int_{-\infty}^{-B} \left\{ \sum_{n=1}^N n^\alpha \Pr(M_n \leq A - y) \right\} d\Pr(X \leq y) \\ &\geq \left\{ \sum_{n=1}^N n^\alpha \Pr(M_n \leq A + B) \right\} \Pr(X \leq -B).\end{aligned}$$

Then, choosing B so that $\Pr(X \leq -B) > 0$, we see that if $\sum n^\alpha \Pr(M_n \leq A)$ converges then $\sum n^\alpha \Pr(M_n \leq A + B)$ converges. Let $A \rightarrow 0+$; we see that the convergence of $\sum n^\alpha \Pr(M_n \leq 0)$ implies that of $\sum n^\alpha \Pr(M_n \leq rB)$ for all positive integral r and hence that of $\sum n^\alpha \Pr(M_n \leq x)$ for all x , $0 < x < \infty$. This completes the proof of the theorem.

The restriction to non-negative integral k in Theorem 3 is unfortunate. This comes about from the use of the derivatives of the generating function

$$\sum_{n=0}^{\infty} \Pr(M_n \leq 0) t^n = \exp \left\{ \sum_{n=1}^{\infty} n^{-1} t^n \Pr(S_n \leq 0) \right\}$$

in [4].

We now go on to consider coefficient sequences of the form $a_n = e^{rn}$, $r > 0$, and shall establish the following two theorems:

THEOREM 4. *Suppose $E|X| < \infty$, $EX = \mu > 0$. In order that*

$$\sum_{n=1}^{\infty} e^{rn} \Pr(S_n \leq x) \sim r^{-1} e^{xr\mu^{-1}} \quad \text{as } x \rightarrow \infty$$

for any r in some interval $0 < r < R$, it is necessary and sufficient that X^- should have an analytic characteristic function.

(The term “analytic characteristic function” is used for a characteristic function which is analytic in a strip containing the origin as an interior point.)

THEOREM 5. *Suppose $E|X| < \infty$, $EX = \mu > 0$. In order that*

$$\sum_{n=1}^{\infty} e^{rn} \Pr(M_n \leq x) \sim r^{-1} e^{xr\mu^{-1}} \quad \text{as } x \rightarrow \infty$$

for any r in some interval $0 < r < R$, it is necessary and sufficient that X^- should have an analytic characteristic function.

The proofs of these theorems, of course, follow markedly similar lines to the proofs of Theorems 1 and 2. A generalized coefficient sequence form along the lines of Theorems 1 and 2 is not, however, convenient in this case.

PROOF OF THEOREMS 4 AND 5. We construct the proofs in parallel fashion as we did with Theorems 1 and 2.

Suppose X^- has an analytic characteristic function. Take β arbitrary with $0 < \beta < \mu$. We consider to begin with

$$\begin{aligned}K_n &= \int_{\beta}^{\mu} e^{-sx} H_n(x) dx & (s > 0) \\ &\leq ne^{-n\beta s} \int_{\beta}^{\mu} H_n(nx) dx \\ &= ne^{-n\beta s} (\mu - \beta) H_n(n\xi)\end{aligned}$$

for some $\xi = \xi(n)$ in $\beta < \xi < \mu$. $(X - \xi)^-$ has an analytic characteristic function and $E(X - \xi) > 0$. It follows from Theorem B of [4] and the inequality

$H_m \leq F_m$ that for any r in some interval $0 < r < R$,

$$(20) \quad \sum e^{rm} H_m(m\xi) < \infty,$$

and hence, considering the totality of series (20) as n varies,

$$(21) \quad H_n(n\xi) = o(e^{-rn}) \quad \text{as } n \rightarrow \infty.$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} e^{rn} K_n &\leq (\mu - \beta) \sum_{n=1}^{\infty} n e^{-n\beta s} H_n(n\xi) e^{rn} \\ &< \infty \quad \text{as } s \rightarrow \mu^{-1}r+ \end{aligned}$$

in view of (21) and so

$$(22) \quad \lim_{s \rightarrow \mu^{-1}r+} (s - \mu^{-1}r) \sum_{n=1}^{\infty} e^{rn} K_n = 0.$$

Next consider

$$L_n = \int_{n\mu}^{\infty} e^{-sx} \{1 - H_n(x)\} dx \quad (s > 0).$$

Using the mean value theorem and the fact that $H_n(nx) \rightarrow 1$ as $n \rightarrow \infty$ for $x > \mu$, we may write

$$L_n = \delta_n \int_{n\mu}^{\infty} e^{-sx} dx = \delta_n s^{-1} e^{-n\mu s},$$

where $\delta_n > 0$ and $\delta_n \rightarrow 0$ as $n \rightarrow \infty$ uniformly in $s > 0$. Now given arbitrary $\epsilon > 0$ we can choose an integer $n_0(\epsilon)$ so large that $\delta_n < \epsilon$ for all $n > n_0$. Then, for $s > \mu^{-1}r$,

$$\begin{aligned} \sum_{n=1}^{\infty} e^{rn} L_n &= s^{-1} \sum_{n=1}^{\infty} \delta_n e^{-n\mu(s-\mu^{-1}r)} \\ &< s^{-1} \sum_{n=1}^{n_0} \delta_n e^{-n\mu(s-\mu^{-1}r)} + \epsilon s^{-1} \sum_{n=1}^{\infty} e^{-n\mu(s-\mu^{-1}r)}, \end{aligned}$$

and since for $s > \mu^{-1}r$,

$$\sum_{n=1}^{\infty} e^{-n\mu(s-\mu^{-1}r)} = e^{-\mu(s-\mu^{-1}r)} / (1 - e^{-\mu(s-\mu^{-1}r)}) \sim 1/\mu(s - \mu^{-1}r) \quad \text{as } s \rightarrow \mu^{-1}r+,$$

it follows that

$$(23) \quad \lim_{s \rightarrow \mu^{-1}r+} (s - \mu^{-1}r) \sum_{n=1}^{\infty} e^{rn} L_n = 0,$$

so that combining (22) and (23),

$$(24) \quad \lim_{s \rightarrow \mu^{-1}r+} (s - \mu^{-1}r) \sum_{n=1}^{\infty} e^{rn} (L_n - K_n) = 0.$$

Now consider the function

$$\begin{aligned} \Phi_{\beta}(x) &= \sum_{n=1}^{\infty} e^{rn} H_n(x) U(x - n\beta) \\ &= \sum_{n=1}^{\infty} e^{rn} U(x - n\mu) - \sum_{n=1}^{\infty} e^{rn} \{U(x - n\mu) - H_n(x)\} U(x - n\beta). \end{aligned}$$

Taking Laplace transforms, we have

$$\Phi_{\beta}^0(s) = s^{-1} \sum_{n=1}^{\infty} e^{-n\mu(s-\mu^{-1}r)} - \sum_{n=1}^{\infty} e^{rn} (L_n - K_n),$$

from which it follows, using (24), that $(s - \mu^{-1}r)\Phi_\beta^0(s) \rightarrow r^{-1}$ as $s \rightarrow \mu^{-1}r+$. Now let

$$(25) \quad \Theta_\beta^0(s - \mu^{-1}r) = \Phi_\beta^0(s);$$

we have $w\Theta_\beta^0(w) \rightarrow r^{-1}$ as $w \rightarrow 0+$, and appealing to a Tauberian theorem of Widder [7], Theorem 4.3, 192, we obtain

$$(26) \quad t^{-1} \int_0^t \Theta_\beta(x) dx \rightarrow r^{-1} \quad \text{as } t \rightarrow \infty.$$

Then, following through exactly the analysis that we used in the case of Theorems 1 and 2 (we are here dealing with the case $\gamma = 0$, $L(t) = 1$, $\alpha\mu^{-\gamma}[\Gamma(\gamma + 2)]^{-1} = r^{-1}$) we find from (26) that

$$(27) \quad \lim_{t \rightarrow \infty} \Theta_\beta(t) = r^{-1}.$$

However, using (25) and the uniqueness theorem for Laplace transformations, $\Theta_\beta(t) = e^{-r\mu^{-1}t}\Phi_\beta(t)$, so that (27) gives

$$(28) \quad \lim_{t \rightarrow \infty} e^{-r\mu^{-1}t}\Phi_\beta(t) = r^{-1}.$$

Now write

$$(29) \quad \sum_{n=1}^{\infty} e^{rn} H_n(x) = \Phi_\beta(x) + \Psi_\beta(x),$$

where

$$(30) \quad \Psi_\beta(x) = \sum_{n=1}^{\infty} e^{rn} H_n(x) \{1 - U(x - n\beta)\},$$

$\sum e^{rn} H_n(x)$ being finite for r in $0 < r < R$ by (20). Also it is clear from (30) and (20) that for all x , $\Psi_\beta(x) \leq \sum_{n=1}^{\infty} e^{rn} H_n(n\beta) < \infty$. We therefore have $e^{-r\mu^{-1}x}\Psi_\beta(x) \rightarrow 0$ as $x \rightarrow \infty$, and hence by (28), $\sum_{n=1}^{\infty} e^{rn} H_n(x) \sim r^{-1}e^{r\mu^{-1}x}$ as $x \rightarrow \infty$. This establishes the sufficiency parts of Theorems 4 and 5.

The necessity parts of the theorems follow readily from Theorem B of [4]. We have $\sum e^{rn} H_n(x) < \infty$ for all x , $-\infty < x < \infty$ and Theorem B gives the result directly in the case $H_n(x) = F_n(x)$. In the case $H_n(x) = G_n(x)$, we take $x > 0$ and make use of the well known relations $G_n(x) = \Pr(M_n \leq x) \geq \Pr(M_n \leq 0) \geq n^{-1} \Pr(S_n \leq 0)$. It is clear that the convergence of $\sum e^{rn} \Pr(M_n \leq x)$ implies the convergence of $\sum n^{-1}e^{rn} \Pr(S_n \leq 0)$ and hence that of $\sum e^{sn} \Pr(S_n \leq 0)$ for $s < r$ and making use of Theorem B again we see that X^- must have an analytic characteristic function. This completes the proofs of both Theorems 4 and 5.

3. Application to a first passage problem. Let X_i , $i=1, 2, 3, \dots$ be independent and identically distributed random variables with $E|X| < \infty$, $EX = \mu > 0$. Write $S_n = \sum_{i=1}^n X_i$ and $M_n = \max_{1 \leq k \leq n} S_k$. Consider a single boundary at x (≥ 0) so that if

$$G_0(x) = 1,$$

$$G_n(x) = \Pr(M_n \leq x), \quad n \geq 1,$$

the probability p_n that the first passage time, $M(x)$, out of the interval $(-\infty, x]$ for the process S_n is n is given by $p_n = G_{n-1}(x) - G_n(x)$, $n \geq 1$.

We introduce the probability generating function $P(\lambda) = \sum_{r=1}^{\infty} \lambda^r p_r$ for the first passage time distribution $\Pr(M(x) = n) = p_n$.

Formally differentiating,

$$P^{(1)}(1) = E[M(x)] = 1 + \sum_{r=1}^{\infty} G_r(x),$$

and for $k > 1$,

$$\begin{aligned} P^{(k)}(1) &= (\alpha)_k \quad (\text{the } k\text{th factorial moment of } M(x)) \\ &= k \sum_{r=k-1}^{\infty} (r)_k G_r(x) = \sum_{r=0}^k s(k, r) E\{[M(x)]^r\}, \end{aligned}$$

where $(r)_k = r(r-1)(r-2)\cdots(r-k+1)$ and $s(k, r)$ are the Stirling numbers of the first kind. It is thus clear that $E\{[M(x)]^r\} < \infty$ for some positive integer r if and only if $\sum n^{r-1} G_n(x) < \infty$. Also, the random variable $M(x)$ has an analytic characteristic function if and only if the radius of convergence of $P(\lambda)$ is greater than unity or equivalently if $\sum e^{rn} G_n(x) < \infty$ for some $r > 0$.

As a particular case of Theorem 2 we obtain for integral $k \geq 1$,

$$kx^{-k} \sum_{n=1}^{\infty} n^{k-1} G_n(x) \rightarrow \mu^{-k} \quad \text{as } x \rightarrow \infty,$$

so long as $E|X^-|^{k+1} < \infty$. Therefore, in view of the above comments, we see that as $x \rightarrow \infty$, $E\{[x^{-1}M(x)]^r\} \rightarrow \mu^{-r}$ for integral $r \geq 1$ so long as $E|X^-|^{r+1} < \infty$. (The $r = 1$ and $r = 2$ cases are included in the results of [2] and [1] respectively). If $E|X^-|^r = \infty$ then Theorem 3 shows us that $\sum n^{r-1} G_n(x)$ diverges and hence that $E\{[x^{-1}M(x)]^r\} = \infty$. If we have the condition that X^- possesses an analytic characteristic function, then it follows from Theorem B of [4] and the inequality, $\Pr(S_n \leq x) \geq \Pr(M_n \leq x) = G_n(x)$, that $M(x)$ possesses an analytic characteristic function. We have, in fact, for $r > 0$ sufficiently small,

$$\begin{aligned} E[e^{rM(x)}] &= \sum_{n=1}^{\infty} e^{rn} P_n \\ &= e^r + (e^r - 1) \sum_{n=1}^{\infty} e^{rn} G_n(x), \end{aligned}$$

so that by Theorem 5,

$$E[e^{rM(x)}] \sim r^{-1}(e^r - 1)e^{r\mu^{-1}x} \quad \text{as } x \rightarrow \infty.$$

The function $r^{-1}(e^r - 1)e^{r\mu^{-1}x}$ is the Laplace-Stieltjes transform of the convolution of a rectangular distribution on the interval $(0, 1)$ and a degenerate distribution at $x\mu^{-1}$. We have therefore obtained the following theorem:

THEOREM 6. *Suppose $EX = \mu > 0$. If for some integral $r \geq 1$, $E|X^-|^{r+1} < \infty$, then*

$$E\{[x^{-1}M(x)]^r\} \rightarrow \mu^{-r} \quad \text{as } x \rightarrow \infty.$$

If X^- possesses an analytic characteristic function then for $r > 0$ sufficiently small

$$E[e^{rM(x)}] \sim r^{-1}(e^r - 1)e^{r\mu^{-1}x} \quad \text{as } x \rightarrow \infty.$$

One can further use a method due to Doob based on the strong law of large numbers to obtain the following result:

THEOREM 7. *Suppose $EX = \mu > 0$. Then,*

$$x^{-1}M(x) \rightarrow_{\text{a.s.}} \mu^{-1} \quad \text{as} \quad x \rightarrow \infty.$$

PROOF. According to the strong law of large numbers, we have for $0 < \epsilon < \mu$ and sufficiently large n , $(\mu - \epsilon)n \leq S_n \leq (\mu + \epsilon)n$ with probability one. In particular, if $n = M(x)$ the left hand side implies $M(x)/x \leq 1/(\mu - \epsilon)$ and if $n = M(x) + 1$ the right side implies $M(x) + 1 \geq x/(\mu + \epsilon)$. Thus, for large x , $1/(\mu + \epsilon) - 1/x \leq M(x)/x \leq 1/(\mu - \epsilon)$ with probability one. The result follows.

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