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On a Property of the Lognormal Distribution

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SUMMARY

In this paper it is established that the lognormal distribution is not determined by its moments. Some brief comments are made on the set of distributions having the same moments as a lognormal distribution.

1. INTRODUCTION

It seems to be generally accepted that all the commonly used distributions in statistics, which possess moments of all orders, are determined by their moments. I will show, however, that this is not the case for the lognormal distribution.

Let the random variable X have the general lognormal distribution. We take $\log(X-a)$ as normally distributed with mean m and variance σ^2 . Then X has frequency function

$$f(x) = \begin{cases} \{\sigma(x-a)(2\pi)^{\frac{1}{2}}\}^{-1} \exp[-\frac{1}{2}\sigma^{-2}\{\log(x-a)-m\}^2] & (a < x < \infty), \\ 0 & (x \leq a). \end{cases}$$

It can readily be shown that this distribution possesses the same moments, namely

$$\alpha_n = \sum_{r=0}^n \binom{n}{r} a^{n-r} \exp(rm + \frac{1}{2}r^2\sigma^2) \quad (n = 0, 1, 2, \dots), \quad (1)$$

as the distribution with frequency function

$$g(x) = \begin{cases} \{\sigma(x-a)(2\pi)^{\frac{1}{2}}\}^{-1} \exp[-\frac{1}{2}\sigma^{-2}\{\log(x-a)-m\}^2] \\ \quad \times (1 + \epsilon \sin[2\pi k \sigma^{-2}\{\log(x-a)-m\}]) & (a < x < \infty), \\ 0 & (x \leq a), \end{cases}$$

where $0 < \epsilon < 1$ and k is a positive integer. Thus the lognormal distribution is not determined by its moments. It is clear that distributions can be chosen possessing the same moments as a lognormal distribution and such that they are indistinguishable from the lognormal for all practical purposes.

2. REMARKS ON THE MOMENT PROBLEM

The theory of the indeterminate moment problem enables us to make some rather interesting remarks.

Applying Theorem 2.13 of Shohat and Tamarkin (1943) to the case of the moment sequence (1), we see that for any real x there exists a purely discrete distribution having x as a point of discontinuity and moment sequence (1).

We now define P as the set of all points of increase of a distribution function F . Applying Theorem 10.42 of Stone (1932) to the case of the moment sequence (1), we

see that there exists a constant $K > a$ such that there are cardinal c distributions, possessing the moment sequence (1) and for which $\inf P$ assumes an arbitrary value μ on the range $-\infty \leq \mu < K$. There is, however, just one distribution with the moment sequence (1) and for which $\inf P = K$.

3. A SPECIAL CASE

It is worth remarking that in the case $a = 0$, $m = 0$, the various determinants which arise out of the theory of the indeterminate moment problem may be expressed in terms of alternants (see for example Muir, 1960, Chapter XI).

Using the notation of Shohat and Tamarkin (1943, p. 6) we have the determinants

$$\begin{aligned}\Delta_n &= |\exp\{\tfrac{1}{2}\sigma^2(i+j)^2\}|_{i,j=0}^n \quad (n \geq 1) \\ &= \exp\{\tfrac{1}{2}n^2\sigma^2(n+1)\} \prod_{r=1}^n (e^{r\sigma^2} - 1)^{n-r+1},\end{aligned}$$

and

$$\begin{aligned}\Delta_n^{(1)} &= |\exp\{\tfrac{1}{2}\sigma^2(i+j+1)^2\}|_{i,j=0}^n \quad (n \geq 1) \\ &= \exp\{\tfrac{1}{2}\sigma^2(n+1)(2n+1)\} \Delta_n.\end{aligned}$$

The corresponding continued fraction (Shohat and Tamarkin, 1943, p. 73) has coefficients l_i ($i = 1, 2, 3, \dots$) given by

$$\begin{aligned}l_1 &= 1, \\ l_{2n+1} &= \{\Delta_{n-1}^{(1)}\}^2 / \{\Delta_{n-1} \Delta_n\} \quad (n = 1, 2, 3, \dots) \\ &= \exp\{\tfrac{1}{2}n(n-1)\sigma^2\} / \prod_{r=1}^n (e^{r\sigma^2} - 1), \\ l_2 &= -\exp(\tfrac{1}{2}\sigma^2), \\ l_{2n} &= -\Delta_{n-1}^2 / \{\Delta_{n-1}^{(1)} \Delta_{n-2}^{(1)}\} \quad (n = 2, 3, 4, \dots) \\ &= \exp\{\tfrac{1}{2}(-n^2 - n + 1)\sigma^2\} \prod_{r=1}^n (e^{r\sigma^2} - 1).\end{aligned}$$

Cases where these may be computed explicitly from the moments are particularly rare.

It is possible, in this case, to obtain the Stieltjes transforms of all the distributions possessing the same moments as the lognormal. I do not include either the calculation or the result here as the calculation is routine and tedious, a particular case of analysis is given in Shohat and Tamarkin (1943, Chapter II), and the result is not in a compact or useful form.

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