An Invariance Principle and Some Convergence Rate Results for Branching Processes

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1. Introduction

Let $Z_0 = 1$, $Z_1, Z_2, \dots$ denote a super-critical Galton-Watson branching process with $1 < EZ_1 = m < \infty$ and $0 < \text{var } Z_1 = \sigma^2 < \infty$. It is well-known that there exists a non-degenerate random variable W such that $\lim_{n \rightarrow \infty} W_n = W$ almost surely, where $W_n = m^{-n} Z_n$ (e. g. Harris [3], p. 13). Also, a central limit analogue holds, namely that, conditional on $Z_n > 0$, $(m^2 - m)^{\frac{1}{2}} \sigma^{-1} Z_n^{-\frac{1}{2}} m^n (W - W_n)$ is asymptotical normal $N(0, 1)$ as $n \rightarrow \infty$ (Heyde [4]). This result is called a central limit analogue by virtue of the analogy with the convergence of $\sigma^{-1} n^{\frac{1}{2}} (\mu - n^{-1} S_n)$ to $N(0, 1)$ for a sum S_n of independent and identically distributed random variables with mean μ and variance σ^2 .

In this paper, we explore the setting of the central limit analogue. An invariance principle in R^∞ is obtained from which it follows as a simple corollary. Also, results are obtained on the rate of convergence to normality. It is shown that, under mild restrictions, there is a geometrically decreasing bound on the L_∞ metric.

2. Results

Let the probability generating function (p.g.f.) of the offspring distribution Z_1 be $F(s) = \sum_{j=0}^{\infty} s^j P(Z_1 = j)$. The p.g.f. $F_n(s)$ of Z_n is the n -th functional iterate of $F(s)$.

For the super-critical process in question, the probability of eventual extinction is q , the unique real number in $[0, 1)$ satisfying $q = F(q)$. Furthermore, in addition to the fact that $Z_n = 0$ for n sufficiently large on the extinction set $[W = 0]$, it is known that

$$Z_n \rightarrow \infty \quad \text{as } n \rightarrow \infty \quad \text{on } [W > 0], \quad (1)$$

so that the process either dies out or becomes arbitrarily large.

The symbols P and E will, for convenience, be used to denote various different probability measures and their associated expectation operators. In addition, we use the symbol Z_n^* to denote the random variable (r. v.) Z_n under the probability measure conditional on $Z_n > 0$. These conveniences should cause no ambiguities.

Let R^∞ denote the space of real sequences $x = (x_1, x_2, \dots)$ with the metric

$$\rho(x, y) = \sup_{n \geq 1} \left\{ \frac{\left| \sum_{j=1}^n (x_j - y_j) \right|}{1 + \left| \sum_{j=1}^n (x_j - y_j) \right|} \right\}.$$

Let $\mathcal{B}$ be the σ -field generated by the open sets of the topology given by ρ and let V be the probability measure on $(R^\infty, \mathcal{B})$ under which the co-ordinates $x_1, x_2, \dots$ of a random element x of R^∞ become independent r.v.'s with x_j distributed as $N(0, \sigma^2 m^{-1-j})$, $j=1, 2, \dots$. Write U for the probability measure on $(R^\infty, \mathcal{B})$ which assigns probability one to the zero sequence $(0, 0, \dots)$ and define, for $j=0, 1, 2, \dots$ and $n=1, 2, 3, \dots$,

$$X_n = (X_{n1}, X_{n2}, \dots) \\ X_{nj} = m^n (W_{n+j} - W_n) (Z_n + 1)^{-\frac{1}{2}},$$

and for $j=1, 2, 3, \dots$

$$Y_{nj} = X_{nj} - X_{n, j-1}.$$

Theorem 1. *Let P_n denote the probability measure on $(R^\infty, \mathcal{B})$ defined by the distributions of $Y_n = (Y_{n1}, Y_{n2}, \dots)$. Then,*

$$P_n \Rightarrow VP(W > 0) + UP(W = 0)$$

as $n \rightarrow \infty$, where $\Rightarrow$ denotes weak convergence of probability measures.

We remark that, as simple corollaries to Theorem 1, $(m^2 - m)^{\frac{1}{2}} \sigma^{-1} Z_n^{-\frac{1}{2}} \cdot m^n (W - W_n)$ conditional on $Z_n > 0$ and $(m^2 - m)^{\frac{1}{2}} \sigma^{-1} m^{-j/2} (m^j - 1)^{-\frac{1}{2}} Z_n^{-\frac{1}{2}} (Z_{j+n} - m^j Z_n)$ conditional on $Z_n > 0$ (fixed j) are both asymptotically $N(0, 1)$. These results have previously been obtained by direct methods by Heyde [4] and by Bühler [2] respectively. They can here be deduced from Theorem 1 using the measurable mapping theorem (Billingsley [1], p. 30) with the mappings

$$h_1(x) = \limsup_{n \rightarrow \infty} \sum_{j=1}^n x_j$$

if this is finite, 0 otherwise and $h_2(x) = x_1 + x_2 + \dots + x_j$ respectively.

The next theorem gives rates of convergence to normality in the limit results cited above.

Theorem 2. *Let $E Z_1^3 < \infty$. Then,*

$$\sup_x |P((m^2 - m)^{\frac{1}{2}} \sigma^{-1} Z_n^{-\frac{1}{2}} m^n (W - W_n) \leq x | Z_n > 0) - \Phi(x)| \\ \leq K \sigma^{-3} (m^2 - m)^{\frac{3}{2}} E(1/Z_n^*)^{\frac{1}{2}} E|W - 1|^3 \quad (2)$$

and

$$\sup_x |P(\sigma_r^{-1} Z_n^{-\frac{1}{2}} (Z_{r+n} - m^r Z_n) \leq x | Z_n > 0) - \Phi(x)| \\ \leq K \sigma_r^{-3} E|Z_r - m^r|^3 E(1/Z_n^*)^{\frac{1}{2}}, \quad (3)$$

where $\Phi(x)$ is the distribution function of $N(0, 1)$, $\sigma_r^2 = \text{var } Z_r = \sigma^2 m^r (m^r - 1) (m^2 - m)^{-1}$ and K is the universal constant in the Berry-Esseen bound ($K < 0.82$; see Zolotarev [6]). Furthermore,

$$E(1/Z_n^*)^{\frac{1}{2}} \leq (E(1/Z_n^*))^{\frac{1}{2}} \leq \{(1 - F(0))^{-1} \gamma^n\}^{\frac{1}{2}} \quad (4)$$

where $\gamma < 1$ is given by

$$\gamma = \int_0^1 (F(s(1-q) + q) - q) s^{-1} (1-q)^{-1} ds,$$

with $\gamma = E Z_1^{-1}$ if $F(0) = 0$.

3. Proof of Theorem 1

Firstly we show that the finite-dimensional distributions of $\{P_n\}$ converge weakly to those of $VP(W>0) + UP(W=0)$.

Let $\mathcal{F}_k$ be the σ -field generated by $Z_0, Z_1, \dots, Z_k$ and for fixed r write

$$\begin{aligned} \phi_n(\theta_1, \theta_2, \dots, \theta_r) &= E \left\{ \exp \left(i \sum_{j=1}^r \theta_j Y_{nj} \right) \right\} \\ &= E \left(\prod_{j=1}^r A_j \right), \end{aligned} \quad (5)$$

where

$$A_j = \exp(i \theta_j m^{-j} (Z_{n+j} - m Z_{n+j-1}) (Z_n + 1)^{-\frac{1}{2}}). \quad (6)$$

Consider the r. v.

$$\begin{aligned} R_r &= \prod_{j=1}^r A_j - \exp \left(-\frac{1}{2} \sigma^2 I(Z_n > 0) \sum_{j=1}^r \theta_j^2 m^{-1-j} \right) \\ &= \sum_{j=1}^r B_j, \end{aligned} \quad (7)$$

where I is the indicator function and for $j=1, 2, \dots, r$, using the convention that a product of the form $\prod_{k=s+1}^s$ is replaced by unity,

$$\begin{aligned} B_j &= \left(\prod_{k=1}^{j-1} A_k \right) \exp \left(-\frac{1}{2} \sigma^2 I(Z_n > 0) \sum_{k=j+1}^r \theta_k^2 m^{-1-k} \right) \\ &\quad \cdot (A_j - \exp(-\frac{1}{2} \sigma^2 I(Z_n > 0) \theta_j^2 m^{-1-j})), \end{aligned}$$

with

$$\begin{aligned} E(B_j | \mathcal{F}_{n+j-1}) &= \left(\prod_{k=1}^{j-1} A_k \right) \exp \left(-\frac{1}{2} \sigma^2 I(Z_n > 0) \sum_{k=j+1}^r \theta_k^2 m^{-1-k} \right) \\ &\quad \cdot (E(A_j | \mathcal{F}_{n+j-1}) - \exp(-\frac{1}{2} \sigma^2 I(Z_n > 0) \theta_j^2 m^{-1-j})). \end{aligned} \quad (8)$$

But, Z_{n+j} is the sum of Z_{n+j-1} independent and identically distributed r.v.'s, each with the distribution of Z_1 , so that from (6),

$$E(A_j | \mathcal{F}_{n+j-1}) = \{ \phi(\theta_j m^{-j} (Z_n + 1)^{-\frac{1}{2}}) \}^{Z_{n+j-1}}, \quad (9)$$

where $\phi(t) = E \{ \exp(i t (Z_1 - m)) \}$. Therefore, $E(A_j | \mathcal{F}_{n+j-1}) \rightarrow 1$ almost surely (a.s.) as $n \rightarrow \infty$ on $[W=0]$, so that from (8),

$$\lim_{n \rightarrow \infty} E(B_j | \mathcal{F}_{n+j-1}) = 0 \quad \text{a.s. on } [W=0], \quad (10)$$

since

$$I(Z_n > 0) \rightarrow I(W > 0) \quad \text{a.s. as } n \rightarrow \infty. \quad (11)$$

Also, we know from the central limit theorem that for each fixed t ,

$$\lim_{n \rightarrow \infty} \{ \phi(t(n+1)^{-\frac{1}{2}}) \}^n = \exp(-\frac{1}{2} \sigma^2 t^2).$$

Therefore, from (9), using (1) and since

$$\lim_{n \rightarrow \infty} Z_n^{-1} Z_{n+j-1} = m^{j-1} \quad \text{a.s. on } [W>0],$$

we have

$$\lim_{n \rightarrow \infty} E(A_j | \mathcal{F}_{n+j-1}) = \exp(-\frac{1}{2} \sigma^2 \theta_j^2 m^{-1-j}) \quad \text{a.s. on } [W>0],$$

so that

$$\lim_{n \rightarrow \infty} E(B_j | \mathcal{F}_{n+j-1}) = 0 \quad \text{a.s. on } [W > 0], \quad (12)$$

using (8) and (11). Clearly $|B_j| \leq 2$ a.s. for each $j = 1, 2, \dots, r$, so that the combination of (10) and (12), with dominated convergence, gives

$$E R_r = E \sum_{j=1}^r B_j = E \sum_{j=1}^r E(B_j | \mathcal{F}_{n+j-1}) \rightarrow 0$$

as $n \rightarrow \infty$. Referring now to (5) and (7), we have

$$\lim_{n \rightarrow \infty} \left\{ \phi_n(\theta_1, \dots, \theta_r) - E \exp \left(-\frac{1}{2} \sigma^2 I(Z_n > 0) \sum_{j=1}^r \theta_j^2 m^{-1-j} \right) \right\} = 0. \quad (13)$$

But,

$$\begin{aligned} \lim_{n \rightarrow \infty} E \exp \left(-\frac{1}{2} \sigma^2 I(Z_n > 0) \sum_{j=1}^r \theta_j^2 m^{-1-j} \right) \\ = P(W=0) + P(W>0) \exp \left(-\frac{1}{2} \sigma^2 \sum_{j=1}^r \theta_j^2 m^{-1-j} \right) \end{aligned} \quad (14)$$

from (11) and dominated convergence. (13) and (14) complete the proof of the convergence of finite-dimensional distributions.

To complete the proof of the theorem, we must show that the sequence $\{P_n\}$ is *tight*. That is, for each $\varepsilon > 0$ there exists a compact set A such that $P_n(A) > 1 - \varepsilon$ for all $n = 1, 2, \dots$.

Consider the sets

$$C = C_r = \left\{ x: \sup_{s \geq k} \left| \sum_{j=k}^s x_j \right| \leq m^{-k/3} \text{ for all } k \geq r \right\}$$

and

$$D = D_M = \left\{ x: \sup_{s \geq 1} \left| \sum_{j=1}^s x_j \right| \leq M \right\}.$$

The set $C \cap D$ has compact closure, for any sequence belonging to it can readily be shown to possess a limit point ([1], p. 217). Tightness will be established if we can show that, given $\varepsilon > 0$, $P_n(C \cap D) > 1 - \varepsilon$ for all n , when M and r are suitably large.

We first note that $C = \bigcap_{k \geq r} A_k$, where

$$A_k = \left\{ x: \sup_{s \geq k} \left| \sum_{j=k}^s x_j \right| \leq m^{-k/3} \right\}.$$

Kolmogorov's inequality for martingales, when applied to the martingale $\left\{ \sum_{j=k}^r Y_{nj}, r = k, k+1, \dots \right\}$ yields the inequality

$$P \left(\max_{k \leq r \leq s} \left| \sum_{j=k}^r Y_{nj} \right| > C \right) \leq C^{-2} \sum_{j=k}^s E Y_{nj}^2. \quad (15)$$

But,

$$\begin{aligned} E(Y_{nj}^2 | \mathcal{F}_{n+j-1}) &= E(m^{-2j}(Z_n + 1)^{-1}(Z_{n+j} - m Z_{n+j-1})^2 | \mathcal{F}_{n+j-1}) \\ &= m^{-2j}(Z_n + 1)^{-1} \sigma^2 Z_{n+j-1}, \end{aligned}$$

and

$$\begin{aligned}
 E Y_{nj}^2 &= E(E(Y_{nj}^2 | \mathcal{F}_{n+j-1})) \\
 &= m^{-2j} \sigma^2 \sum_{k \geq 0} E((Z_n + 1)^{-1} Z_{n+j-1} | Z_n = k) P(Z_n = k) \\
 &= m^{-2j} \sigma^2 \sum_{k \geq 0} k(k+1)^{-1} E Z_{j-1} P(Z_n = k) \\
 &\leq \sigma^2 m^{-1-j}.
 \end{aligned}$$

Therefore, from (15),

$$P\left(\max_{k \leq r \leq s} \left| \sum_{j=k}^r v_{nj} \right| > C\right) \leq \sigma^2 m^{-k} C^{-2} (m-1)^{-1}. \quad (16)$$

By applying the inequality (16) to A'_k and D' (the prime denoting complementation), we obtain

$$P_n(A'_k) \leq \sigma^2 (m-1)^{-1} m^{-k/3}$$

and

$$P_n(D') \leq \sigma^2 m^{-1} (m-1)^{-1} M^{-2},$$

for all $n=1, 2, \dots$. It now follows that

$$\begin{aligned}
 P_n(C) &= 1 - P_n\left(\bigcup_{k \geq r} A'_k\right) \\
 &\geq 1 - \sum_{k \geq r} P_n(A'_k) \\
 &\geq 1 - \sigma^2 (m-1)^{-1} (1 - m^{-\frac{1}{3}})^{-1} m^{-r/3}
 \end{aligned} \quad (17)$$

and

$$P_n(D) \geq 1 - \sigma^2 (m-1)^{-1} m^{-1} M^{-2}, \quad (18)$$

for all $n=1, 2, \dots$. Thus, from (17) and (18), $P_n(C \cap D) > 1 - \varepsilon$ for all n if r and M are chosen suitably large, so that tightness is established. The proof of the theorem is thus complete.

It is worth remarking that the result of Theorem 1 can be generalized to cover the case where Z_1 belongs to the domain of attraction of a stable law of index α , $1 < \alpha \leq 2$. This provides a complete extension of the result of Theorem 3 of Heyde [4]. Theorem 1 has been given in the less general form in the interests of a unified exposition in this paper. In order to obtain the general form we would use

$$X_{nj} = m^n b_{Z_{n+1}}^{-1} (W_{n+j} - W_n),$$

in the notation of [4].

4. Proof of Theorem 2

We first need the following lemma.

Lemma. *Let Y_i , $i=1, 2, 3, \dots$ be independent and identically distributed r.v.'s with $E Y_1 = 0$, $\text{var } Y_1 = \alpha^2$ and $E |Y_1|^3 < \infty$. Let N be a positive integer-valued r.v. which is independent of the $\{Y_i\}$. Then,*

$$\sup_x |P(\alpha^{-1} N^{-\frac{1}{3}} (Y_1 + \dots + Y_N) \leq x) - \Phi(x)| \leq C E(N^{-\frac{1}{3}}),$$

where $C = K \alpha^{-3} E |Y_1|^3$, K being the universal constant of the Berry-Esseen bound.

Proof. Using the Berry-Esseen bound,

$$\begin{aligned} -CE(N^{-\frac{1}{2}}) &= -C \sum_{j=1}^{\infty} j^{-\frac{1}{2}} P(N=j) \\ &\leq \sum_{j=1}^{\infty} \{P(\alpha^{-1} j^{-\frac{1}{2}} (Y_1 + \cdots + Y_j) \leq x) - \Phi(x)\} P(N=j) \\ &\leq CE(N^{-\frac{1}{2}}). \end{aligned} \quad (19)$$

But, since N is independent of the $\{Y_i\}$,

$$\begin{aligned} &\sum_{j=1}^{\infty} \{P(\alpha^{-1} j^{-\frac{1}{2}} (Y_1 + \cdots + Y_j) \leq x) - \Phi(x)\} P(N=j) \\ &= \sum_{j=1}^{\infty} \{P(\alpha^{-1} j^{-\frac{1}{2}} (Y_1 + \cdots + Y_j) \leq x | N=j) - \Phi(x)\} P(N=j) \\ &= P(\alpha^{-1} N^{-\frac{1}{2}} (Y_1 + \cdots + Y_N) \leq x) - \Phi(x), \end{aligned} \quad (20)$$

and the required result follows from (19) and (20).

We now proceed with the proof of the theorem. Firstly we establish (2) and (3).

Heyde [4] has shown that, conditional on $Z_n > 0$, $m^n Z_n^{-\frac{1}{2}} (W - W_n)$ has the same distribution as $(Z_n^*)^{-\frac{1}{2}} (U_1 + \cdots + U_{Z_n^*})$, where the U_i are independent of Z_n^* and are independent and identically distributed, each with the distribution of $W - 1$. Furthermore, $\text{var } W = \sigma^2 (m^2 - m)^{-1}$, and hence

$$\begin{aligned} P((m^2 - m)^{\frac{1}{2}} \sigma^{-1} Z_n^{-\frac{1}{2}} m^n (W - W_n) \leq x | Z_n > 0) \\ = P((\text{var } U_1)^{-\frac{1}{2}} (Z_n^*)^{-\frac{1}{2}} (U_1 + \cdots + U_{Z_n^*}) \leq x), \end{aligned}$$

so that (2) follows immediately from the lemma. (3) is obtained similarly by noting that, conditional on $Z_n > 0$, $Z_n^{-\frac{1}{2}} (Z_{r+n} - m^r Z_n)$ has the same distribution as $(Z_n^*)^{-\frac{1}{2}} (V_1 + \cdots + V_{Z_n^*})$, where the V_i are independent of Z_n^* and are independent and identically distributed, each with the distribution of $Z_r - m^r$.

Next, in order to obtain (4) we note that

$$E(Z_n^*)^{-\frac{1}{2}} \leq (E(Z_n^*)^{-1})^{\frac{1}{2}}, \quad (21)$$

by a standard moment inequality, so that it suffices to study $E(Z_n^*)^{-1}$, which is given by

$$E(Z_n^*)^{-1} = \int_0^1 (F_n(s) - F_n(0)) s^{-1} (1 - F_n(0))^{-1} ds. \quad (22)$$

Now define a p.g.f.

$$H(s) = (1 - q)^{-1} (F(s(1 - q) + q) - q), \quad 0 \leq s \leq 1.$$

Then, if $H_n(s)$ is the n -th functional iterate of $H(s)$, it is easily seen that

$$H_n(s) = (1 - q)^{-1} (F_n(s(1 - q) + q) - q),$$

and $H_n(s)$ is the p.g.f. of a super-critical Galton-Watson process $\{Y_n\}$ (say) with $P(Y_n = 0) = 0$ for each n . The growth properties of $\{Y_n\}$ are closely related to those of $\{Z_n\}$; in particular $E Y_n = E Z_n = m^n$. Also, we note that

$$U(s) = F_n(s(1 - q) + q) - (1 - q) F_n(s)$$

satisfies

$$U(0) = F_n(q) - (1 - q) F_n(0) = q - (1 - q) F_n(0)$$

and

$$U'(s) = (1-q)(F'_n(s(1-q)+q) - F'_n(s)) \geq 0$$

since $s(1-q)+q \geq s$ ($0 \leq s \leq 1$) and $F'_n(t)$ is monotone increasing in t . Thus for $0 \leq s \leq 1$,

$$F_n(s(1-q)+q) - (1-q)F_n(s) \geq q - (1-q)F_n(0),$$

that is,

$$H_n(s) \geq F_n(s) - F_n(0).$$

Consequently, from (22),

$$\begin{aligned} E(Z_n^*)^{-1} &\leq (1-F_n(0))^{-1} \int_0^1 s^{-1} H_n(s) ds \\ &= (1-F_n(0))^{-1} E(Y_n^{-1}). \end{aligned} \quad (23)$$

Now let $\mathcal{F}_n$ be the σ -field generated by $Y_1, \dots, Y_n$, and note that $Y_n \geq 1$ for each n . Then,

$$\begin{aligned} E(Y_n^{-1}) &= E(Y_{n-1}^{-1} E(Y_{n-1} Y_n^{-1} | \mathcal{F}_{n-1})) \\ &= E(Y_{n-1}^{-1} E(Y_{n-1}(Y_1^{(1)} + \dots + Y_1^{(Y_{n-1})})^{-1} | \mathcal{F}_{n-1})), \end{aligned} \quad (24)$$

where the $Y_1^{(i)}$ are independent of Y_{n-1} and are independent and identically distributed, each with the distribution of Y_1 . Furthermore, using the harmonic mean-arithmetic mean inequality,

$$Y_{n-1}(Y_1^{(1)} + \dots + Y_1^{(Y_{n-1})})^{-1} \leq Y_{n-1}^{-1}((Y_1^{(1)})^{-1} + \dots + (Y_1^{(Y_{n-1})})^{-1}),$$

so that

$$E(Y_{n-1}(Y_1^{(1)} + \dots + Y_1^{(Y_{n-1})})^{-1} | \mathcal{F}_{n-1}) \leq E(Y_1^{-1}).$$

Hence, from (24),

$$E(Y_n^{-1}) \leq E(Y_1^{-1}) E(Y_{n-1}^{-1}),$$

so that

$$E(Y_n^{-1}) \leq (E(Y_1^{-1}))^n. \quad (25)$$

(4) then follows from (21), (23) and (25) by setting $\gamma = E(Y_1^{-1})$, which is in turn given by

$$\gamma = E(Y_1^{-1}) = \int_0^1 s^{-1} (1-q)^{-1} (F(s(1-q)+q) - q) ds,$$

with $\gamma = E(Z_1^{-1})$ when $P(Z_1=0)=0$, for then $q=0$. This completes the proof of Theorem 2.

5. Remarks on Theorem 2

The bound (4) of Theorem 2 appears to be rather crude and with reference to it we note that, by Jensen's inequality,

$$E(Y_1^{-1}) \geq (E Y_1)^{-1} = m^{-1}.$$

How, if $E(W^{-1} | W > 0) < \infty$, it is plausible that under suitable conditions, $E(Z_n^*)^{-1} \sim m^{-n} E(W^{-1} | W > 0)$, since $\lim_{n \rightarrow \infty} m^n Z_n^{-1} = W^{-1}$ a.s. on $[W > 0]$. However, it emerges that $E(W^{-1} | W > 0)$ is often not finite and there are wide ranging possibilities for the asymptotic behaviour of $E(Z_n^*)^{-1}$. For example, take

$$F(s) = s(m - (m-1)s^k)^{-1/k}, \quad m > 1,$$

where k is a positive integer. Then,

$$F_n(s) = s(m^n - (m^n - 1)s^k)^{-1/k},$$

so that

$$\begin{aligned} E(Z_n^*)^{-1} &= E(Z_n^{-1}) = \int_0^1 s^{-1} F_n(s) ds \\ &= \int_0^1 (m^n - (m^n - 1)s^k)^{-1/k} ds, \end{aligned}$$

and, putting $y = 1 - s^k(1 - m^{-n})$, we obtain

$$E(Z_n^{-1}) = \frac{1}{k(m^n - 1)^{1/k}} \int_{m^{-n}}^1 \frac{dy}{y^{1/k}(1-y)^{(k-1)/k}}.$$

Thus, if $k = 1$,

$$E(Z_n^{-1}) = (m^n - 1)^{-1} n \log m,$$

and if $k > 1$,

$$\begin{aligned} E(Z_n^{-1}) &\sim k^{-1} m^{-n/k} B(1 - k^{-1}, k^{-1}) \\ &\sim k^{-1} m^{-n/k} \pi (\sin(\pi/k))^{-1} \end{aligned}$$

as $n \rightarrow \infty$ since

$$B(1 - k^{-1}, k^{-1}) = \Gamma(k^{-1}) \Gamma(1 - k^{-1}) = \pi (\sin(\pi/k))^{-1}.$$

In conclusion, we make two observations. Firstly, Theorem 2 may be extended to deal with the case $EZ_1^{2+\delta} < \infty$, $0 < \delta \leq 1$, by using results of Ibragimov [5]. In this case we obtain bounds of the form $c(E(Z_n^*)^{-1})^{\delta/2}$ in (2) and (3), c denoting a positive constant. We have not framed Theorem 2 in this more general context because an explicit form for c is not available.

Secondly, an improved rate of convergence to normality can be obtained by conditioning on the value of Z_n as well as on $[Z_n > 0]$. In this case it is apparent that, after minor modifications in the foregoing proof, the bound in (2) and (3) becomes $O((Z_n^*)^{-\frac{1}{2}})$, which is $O(m^{-\frac{1}{2}n})$ as $n \rightarrow \infty$.

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