A RATE OF CONVERGENCE RESULT FOR THE SUPER-CRITICAL GALTON-WATSON PROCESS

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Let $Z_0 = 1, Z_1, Z_2, \dots$ denote a super-critical Galton-Watson process whose non-degenerate offspring distribution has probability generating function $F(s) = \sum_{j=0}^{\infty} s^j P(Z_1 = j)$, $0 \leq s \leq 1$, where $1 < m = EZ_1 < \infty$. The Galton-Watson process evolves in such a way that the generating function $F_n(s)$ of Z_n is the n th functional iterate of $F(s)$. The convergence problem for Z_n , when appropriately normed, has been studied by quite a number of authors; for an ultimate form see Heyde [2]. However, no information has previously been obtained on the rate of such convergence. We shall here suppose that $EZ_1^2 < \infty$ in which case $W_n = m^{-n}Z_n$ converges almost surely to a non-degenerate random variable W as $n \rightarrow \infty$ (Harris [1], p. 13). It is our object to establish the following result on the rate of convergence of W_n to W .

Theorem. Let $\text{var } Z_1 = \sigma^2 < \infty$. As $n \rightarrow \infty$, $m^{n/2}(W - W_n)$ converges in distribution. The limit law is given by the characteristic function relation

$$\lim_{n \rightarrow \infty} E[\exp\{it m^{n/2}(W - W_n)\}] = E[\exp\{-\frac{1}{2}t^2\sigma^2(m^2 - m)^{-1}W\}].$$

Proof. Firstly, we take $r > n$ and consider

$$\begin{aligned} & E[\exp\{it m^{n/2}(W_r - W_n)\}] \\ (1) \quad &= \sum_{j=0}^{\infty} E[\exp\{it m^{n/2}(m^{-r}Z_r - m^{-n}Z_n)\} \mid Z_n = j] P(Z_n = j) \\ &= \sum_{j=0}^{\infty} \exp\{-it m^{-n/2}j\} [E[\exp\{it m^{n/2-r}Z_{r-n}\}]]^j P(Z_n = j) \\ &= E[\exp\{-it m^{-n/2}\}] E[\exp\{it m^{n/2-r}Z_{r-n}\}]^{Z_n}. \end{aligned}$$

Next, we let $r \rightarrow \infty$ in (1) keeping n fixed. Given $\varepsilon > 0$ we can choose N so large that $\sum_{j=N+1}^{\infty} P(Z_n = j) < \varepsilon$. Also, since W_{r-n} converges in distribution to W , $E[\exp\{it m^{n/2-r}Z_{r-n}\}]$ converges to $E[\exp\{it m^{-n/2}W\}]$, uniformly in any finite t interval, so we can find R so large that

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$$\left| E[\exp\{it m^{n/2-r} Z_{r-n}\}] - E[\exp\{it m^{-n/2} W\}] \right| < \varepsilon$$

for $r > R$. Consequently,

$$\begin{aligned} & \sum_{j=0}^{\infty} \exp\{-it m^{-n/2} j\} [E[\exp\{it m^{n/2-r} Z_{r-n}\}]]^j P(Z_n = j) \\ &= \sum_{j=0}^{\infty} \exp\{-it m^{-n/2} j\} [E[\exp\{it m^{-n/2} W\}]]^j P(Z_n = j) \\ &+ \sum_{j=0}^R \exp\{-it m^{-n/2} j\} \{[E[\exp\{it m^{n/2-r} Z_{r-n}\}]]^j - [E[\exp\{it m^{-n/2} W\}]]^j\} P(Z_n = j) \\ &+ \sum_{j=R+1}^{\infty} \exp\{-it m^{-n/2} j\} \{[E[\exp\{it m^{n/2-r} Z_{r-n}\}]]^j - [E[\exp\{it m^{-n/2} W\}]]^j\} P(Z_n = j), \end{aligned}$$

and

$$\begin{aligned} & \left| \sum_{j=R+1}^{\infty} \exp\{-it m^{-n/2} j\} \{[E[\exp\{it m^{n/2-r} Z_{r-n}\}]]^j - [E[\exp\{it m^{-n/2} W\}]]^j\} P(Z_n = j) \right| \\ & \leq 2 \sum_{j=R+1}^{\infty} P(Z_n = j) \leq 2\varepsilon, \end{aligned}$$

while for $r > R$,

$$\begin{aligned} & \left| \sum_{j=0}^R \exp\{it m^{-n/2} j\} \{[E[\exp\{it m^{n/2-r} Z_{r-n}\}]]^j - [E[\exp\{it m^{-n/2} W\}]]^j\} P(Z_n = j) \right| \\ & \leq \sum_{j=0}^R \left| [E[\exp\{it m^{n/2-r} Z_{r-n}\}]]^j - [E[\exp\{it m^{-n/2} W\}]]^j \right| P(Z_n = j) \\ & \leq \varepsilon \sum_{j=0}^R j P(Z_n = j) \leq \varepsilon m^n \end{aligned}$$

since $EZ_n = m^n$. However, ε is arbitrary and n fixed so that

$$\begin{aligned} E[\exp\{it m^{n/2}(W - W_n)\}] &= \lim_{r \rightarrow \infty} E[\exp\{it m^{n/2}(W_r - W_n)\}] \\ (2) \quad &= \sum_{j=0}^{\infty} \exp\{-it m^{-n/2} j\} [E[\exp\{it m^{-n/2} W\}]]^j P(Z_n = j) \\ &= E[E[\exp\{it m^{-n/2}(W - 1)\}]]^{Z_n}. \end{aligned}$$

Now,

$$\begin{aligned} & E[E[\exp\{it m^{-n/2}(W - 1)\}]]^{Z_n} \\ &= \int_0^{\infty} [E[\exp\{it m^{-n/2}(W - 1)\}]]^{xm^n} dP(m^{-n}Z_n \leq x) \end{aligned}$$

$$(3) \quad = P(Z_n = 0 + m^n \log E[\exp\{itm^{-n/2}(W-1)\}]) \int_0^\infty P(m^{-n}Z_n > x) \\ \times [E[\exp\{itm^{-n/2}(W-1)\}]]^{xmn} dx,$$

using integration by parts. Furthermore, under the hypotheses of the theorem, W has mean 1 and variance $\sigma^2/(m^2 - m)$ ([1], p. 13) so we may expand $E[\exp\{itm^{-n/2}(W-1)\}]$ in the form

$$(4) \quad E[\exp\{itm^{-n/2}(W-1)\}] = 1 - \frac{1}{2}t^2\sigma^2m^{-n}(m^2 - m)^{-1} + o(t^2m^{-n}).$$

Thus,

$$(5) \quad m^n \log E[\exp\{itm^{-n/2}(W-1)\}] \rightarrow -\frac{1}{2}t^2\sigma^2(m^2 - m)^{-1}$$

and

$$(6) \quad [E[\exp\{itm^{-n/2}(W-1)\}]]^{m^n} \rightarrow \exp\{-\frac{1}{2}t^2\sigma^2(m^2 - m)^{-1}\}$$

as $n \rightarrow \infty$. Also, using (4), we have for fixed t and suitably large n ,

$$\begin{aligned} |E[\exp\{itm^{-n/2}(W-1)\}]|^{m^n} &< \exp\{-\frac{1}{2}t^2\sigma^2(m^2 - m)^{-1} + o(t^2)\} \\ &< \exp\{-\frac{1}{4}t^2\sigma^2(m^2 - m)^{-1}\}, \end{aligned}$$

so that the integrand in the rightmost term of (3) is bounded in absolute value by $\exp\{-\frac{1}{4}t^2\sigma^2x(m^2 - m)^{-1}\}$. It then follows from Fatou's lemma together with (5), (6) and since $P(m^{-n}Z_n > x) \rightarrow P(W > x)$ as $n \rightarrow \infty$, that

$$\begin{aligned} \lim_{n \rightarrow \infty} E[\exp\{itm^{n/2}(W - W_n)\}] \\ = P(W = 0) - \frac{1}{2}t^2\sigma^2(m^2 - m)^{-1} \int_0^\infty P(W > x) \exp\{-\frac{1}{2}t^2\sigma^2x(m^2 - m)^{-1}\} dx \\ = \int_0^\infty \exp\{-\frac{1}{2}t^2\sigma^2x(m^2 - m)^{-1}\} dP(W \leq x) \\ = E[\exp\{-\frac{1}{2}t^2\sigma^2(m^2 - m)^{-1}W\}], \end{aligned}$$

again using integration by parts. This completes the proof of the theorem.

Unfortunately, the distribution function corresponding to the characteristic function $E[\exp\{-\frac{1}{2}t^2\sigma^2(m^2 - m)^{-1}W\}]$ does not seem to have a useful general representation. However, in certain particular cases it may be recognised simply.

For example, in the case of a fractional linear generating function for the offspring distribution ([1], pp. 9, 17) it is easily found to have an atom at zero and a Laplace density elsewhere.

References

- [1] HARRIS, T. E. (1963) *The Theory of Branching Processes*. Springer-Verlag, Berlin.
- [2] HEYDE, C. C. (1970) Extension of a result of Seneta for the super-critical Galton-Watson process. *Ann. Math. Statist.* **41** (in press).