On the Implication of a Certain Rate of Convergence to Normality

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1. Introduction

Consider a sequence of independent and identically distributed random variables X_i , $i = 1, 2, 3, \dots$ which belong to the domain of attraction of the normal distribution. Since this last property implies $E|X_i| < \infty$, we shall suppose for convenience that $EX_i = 0$. Write

$$S_n = \sum_{i=1}^n X_i, \quad n \geq 1,$$

and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}u^2} du.$$

We shall establish the following main theorem.

Theorem 1. *If there is a monotone sequence of normalizing constants $\{B_n\}$ with $B_n \rightarrow \infty$ as $n \rightarrow \infty$ such that*

$$\sum_{n=3}^{\infty} n^{-1} (\log \log n) \sup_x |Pr(S_n \leq B_n x) - \Phi(x)| < \infty, \quad (1)$$

then $EX_i^2 < \infty$.

This result extends that contained in Theorem 3.1 of Ibragimov [7] where it is shown that if

$$\sup_x |Pr(S_n \leq B_n x) - \Phi(x)| = O(n^{-\delta})$$

for some $\delta > 0$, then $EX_i^2 < \infty$. It provides us with a converse to a result of Heyde [4]. Furthermore, it provides an interesting elucidation of a result of Petrov [8] concerning iterated logarithm behaviour which was proved without explicit assumptions regarding the existence of moments. Theorem 1 is proved in Section 2 while Section 3 contains a discussion of its applications.

2. Proof of Theorem 1

Suppose firstly that the X_i are symmetric. Since

$$1 - \Phi(x) \sim \frac{1}{x\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

as $x \rightarrow \infty$, we have

$$\sum_{n=3}^{\infty} n^{-1} (\log \log n) \{1 - \Phi[(2 + \varepsilon) \log \log n]^{\frac{1}{2}}\} < \infty$$

for any $\varepsilon > 0$ so that, writing $t_n = [(2 + \varepsilon) \log \log n]^{\frac{1}{2}}$, it follows from (1) that

$$\sum_{n=3}^{\infty} n^{-1} (\log \log n) \Pr(|S_n| > B_n t_n) < \infty. \quad (2)$$

But, for symmetric random variables,

$$\Pr(|S_n| > B_n t_n) \geq \frac{1}{2} \Pr\left(\max_{1 \leq k \leq n} |X_k| > B_n t_n\right), \quad (3)$$

(e.g. Feller [2], II, 147) and, from Bonferroni's inequalities (e.g. [2], I, 100) we have

$$\begin{aligned} n \Pr(|X_i| > B_n t_n) [1 - \tfrac{1}{2} (n-1) \Pr(|X_i| > B_n t_n)] \\ \leq \Pr\left(\max_{1 \leq k \leq n} |X_k| > B_n t_n\right) \leq n \Pr(|X_i| > B_n t_n), \end{aligned}$$

so that

$$\Pr\left(\max_{1 \leq k \leq n} |X_k| > B_n t_n\right) \sim n \Pr(|X_i| > B_n t_n) \quad (4)$$

as $n \rightarrow \infty$ and hence, from (2), (3) and (4),

$$\sum_{n=3}^{\infty} (\log \log n) \Pr(|X_i| > B_n t_n) < \infty. \quad (5)$$

Now suppose that $EX_i^2 = \infty$ so that

$$K(t) = \int_{|x| \leq 1} x^2 d\Pr(X_i \leq x) \rightarrow \infty$$

as $t \rightarrow 0$. Since the X_i belong to the domain of attraction of the normal distribution, $K(t)$ is slowly varying as $t \rightarrow 0$ (e.g. [2], II, 303) and $n B_n^{-2} K(B_n^{-1}) \rightarrow 1$ as $n \rightarrow \infty$ (e.g. [2], II, 304–305).

Then, we have

$$\begin{aligned} \int_{|x| \leq B_n t_n} x^2 d\Pr(X_i \leq x) &\leq \int_{|x| \leq B_3 t_3} x^2 d\Pr(X_i \leq x) \\ &\quad + \sum_{k=3}^n B_{k+1}^2 t_{k+1}^2 \Pr(B_k t_k < |X_i| \leq B_{k+1} t_{k+1}) \\ &\leq \int_{|x| \leq B_3 t_3} x^2 d\Pr(X_i \leq x) \\ &\quad + C \sum_{k=3}^n K(B_k^{-1}) k \log \log k \Pr(B_k t_k < |X_i| \leq B_{k+1} t_{k+1}) \end{aligned} \quad (6)$$

for some positive constant C . Furthermore, it is easily shown, using (5), that

$$\sum_{k=3}^{\infty} k (\log \log k) \Pr(B_k t_k < |X_i| \leq B_{k+1} t_{k+1}) < \infty. \quad (7)$$

Thus, using (6), (7) and by virtue of our assumption that $K(t) \rightarrow \infty$ as $t \rightarrow 0$, we have from the Kronecker lemma that

$$n B_n^{-2} \int_{|x| \leq B_n t_n} x^2 d\Pr(X_i \leq x) \sim [K(B_n^{-1})]^{-1} K(B_n^{-1} t_n^{-1}) \rightarrow 0$$

as $n \rightarrow \infty$. This establishes the required contradiction since by definition,

$$[K(B_n^{-1})]^{-1} K(B_n^{-1} t_n^{-1}) \geq 1.$$

It follows that (5) is only possible in the case where $K(B_n^{-1})$ approaches a positive constant as $n \rightarrow \infty$ in which case it is easily seen that $EX_i^2 < \infty$. This completes the proof for the case of symmetric X_i .

In order to deal with the general case, consider the sequence $Y_i, i = 1, 2, 3, \dots$ of independent, symmetrized random variables; each Y_i having the distribution of the difference between two independent X_i 's. Write

$$G_n(x) = Pr(Y_1 + \dots + Y_n \leq x B_n \sqrt{2}), \quad F_n(x) = Pr(S_n \leq B_n x).$$

Then, using $*$ to denote convolution,

$$G_n(x) = F_n(x \sqrt{2}) * (1 - F_n(-x \sqrt{2} - 0)),$$

and in view of (1) we obtain

$$\begin{aligned} & \sum_{n=3}^{\infty} n^{-1} (\log \log n) \sup_x |G_n(x) - \Phi(x)| \\ &= \sum_{n=3}^{\infty} n^{-1} (\log \log n) \sup_x |F_n(x \sqrt{2}) * (1 - F_n(-x \sqrt{2} - 0)) \\ & \quad - \Phi(x \sqrt{2}) * (1 - \Phi(-x \sqrt{2}))| \\ &\leq \sum_{n=3}^{\infty} n^{-1} (\log \log n) \sup_x |F_n(x \sqrt{2}) * (1 - F_n(-x \sqrt{2} - 0)) \\ & \quad - \Phi(x \sqrt{2}) * (1 - F_n(-x \sqrt{2} - 0))| \\ & \quad + \sum_{n=3}^{\infty} n^{-1} (\log \log n) \sup_x |\Phi(x \sqrt{2}) * (1 - F_n(-x \sqrt{2} - 0)) \\ & \quad - \Phi(x \sqrt{2}) * (1 - \Phi(-x \sqrt{2}))| < \infty. \end{aligned}$$

Thus, our result for the symmetric case gives $EY_i^2 < \infty$ which implies $EX_i^2 < \infty$. This completes the proof of the theorem.

Remark 1. The following result of Davis [1] is worth noting in connection with Theorem 1. If $EX_i = 0$, $EX_i^2 = 1$, $EX_i^2 \log^+ \log^+ |X_i| < \infty$ and

$$\beta_n^2 = \int_{|x| < \sqrt{n}} x^2 d Pr(X_i \leq x) - \left(\int_{|x| < \sqrt{n}} x d Pr(X_i \leq x) \right)^2,$$

then

$$\sum_{n=3}^{\infty} n^{-1} (\log \log n) \sup_x |Pr(S_n \leq x \beta_n \sqrt{n}) - \Phi(x)| < \infty.$$

(Here $\log^+ x$ denotes $\max(0, \log x)$.)

Remark 2. It is conjectured that an improved version of Theorem 1 will hold with the $\log \log n$ term omitted in (1). For background to this conjecture see Heyde [5].

3. Applications of Theorem 1

Firstly we use Theorem 1 to elucidate a result of Heyde [4].

Theorem 2. Take $0 < \delta < 1$. Let $E|X_i|^{2+\delta} < \infty$ and write $B_n^2 = nEX_i^2(1 + \varepsilon_n)$. Then, if $\sum_{n=1}^{\infty} n^{-1+\delta/2} |\varepsilon_n| < \infty$, we obtain

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} \sup_x |Pr(S_n \leq B_n x) - \Phi(x)| < \infty. \quad (8)$$

Conversely, if (8) holds for some monotone sequence $\{B_n\}$, and the X_i belong to the domain of attraction of the normal distribution, then $EX_i^2 < \infty$ and $B_n^2 = nEX_i^2(1 + \varepsilon_n)$ where $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. If, in addition, $\sum_{n=1}^{\infty} n^{-1+\delta/2} |\varepsilon_n| < \infty$, then $E|X_i|^{2+\delta} < \infty$.

Proof. Write $EX_i^2 = \sigma^2 (< \infty)$. If $E|X_i|^{2+\delta} < \infty$, we obtain from the theorem of [4] that

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} \sup_x |Pr(S_n \leq x \sigma \sqrt{n}) - \Phi(x)| < \infty. \quad (9)$$

Furthermore,

$$\begin{aligned} & Pr(S_n \leq x \sigma \sqrt{n(1 + \varepsilon_n)}) - \Phi(x) \\ &= Pr(S_n \leq (x \sqrt{1 + \varepsilon_n} \sigma \sqrt{n}) - \Phi(x \sqrt{1 + \varepsilon_n}) + \Phi(x \sqrt{1 + \varepsilon_n}) - \Phi(x), \end{aligned} \quad (10)$$

while expanding by Taylor's theorem gives

$$\Phi(x \sqrt{1 + \varepsilon_n}) - \Phi(x) = \frac{1}{2\sqrt{2\pi}} \varepsilon_n x e^{-\frac{1}{2}x^2} + O(\varepsilon_n^2). \quad (11)$$

Thus, upon taking sup with respect to x in (10) and using (11), we obtain

$$\sup_x |Pr(S_n \leq x \sigma \sqrt{n(1 + \varepsilon_n)}) - \Phi(x)| \leq \sup_x |Pr(S_n \leq x \sigma \sqrt{n}) - \Phi(x)| + C |\varepsilon_n|, \quad (12)$$

for some positive constant C . The result of the first part of the theorem then follows from (9) and (12).

Now suppose that (8) holds for some monotone sequence $\{B_n\}$, and the X_i belong to the domain of attraction of the normal distribution. It follows from Theorem 1 that $EX_i^2 = \sigma^2 < \infty$, and consequently B_n must be of the form $B_n = \sigma \sqrt{n(1 + \varepsilon_n)}$ where $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Further, the equation (10) can be rewritten to yield

$$\sup_x |Pr(S_n \leq x \sigma \sqrt{n}) - \Phi(x)| \leq \sup_x |Pr(S_n \leq x \sigma \sqrt{n(1 + \varepsilon_n)}) - \Phi(x)| + C' |\varepsilon_n|,$$

where C' is a positive constant, so that, if $\sum_{n=1}^{\infty} n^{-1+\delta/2} |\varepsilon_n| < \infty$, (9) holds. The result $E|X_i|^{2+\delta} < \infty$ then follows from the theorem of [4]. This completes the proof.

Our next application of Theorem 1 concerns its relation to questions of behaviour of iterated logarithm type. It is well-known that

$$Pr(\limsup_{n \rightarrow \infty} (2n \log \log n)^{-\frac{1}{2}} S_n = 1) = 1$$

if and only if $EX_i = 0$, $EX_i^2 = 1$ (Hartman and Wintner [3] obtained the sufficiency part and Strassen [9] the necessity). However, it has been shown by Freedman (see [9]) that there exist symmetric random variables X_i with $EX_i^2 = \infty$ or even $E|X_i| = \infty$ for which it is possible to choose a monotone sequence $\{b_n\}$ of positive constants with $b_n \rightarrow \infty$ as $n \rightarrow \infty$ such that

$$Pr(\limsup_{n \rightarrow \infty} b_n^{-1} S_n = 1) = 1. \quad (13)$$

Furthermore, Heyde [6] has shown that the behaviour (13) is only possible if the X_i belong to the domain of partial attraction of the normal distribution. Also, the construction of Freedman's is quite inexplicit and it is possible to obtain very little idea of the generality of the circumstances under which behaviour of the type (13) is possible when $EX_i^2 = \infty$.

Now the following theorem has recently been obtained by Petrov [8].

Theorem. *Suppose that the independent random variables X_i , $i = 1, 2, 3, \dots$ have symmetric distributions. If there exists a sequence of constants $\{C_n\}$ such that*

$$C_n \uparrow \infty, \quad C_n^{-1} C_{n+1} \rightarrow 1$$

and

$$\sup_x \left| Pr \left(\sum_{i=1}^n X_i < C_n^{\frac{1}{2}} x \right) - \Phi(x) \right| = O((\log C_n)^{-1-\delta}) \quad (14)$$

as $n \rightarrow \infty$ for some $\delta > 0$, then

$$Pr \left(\limsup_{n \rightarrow \infty} (2 C_n \log \log C_n)^{-\frac{1}{2}} \sum_{i=1}^n X_i = 1 \right) = 1.$$

This theorem involves no explicit assumptions regarding the existence of moments and it is an interesting question as to whether it can be used to generate results of the type of (13) for identically distributed random variables with $EX_i^2 = \infty$. The answer to this question is in the negative. This follows immediately from Theorem 1 upon noting that, in the case of identically distributed X_i ,

$$\log C_n \sim \frac{1}{2} \log n$$

as $n \rightarrow \infty$.

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(Received August 18, 1969)