ON THE CONVERSE TO THE ITERATED LOGARITHM LAW

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1. Introduction

Let $X_i, i = 1, 2, 3, \dots$ be a sequence of independent and identically distributed random variables with law $\mathcal{L}(X)$ and write $S_n = \sum_{i=1}^n X_i$. If $EX = 0$ and $EX^2 = \sigma^2 < \infty$, the law of the iterated logarithm (Hartman and Wintner [1]) tells us that

$$\Pr(\limsup_{n \rightarrow \infty} (2\sigma n \log \log n)^{-\frac{1}{2}} |S_n| = 1) = 1.$$

Furthermore, it has recently been shown by Strassen [4] that if

$$\Pr(\limsup_{n \rightarrow \infty} (n \log \log n)^{-\frac{1}{2}} |S_n| < \infty) = 1,$$

then $EX = 0, EX^2 < \infty$. The proof of Strassen, however, utilizes deep results of Skorokhod involving the use of an associated Wiener process and it is the object of the present paper to develop a direct approach to this result. As a by-product, it will be clear how the results of Stone [3] can be extended.

2. Results

Theorem. If $EX^2 = \infty$, then

$$\Pr(\limsup_{n \rightarrow \infty} (n \log \log n)^{-\frac{1}{2}} |S_n| = \infty) = 1.$$

Proof. Firstly suppose that $\sum \Pr[|X| > (n \log \log n)^{\frac{1}{2}}] = \infty$. Then, for $N > 0$,

$$\begin{aligned} \frac{|X_n|}{N[n \log \log n]^{\frac{1}{2}}} &= \frac{|S_n - S_{n-1}|}{N[n \log \log n]^{\frac{1}{2}}} \leq \frac{|S_n|}{N[n \log \log n]^{\frac{1}{2}}} + \frac{|S_{n-1}|}{N[n \log \log n]^{\frac{1}{2}}} \\ &\leq \frac{|S_n|}{N[n \log \log n]^{\frac{1}{2}}} + \frac{|S_{n-1}|}{N[(n-1) \log \log (n-1)]^{\frac{1}{2}}}, \end{aligned}$$

so that if $\Pr(|S_n| > N(n \log \log n)^{\frac{1}{2}} \text{ i.o.}) = 0$, then we must have

$$\Pr(|X_n| > 2N(n \log \log n)^{\frac{1}{2}} \text{ i.o.}) = 0$$

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and hence, from the Borel-Cantelli Lemma (the X_i 's being independent), $\sum \Pr[|X| > 2N(n \log \log n)^{\frac{1}{2}}] < \infty$. ("i.o." means infinitely often). This clearly contradicts the condition $\sum \Pr[|X| > (n \log \log n)^{\frac{1}{2}}] = \infty$ so, by appeal to the zero-one law, $\Pr(|S_n| > N(n \log \log n)^{\frac{1}{2}} \text{ i.o.}) = 1$. Since N is arbitrarily large, the result of the theorem follows. Consequently, it remains to consider the case where $\sum \Pr[|X| > (n \log \log n)^{\frac{1}{2}}] < \infty$ but $\sum \Pr(|X| > n^{\frac{1}{2}}) = \infty$, this last condition being equivalent to $EX^2 = \infty$. We note, in particular, that $\sum \Pr(|X| > n^{1/(2-\delta)}) < \infty$; that is, $E|X|^{2-\delta} < \infty$, every $\delta > 0$. Furthermore, if $EX = \mu \neq 0$, the truth of the theorem is evident from the strong law of large numbers so we may subsequently restrict ourselves to the case $EX = 0$. We shall proceed by developing a modification of the argument of Stone [3].

Write F for the distribution function of X . Also, for a positive constant K , let H be the distribution function which results from truncating X at $-K, +K$ and then centering to make the mean zero. Then, there are positive constants a and b ($a + b = 1$) such that $F = aG + bH$, G being a distribution function. Of course, $b = \Pr(|X| \leq K)$. Let $Y_1, Y_2, \dots$ be independent random variables with common distribution function G ; $Z_1, Z_2, \dots$ be independent random variables with common distribution function H and write $T_n = \sum_{i=1}^n Y_i$, $U_n = \sum_{i=1}^n Z_i$. Also, $\xi_1, \xi_2, \dots$ are independent and identically distributed such that $\Pr(\xi_k = 1) = a$ and $\Pr(\xi_k = 0) = b$. The Y_k 's, Z_k 's and ξ_k 's are taken as mutually independent. We write also, $j(n) = \sum_{i=1}^n \xi_i$, $k(n) = n - j(n)$ and $V_n = T_{j(n)} + U_{k(n)}$. Clearly V_n has the same probabilistic structure as S_n , so in order to obtain the desired result we have just to show that for any positive N ,

$$\Pr[(n \log \log n)^{-\frac{1}{2}} |V_n| \geq N \text{ i.o.}] = 1.$$

Now, since $EX = 0$ and $EZ_i = 0$, we have $EY_i = 0$ and consequently $\Pr(|T_n| \leq \varepsilon(n \log \log n)^{\frac{1}{2}} \text{ i.o.}) = 1$ for any fixed $\varepsilon > 0$. Define events A_n , $n \geq 3$, by

$$A_n = \{j(n) \leq \frac{1}{2}(a+1)n; |T_{j(n)}| \leq \varepsilon(n \log \log n)^{\frac{1}{2}}\}.$$

Then, $\Pr(A_n \text{ i.o.}) = 1$, for $n^{-1}j(n) \xrightarrow{\text{a.s.}} a$ as $n \rightarrow \infty$ and consequently,

$$\Pr(j(n) > \frac{1}{2}(a+1)n; |T_{j(n)}| \leq \varepsilon(n \log \log n)^{\frac{1}{2}} \text{ i.o.}) = 0.$$

("a.s." denotes almost sure convergence). Furthermore,

$$\begin{aligned} \Pr[(n \log \log n)^{-\frac{1}{2}} |V_n| \geq N \text{ i.o.}] &= \Pr(|T_{j(n)} + U_{k(n)}| \geq N(n \log \log n)^{\frac{1}{2}} \text{ i.o.}) \\ &= \Pr(A_n; |T_{j(n)} + U_{k(n)}| \geq N(n \log \log n)^{\frac{1}{2}} \text{ i.o.}) \\ &\geq \Pr(A_n; |U_{k(n)}| \geq (N + \varepsilon)(n \log \log n)^{\frac{1}{2}} \text{ i.o.}). \end{aligned}$$

Also, when A_n holds, $k(n) \geq \frac{1}{2}nb$, and given a small positive η , we have $(1 + \eta) \log \log k(n) > \log \log n$ for n sufficiently large. Thus,

$$\begin{aligned}
(1) \quad & \Pr(|V_n| \geq N(n \log \log n)^{\frac{1}{2}} \text{ i.o.}) \\
& \geq \Pr(A_n; |U_{k(n)}| \geq (N + \varepsilon)[(1 + \eta)2b^{-1}k(n) \log \log k(n)]^{\frac{1}{2}} \text{ i.o.}),
\end{aligned}$$

and the problem is to show that this right hand probability is unity.

Let $\sigma^2 = \text{var } Z_i = EZ_i^2$. Then, using the law of the iterated logarithm,

$$\Pr(|U_{k(n)}| \geq (N + \varepsilon)[(1 + \eta)2b^{-1}k(n) \log \log k(n)]^{\frac{1}{2}} \text{ i.o.}) = 1$$

provided $(N + \varepsilon)^2(1 + \eta) < \sigma b$. In order to proceed further, we show that $(n \log \log n)^{-\frac{1}{2}}T_n \xrightarrow{P} 0$ as $n \rightarrow \infty$. (“ P ” denotes convergence in probability).

Firstly, we recall that $\sum \Pr[|X| > (n \log \log n)^{\frac{1}{2}}] < \infty$, and consequently $n \Pr[|X| > (n \log \log n)^{\frac{1}{2}}] \rightarrow 0$ as $n \rightarrow \infty$. Thus, using the degenerate convergence criterion (Loève [2], 317), we shall have $(n \log \log n)^{-\frac{1}{2}}S_n \xrightarrow{P} 0$ provided we can show that

$$(i) \quad (\log \log n)^{-1} \int_{|x| \leq (n \log \log n)^{\frac{1}{2}}} x^2 dF(x) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and

$$(ii) \quad n^{\frac{1}{2}}(\log \log n)^{-\frac{1}{2}} \int_{|x| \leq (n \log \log n)^{\frac{1}{2}}} x dF(x) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

For $n \geq 3$ we have, using integration by parts,

$$\begin{aligned}
\int_{|x| \leq (n \log \log n)^{\frac{1}{2}}} x^2 dF(x) &= - \int_0^{(n \log \log n)^{\frac{1}{2}}} x^2 d\Pr(|X| > x) \\
&\leq 2 \int_0^{(n \log \log n)^{\frac{1}{2}}} x \Pr(|X| > x) dx \\
&= 2 \int_0^{(3 \log \log 3)^{\frac{1}{2}}} x \Pr(|X| > x) dx \\
&\quad + 2 \sum_{k=3}^{n-1} \int_{[k \log \log k]^{\frac{1}{2}}}^{[(k+1) \log \log (k+1)]^{\frac{1}{2}}} x \Pr(|X| > x) dx \\
&\leq 2 \int_0^{(3 \log \log 3)^{\frac{1}{2}}} x \Pr(|X| > x) dx \\
&\quad + \sum_{k=3}^{n-1} \Pr[|X| > (k \log \log k)^{\frac{1}{2}}] [(k+1) \log \log (k+1) - k \log \log k] \\
&\leq 2 \int_0^{(3 \log \log 3)^{\frac{1}{2}}} x \Pr(|X| > x) dx + C \sum_{k=3}^{n-1} \log \log k \Pr[|X| > (k \log \log k)^{\frac{1}{2}}]
\end{aligned}$$

for some positive constant C . Consequently, since $\sum \Pr[|X| > (n \log \log n)^{\frac{1}{2}}] < \infty$, we have from the Kronecker Lemma (e.g. Loève [2], 238) that

$$(\log \log n)^{-1} \int_{|x| \leq (n \log \log n)^{\frac{1}{2}}} x^2 dF(x) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and (i) is established.

In the case of (ii) we have, since $EX = 0$,

$$\begin{aligned} \left| \int_{|x| \leq (n \log \log n)^{\frac{1}{2}}} x dF(x) \right| &= \left| \int_{|x| > (n \log \log n)^{\frac{1}{2}}} x dF(x) \right| \\ &\leq \int_{|x| > (n \log \log n)^{\frac{1}{2}}} |x| dF(x) \\ &= (n \log \log n)^{\frac{1}{2}} \Pr[|X| > (n \log \log n)^{\frac{1}{2}}] + \int_{(n \log \log n)^{\frac{1}{2}}}^{\infty} \Pr(|X| > x) dx \\ &= (n \log \log n)^{\frac{1}{2}} \Pr[|X| > (n \log \log n)^{\frac{1}{2}}] + \sum_{k=n}^{\infty} \int_{[k \log \log k]^{\frac{1}{2}}}^{[(k+1) \log \log (k+1)]^{\frac{1}{2}}} \Pr(|X| > x) dx \\ &\leq (n \log \log n)^{\frac{1}{2}} \Pr[|X| > (n \log \log n)^{\frac{1}{2}}] \\ &\quad + \sum_{k=n}^{\infty} \Pr[|X| > (k \log \log k)^{\frac{1}{2}}] \{[(k+1) \log \log (k+1)]^{\frac{1}{2}} - [k \log \log k]^{\frac{1}{2}}\} \\ &\leq (n \log \log n)^{\frac{1}{2}} \Pr[|X| > (n \log \log n)^{\frac{1}{2}}] \\ &\quad + C \sum_{k=n}^{\infty} k^{-\frac{1}{2}} (\log \log k)^{\frac{1}{2}} \Pr[|X| > (k \log \log k)^{\frac{1}{2}}] \\ &\leq (n \log \log n)^{\frac{1}{2}} \Pr[|X| > (n \log \log n)^{\frac{1}{2}}] \\ &\quad + C' n^{-\frac{1}{2}} (\log \log n)^{\frac{1}{2}} \sum_{k=n}^{\infty} \Pr[|X| > (k \log \log k)^{\frac{1}{2}}], \end{aligned}$$

C, C' being suitably chosen positive constants. Since $\sum \Pr[|X| > (n \log \log n)^{\frac{1}{2}}] < \infty$, it follows immediately that (ii) is satisfied. We therefore have $(n \log \log n)^{-\frac{1}{2}} S_n \xrightarrow{P} 0$ and consequently that $(n \log \log n)^{-\frac{1}{2}} T_n \xrightarrow{P} 0$.

Next we return to the main problem. Define events B_n by

$$B_n = \{|U_{k(n)}| \geq (N + \varepsilon)[(1 + \eta)2b^{-1}k(n) \log \log k(n)]^{\frac{1}{2}}\}.$$

Then from (1), our problem is to show that $\Pr(A_n \cap B_n \text{ i.o.}) = 1$, or equivalently, that $\lim_{n \rightarrow \infty} \Pr[\bigcup_{m=n}^{\infty} (A_m \cap B_m)] = 1$.

Let $\bar{A}$ denote the complement of A . We have

$$\begin{aligned} \Pr\left[\bigcup_{m=n}^{\infty} (A_m \cap B_m)\right] &= \Pr\left[\bigcup_{m=n}^{\infty} \left\{ \bigcap_{l=n}^{m-1} (\bar{A}_l \cap B_l) \cap (A_m \cap B_m) \right\}\right] \\ &= \sum_{m=n}^{\infty} \Pr\left[\bigcap_{l=n}^{m-1} (\bar{A}_l \cap B_l) \cap (A_m \cap B_m)\right] \end{aligned}$$

$$\begin{aligned}
&\geq \sum_{m=n}^{\infty} \Pr \left[\bigcap_{l=n}^{m-1} \bar{B}_l \cap B_m \cap A_m \right] \\
&= \sum_{m=n}^{\infty} \sum_{i=1}^{\frac{1}{2}(a+1)m} \Pr \left[\bigcap_{l=n}^{m-1} \bar{B}_l \cap B_m \cap A_m \cap \{j(m) = i\} \right] \\
&\geq \sum_{m=n}^{\infty} \sum_{i=\delta m}^{\frac{1}{2}(a+1)m} \Pr \left[\bigcap_{l=n}^{m-1} \bar{B}_l \cap B_m \cap \{j(m) = i\} \right] \Pr [A_m \cap \{j(m) = i\}],
\end{aligned}$$

for arbitrary δ , $0 < \delta < a$, the $\{Y_i\}$ and $\{Z_i\}$ being mutually independent. Now, since $(n \log \log n)^{-\frac{1}{2}} T_n \xrightarrow{P} 0$, $\Pr[|T_n| \leq \varepsilon(n \log \log n)^{\frac{1}{2}}] \rightarrow 1$ as $n \rightarrow \infty$. Thus, we can find an integer I and a constant $K > 0$ such that

$$\Pr[A_m \cap \{j(m) = i\}] \geq K \quad \text{for all } m \geq n \geq I, \delta m \leq i \leq \frac{1}{2}(a+1)m.$$

Then, for $n \geq I$,

$$\begin{aligned}
(2) \quad \Pr \left[\bigcup_{m=n}^{\infty} (A_m \cap B_m) \right] &\geq K \sum_{m=n}^{\infty} \sum_{i=\delta m}^{\frac{1}{2}(a+1)m} \Pr \left[\bigcap_{l=n}^{m-1} \bar{B}_l \cap B_m \cap \{j(m) = i\} \right] \\
&= K \sum_{m=n}^{\infty} \Pr \left[\bigcap_{l=n}^{m-1} \bar{B}_l \cap B_m \cap \{\delta m \leq j(m) \leq \frac{1}{2}(a+1)m\} \right] \\
&\geq K \sum_{m=n}^{\infty} \Pr \left[\bigcap_{l=n}^{m-1} \bar{B}_l \cap B_m \right] - K \sum_{m=n}^{\infty} \Pr [\overline{\{\delta m \leq j(m) \leq \frac{1}{2}(a+1)m\}}].
\end{aligned}$$

Now

$$\begin{aligned}
\Pr [\overline{\{\delta m \leq j(m) \leq \frac{1}{2}(a+1)m\}}] &= \Pr [j(m) < \delta m] + \Pr [j(m) > \frac{1}{2}(a+1)m] \\
&= \Pr [k(m) > (1-\delta)m] + \Pr [j(m) > \frac{1}{2}(a+1)m] \\
&\leq e^{-(1-\delta)mt_1} E[e^{t_1 k(m)}] + e^{-\frac{1}{2}(a+1)mt_2} E[e^{t_2 j(m)}] \\
&= [e^{-t_1(1-\delta)}(ae^{t_1} + b)]^m + [e^{-\frac{1}{2}(a+1)t_2}(a - be^{t_2})]^m \\
&= [\xi(t_1)]^m + [\eta(t_2)]^m,
\end{aligned}$$

say, where t_1 and t_2 are positive and chosen so small that $\xi(t_1) < 1$, $\eta(t_2) < 1$. Then,

$$\begin{aligned}
\sum_{m=n}^{\infty} \Pr [\overline{\{\delta m \leq j(m) \leq \frac{1}{2}(a+1)m\}}] &\leq \frac{[\xi(t_1)]^n}{1 - \xi(t_1)} + \frac{[\eta(t_2)]^m}{1 - \eta(t_2)} \\
&\rightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned}$$

Consequently, from (2),

$$\begin{aligned}
\lim_{n \rightarrow \infty} \Pr \left[\bigcup_{m=n}^{\infty} (A_m \cap B_m) \right] &\geq K \lim_{n \rightarrow \infty} \sum_{m=n}^{\infty} \Pr \left[\bigcup_{l=n}^{m-1} (\bar{B}_l \cap B_m) \right] \\
&= K \lim_{n \rightarrow \infty} \Pr \left[\bigcup_{m=n}^{\infty} B_n \right] = K
\end{aligned}$$

for $(N + \varepsilon)^2(1 + \eta) < \sigma b$ since $\Pr(B_n \text{ i.o.}) = 1$ under this condition. It then follows, from the zero-one law, that $\Pr(|S_n| > N(n \log \log n)^{\frac{1}{2}} \text{ i.o.}) = 1$ provided that $(N + \varepsilon)^2(1 + \eta) < \sigma b$. However, $EX^2 = \infty$, so σ can be made arbitrarily large by increasing K and, correspondingly, b tends towards unity. ε and η are arbitrarily small so in fact the relation $\Pr(|S_n| > N(n \log \log n)^{\frac{1}{2}} \text{ i.o.}) = 1$ holds no matter how large be $N > 0$. This establishes the result of the theorem.

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