

MODEL ENDGAME ANALYSIS

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Abstract A reference model of Fallible Endgame Play has been implemented and exercised with the chess engine WILHELM. Various experiments have demonstrated the value of the model and the robustness of decisions based on it. Experimental results have also been compared with the theoretical predictions of a Markov model of the endgame and found to be in close agreement.

Keywords: chess, endgame, experiment, fallibility, Markov, model, theory

1. Introduction

In Haworth (2003), a reference model of fallible endgame play was defined in terms of a spectrum of Reference Endgame Players (REPs) R_c . The REPs are defined as choosing their moves stochastically, using only successor positions' values and depths from an endgame table (EGT). Exploring here the parameters of the model and various opponent-sensitive uses of the REPs including choice of move, we report on:

- a) the robustness of decisions based on the model,
given that various parameters of the model may be changed,
- b) the apparent competence of reference player R_{20} in an $R_{20} - R_{\infty}$ match,
- c) the distribution of game lengths in that match versus Markov theory,
- d) the probability of beating a 50-move draw claim versus Markov theory,
- e) the apparent competence of carbon and silicon players over the board.

In Section 2, we revisit the basic concepts and theory of the REP model, while in Section 3, we describe the REP implementation in WILHELM (Andrist, 2003). In Sections 4 to 7, we focus on the five topics above. Section 8 summarises and notes some questions arising from this work.

2. The Reference Endgame Player Model

A nominated endgame, e.g., chess' KQKR, is considered to be a system with a finite set of states $\{s_i\}$ numbered from 0 to $ns-1$.¹ Each state $s(val, d)$ is an *equivalence class* of positions of the same theoretical value val and depth d . Higher-numbered states are assumed to be less attractive to the side to move, which is taken to be White. Thus, for KQKR with the DTC² metric, we have maxDTCs (1-0) $n_w=31$, (0-1) $n_B=3$, and $ns=37$ states in total:

- $s_i, i=0$: a 1-0 win, i.e., for White, not requiring a winner's move³,
- $s_i, 1 \leq i \leq 31$: 1-0 wins of depth i ,
- $s_i, i=32$: theoretical draw, either in the endgame or a subgame,
- $s_i, 33 \leq i \leq 35$: 0-1 wins, i.e., for Black, of depth $36-i$
- $s_i, i=36$: a 0-1 win not requiring a winner's move.

The REP R_c in position P chooses stochastically from moves which each have a probability proportional to a Preference⁴, $S_c(val_s, d_s)$, where s is the move's destination state with theoretical value val_s and win/loss depth d_s . Each move-choice by R_c is independent of previous move-choices.

We require that $\{R_c\}$ is a spectrum of players, ranging linearly from the metric-infallible player R_∞ via the random player R_0 to the anti-infallible player $R_{-\infty}$. To ensure this, the function $S_c(val, d)$ is required to meet some natural criteria, as described more fully and formally in Haworth (2003) and in Appendix B.

Here, we choose, as an $S_c(val, d)$ function meeting those criteria:

$$\begin{aligned} S_c(win, d) &\equiv (d + \kappa)^{-c} \text{ with } \kappa > 0 \text{ to ensure that } S_c \text{ is finite,} \\ S_c(draw) &\equiv S_c(win, n_1) \equiv S_c(loss, n_2) \text{ with } n_1 > n_w \text{ and } n_2 > n_B, \\ S_c(loss, d) &\equiv \lambda \cdot (d + \kappa)^c, \lambda \text{ being defined by } n_1 \text{ and } n_2 \text{ above.} \end{aligned}$$

This ensures, as required, that R_0 prefers no move to any other, that R_c with $c > 0$ prefers better moves to worse moves, and that as $c \rightarrow \infty$, the R_c increase in competence and tend to infallibility in terms of the chosen metric.

Although the R_c have no game-specific knowledge, the general REP model allows moves to be given a prior, ancillary, weighting v_m based on such considerations (Jansen, 1992). Thus, $v_m = 0$, as used in this paper, prevents a move being chosen and $v_m > 1$ makes it more likely to be chosen.

The probability $T_c(i)$ of moving to state s_i is therefore:

$$T_c(i) \equiv S_c(s_i) \cdot \sum_{\text{moves_to_state_i}} v_m / \sum_{\text{all_moves}} v_m \cdot S_c(s_{\text{move}})$$

¹ For convenience, Appendix A summarises the key acronyms, notation, and terms.

² DTC \equiv DTC(onversion) \equiv Depth to Conversion, i.e., to mate and/or change of material.

³ i.e., mate, achieved conversion to won subgame, or loser forced to convert on next move.

⁴ For convenience and clarity, the Preference Function $S_c(val_s, d_s)$ may be signified by the more compact notations $S_c(val, d)$ or merely $S_c(s)$ if the context allows.

3. Implementing the REP Model

The second author has implemented in WILHELM (Andrist, 2003) a subset of the REP model which is sufficient to provide the results of this paper.

Ancillary weightings v_m are restricted to 1 and 0. $v_m = 0$ is, if relevant, applied to all moves to a state s rather than to specific moves: it can be used to exclude moves losing theoretical value, and/or to emulate a search horizon of H moves, within which a player will win or not lose if possible.

WILHELM offers five agents based on the REP model: these are, as defined below, the *Player*, *Analyser*, *Predator*, *Emulator*, and *Predictor*. A predefined number of games may be played between any two of WILHELM, *Player*, *Predator*, *Emulator* and an infallible player with endgame data. WILHELM also supports the creation of Markov matrices, see Section 5.

3.1 The Player

The *Player* is an REP R_c of competence c , and therefore chooses its moves stochastically using a validated (pseudo-)random number generator in conjunction with the function $S_c(val, d)$ defined earlier.

3.2 The Analyser

Let us imagine that an unknown fallible opponent is actually going to play as an R_c with probability $p(d) \cdot \delta x$ that $c \in (x, x + \delta x)$: $\int p(x) dx = 1$.

The *Analyser* attempts to identify the actual, underlying c of the R_c which it observes. For computational reasons, the *Analyser* must assume that c is a value from a finite set $\{c_j\}$ and that $c = c_j$ with initial probability $pc_{0,j}$.

Here, the c_j are regularly spaced in $[c_{min}, c_{max}]$ as follows:

$$c_{min} = c_1, c_j = c_1 + (j-1) \cdot c\delta \text{ and } c_{max} = c_1 + (n-1) \cdot c\delta, \text{ i.e. } c = c_{min}(c\delta)c_{max}.$$

The notation $c = c_{min}(c\delta)c_{max}$ is used to denote this set of possible values c . The initial probabilities $pc_{0,j}$ may be $1/n$, the usual ‘know nothing’ uniform distribution, or may be based on previous experience or hypothesis. They are modified, given a move to state s_{next} , by Bayesian inference:

$$T_j(next) = \text{Prob}[\text{move to state } s_{next} \mid c = c_j], \text{ and}$$

$$pc_{i+1,j} = pc_{i,j} \cdot T_j(next) / \sum_k [pc_{i,k} \cdot T_k(next)].$$

$$\text{Thus, the new Expected}[c] = \sum_j pc_{i+1,j} \cdot c_j.$$

In Subsection 4.1, we investigate what values should be chosen for the parameters c_{min} , $c\delta$ and c_{max} so that the errors of discrete approximation are acceptably small.

3.3 The Predator

On the basis of what the *Predator* has learned from the *Analysers* about its opponent, it chooses its move to best challenge the opponent, i.e., to optimise the expected value and depth of the position after a sequence of moves. As winning attacker, it seeks to minimise expected depth; as losing defender, it seeks to maximise expected depth. In a draw situation, it seeks to finesse a win.

Different moves by the *Predator* create different sets of move-choices for the fallible opponent. These in turn lead to different expectations of theoretical value and depth after the opponent's moves.

The predator implementation in WILHELM chooses its move on the basis of only a 2-ply search. It may be that deeper searches will be worthwhile, particularly in the draw situation.

3.4 The Emulator

The *Emulator* E_c is conceived as a practice opponent with a 'designer' level of competence tailorable to the requirements of the practising player. An REP R_c will exhibit an apparent competence c' varying, perhaps widely, above and below c because it chooses its moves stochastically. In contrast, the *Emulator* E_c chooses a move which exhibits to an *Analysers* an apparent competence c'' as close to c as possible.

The reference *Analysers* is defined as initially assuming the *Emulator* is an R_x , $x = 0(1)2c$, where $x = x_j$ with initial probability $1/(2c+1)$.

The *Emulator* E_c therefore opposes a practising player with a more consistent competence c than would R_c , albeit with some loss of variety in its choice of moves. The value c can be chosen to provide a suitable challenge in the practice session.

The practising player may also have their apparent competence assessed by the *Analysers*.

3.5 The Predictor

The *Predictor* is advised of the apparent competence c of the opponent. It then predicts how long it will take to win, or what its chances are of turning a draw into a win, using data from an *Analysers* and from a Markov model of the endgame. This model is defined in Section 5.

4. Robustness of the Model

The two famous Browne-BELLE KQKR exhibition games have already been studied using the REP model (Haworth, 2003). Browne’s apparent competence c was assessed by an *Analys*er, and BELLE’s moves as Black were compared with the decisions of a *Predator* using the *Analys*er’s output.

In that analysis, the following six choices were made:

- $c_{min} = 0$, $c\delta = 1$, $c_{max} = 50$; all c_j were deemed equally likely,
- $\kappa = 0^+$ (i.e., arbitrarily small, effectively zero) and metric = DTC.

The following question therefore arises: to what extent are the conclusions of the *Analys*er and the choices made by the *Predator* affected by these six choices? Our first studies addressed this question.

4.1 The Effect of Numerical Approximation

Browne-BELLE game 1 was first reanalysed, this time with $\kappa = 1$, and:

$c_{min} = 0$, $c_{max} = 50$ and $c\delta$ in turn set to 0.01, 0.1, 1, 2, 5 and 10.

Figure 1 takes the *Analys*er with $c\delta = 0.01$ as a benchmark, and shows how the choice of $c\delta$ affected the *Analys*er’s inferences during play.

It may be shown the *Analys*er’s Bayesian calculation is a discrete approximation to a calculable integral: the theory of integration therefore guarantees that this calculation will converge as $c\delta \rightarrow 0$. We judge that the error is ignorable with $c\delta = 1$ and that no smaller $c\delta$ is needed.

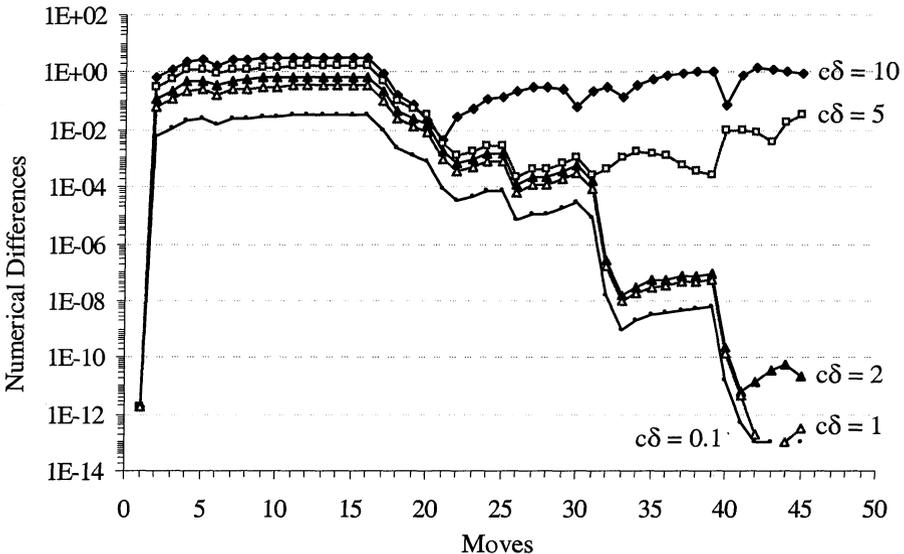


Figure 1. Differences in c -estimation, relative to the $c\delta = 0.01$ estimate.

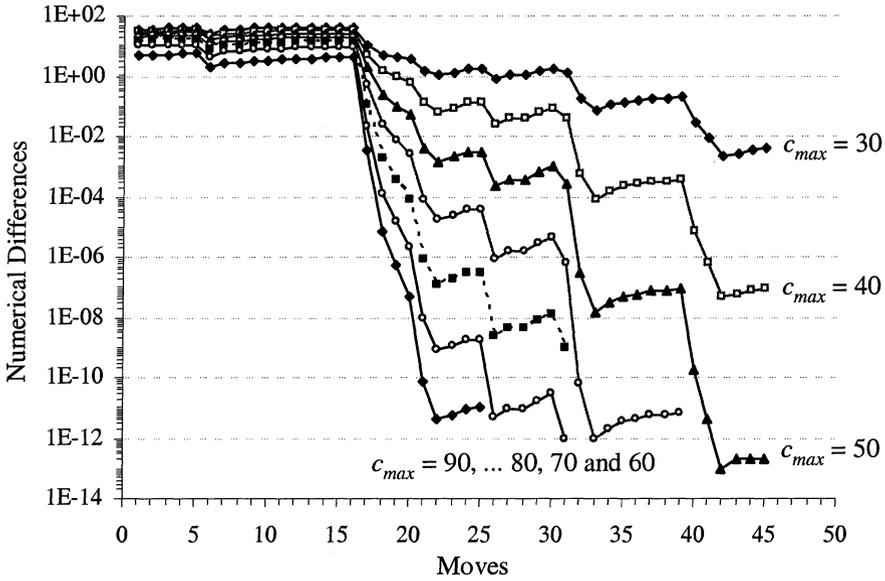


Figure 2. Differences in c -estimation, relative to the $c_{max} = 100$ estimate.

The analysis of the game was then repeated with:

$c\delta = 1$, $c_{min} = 0$ and c_{max} in turn set to 100, 90, 80, 70, 60, 50, 40 and 30.

Again, intuitively, we would expect the error introduced by a finite c_{max} to reduce as $c_{max} \rightarrow \infty$. Figure 2 shows that this is indeed the case and that, with Browne's apparent $c \approx 20$, $c_{max} = 50$ is conservative enough. However, it may need to be larger for easier endgames.

We assume that our opponent has positive apparent competence c and that the *Analyser* is correct in taking $c_{min} = 0$ as a lower bound on c .

4.2 The Effect of κ

Given the requirements on $S_c(val, d)$, it may be shown⁵ that, as κ increases, R_c progressively loses its ability to differentiate between better and worse moves, that R_c 's expectation of state and theoretical value do not improve and that $R_c \rightarrow R_0$. Thus, for a given set of observations, if the *Analyser* assumes a greater κ , it will infer an increasing apparent competence c .

In this paper, we choose a fixed $\kappa = 1$ throughout, as it were, recognising the next move in the line contemplated. We have not tested the effect of different κ on a *Predator*'s choices of move, but assume it is not great. There seems little reason to choose one value of κ over another.

⁵ The proof is by elementary algebra and in the style of Theorem 3 (Haworth, 2003).

4.3 The Effect of the Initial Probability Assumption

The usual, neutral, initial stance is a *know nothing* one, assuming that c is uniformly distributed in a conservatively-wide interval $[c_{min}, c_{max}]$. However, it is clear that had BELLE been using the REP model, it could have started game two with its perception of Browne as learned from game one, just as Browne started that game with his revised perception of KQKR. Also, one might have a perception of the competence c likely to be demonstrated by the opponent with the given endgame force – and choose this to be the midpoint of a $[c_{min}, c_{max}]$ range with a normal distribution.

Bayesian theory, see Subsection 3.2, shows that the initial, assumed non-zero probabilities continue to appear explicitly in the calculation of subsequent, inferred probabilities. We therefore conclude that initial probabilities have some effect on the inferred probabilities.

4.4 The Effect of the Chosen Metric

The metric Depth to Conversion (DTC) was chosen because *conversion* is an obvious intermediate goal in most positions. The adoption of DTC is however a chessic decision.

Our analysis of the Browne-BELLE games shows that the *Predator* would never have made a DTC-suboptimal move-choice for Black. It is reasonable to assume that, had DTM(ate) been the chosen metric, it would never have chosen a DTM-suboptimal move.

However, different metrics occasionally define different subsets of moves as metric-optimal. Where this occurs, the *Predator* might well choose a different move in its tracking of the Browne-BELLE games.

5. A Markov Model of the Endgame

Let us assume that the Preference Function $S_c(val, d)$ is fixed, e.g., as the function defined here with $\kappa = 1$.

Given a position P in state s_i , we can calculate the probability of R_c choosing move m to some position P' in state s_j . We may therefore calculate the probability, $T_c(j)$ of moving from position P to state s_j . Averaging this across the endgame over all such positions P in state s_i , we may derive the probability $m_{i,j}$ of a state transition $s_i \rightarrow s_j$ assuming initial state s_i .

The $\{m_{i,j}\}$ define a Markov matrix $\mathbf{M}_c = [m_{i,j}]$ for player R_c . This matrix, and the predictions which may be derived from it, provide a characterisation of the endgame as a whole.

Let us assume that the initial position is 1-0, in state s_i , and that R_c does not concede the win. From the matrix, we may derive predictions such as:

- the probability of R_c winning on or before move m ,
- the expected number of moves required for R_c to achieve the win.
this is L_i in the solution of $(\mathbf{I} - \mathbf{M}_c) \cdot \underline{\mathbf{L}} = \underline{\mathbf{U}}^6$, q.v. (Haworth, 2003).

These theoretical predictions have been computed and are compared with the results of the extensive experiment described in the next section.

6. An Experiment with R_{20}

Echoing Browne-BELLE, a model KQKR match was staged between the fallible attacker R_{20} and the infallible defender R_∞ . It was assumed that R_{20} would not concede the win but eventually secure it as theory predicts. The game-specific repetition and 50-move drawing rules were assumed not to be in force. Table 1 summarises the results of this experiment.

1,000 games were played from each of the two maxDTC KQKR positions used in the Browne-BELLE match. Games ended when conversion was achieved by White. The purpose of the experiment was to observe:

- the distribution of the c inferred by an *Analyser*⁷ at the end of each game with the assumed probability of c_i set to 1/51 at start of each game,
- the distribution of the lengths of the games, and
- the trend in the *Analyser*'s inferred c , ignoring game-starts after the first.

KQKR: R_{20} - R_∞	Position 1	Position 2	Overall
Min., end-of-game apparent c	15.06	14.73	14.73
Max., end-of-game apparent c	35.66	40.71	40.71
Mean, end-of-game apparent c	21.318	21.620	21.469
St. Dev., end-of-game apparent c	3.345	3.695	3.524
St. Dev of the Mean apparent c	0.106	0.117	0.079
Mean c - 20 /Stdev_mean	12.43	13.85	18.59
Min. moves, m , to conversion	37	37	37
Maximum moves, m	395	325	395
Mean moves, m	96.88	94.31	95.60
St. Dev., m	102.951	102.273	102.587
St. Dev., mean of m	3.256	3.234	2.294

Table 1. Statistical Analysis of the 2,000-game experiment.

⁶ \mathbf{I} is the Identity matrix; $\underline{\mathbf{U}}$ is a vector where each element is the unit '1'.

⁷ using $c_{min} = 0$, $c\delta = 1$ and $c_{max} = 50$ as found adequate in Section 4.1.

6.1 R_{20} 's Apparent c after One Game

Figure 3 shows the distribution of the apparent c as inferred at the end of each, single game: the mean c is 21.50 ± 0.08^8 . This rather surprised us, being more distant from the actual $c = 20$ than expected. The reason is that the mean of {end-of-game estimated c } is not statistically the best way to estimate the underlying c , a task we revisit in Subsection 6.4.

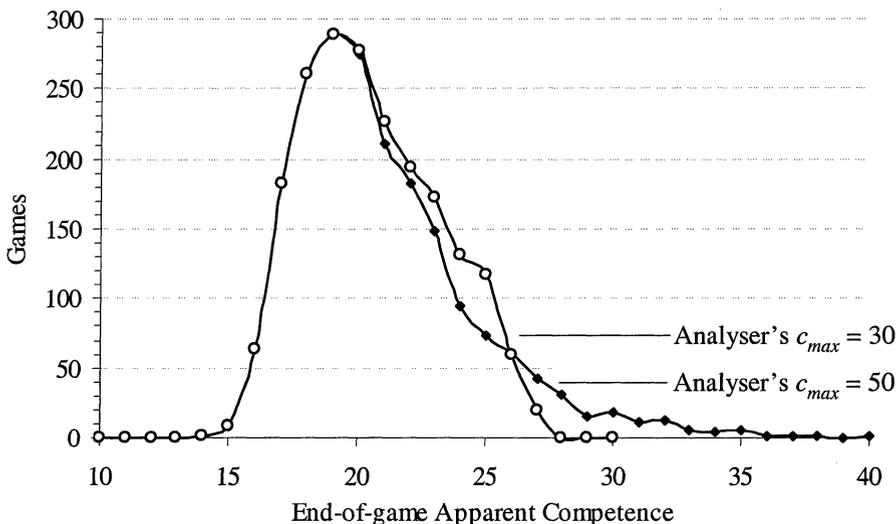


Figure 3. Distribution of apparent c as inferred after one game.

6.2 Game-Length Statistics

Starting from the two positions with (maximum) DTC depth 31, and taken over the 2,000 games, the mean number of moves required for conversion is 95.60 ± 2.29 . Figure 4 shows the distribution of the experiment's game lengths in comparison with the predictions of the Markov model.

Figure 5 shows the Markov-model predictions for the expected number of moves to conversion, for $c = 20, 21,$ and 22 and starting at any depth. Note that it shows that the main barriers to progress seem to be between depths 17 and 26 rather than at the greatest depths.

From depth 31, the moves predicted are 97.20 for $c = 20,$ 83.70 for $c = 21$ and 74.16 for $c = 22$. The experimental results are therefore in close agreement with these predictions, indicating a c of ~ 20.1 .

⁸ Mean end-of-game apparent c is still 21.04 when the *Analysers'* c_{max} is 30 rather than 50.

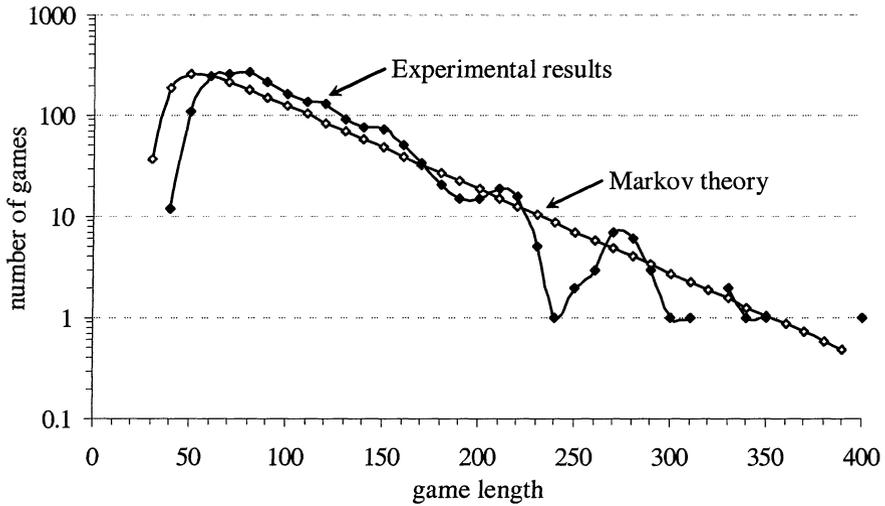


Figure 4. Distribution of game lengths in the R_c - R_∞ KQKR match.

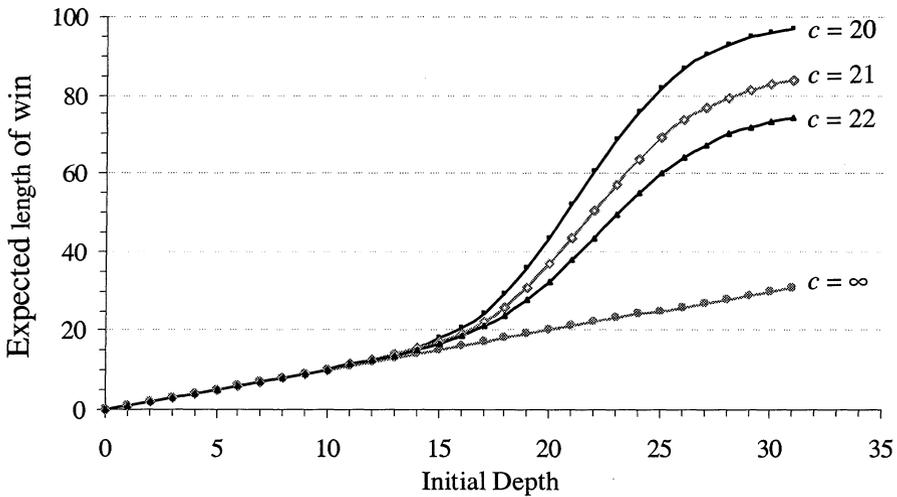


Figure 5. Expected[moves to conversion in an R_c - R_∞ KQKR game].

6.3 The Probability of Winning

The games were played without the 50-move rule but the Markov model allows us to calculate the probability of winning from depth d on or before move 50, before a possible draw claim by the opponent. It is the probability of being in state 0 after 50 moves, namely the element $\mathbf{M}_c^{50}[d, 0]$ of \mathbf{M}_c^{50} .

Figure 6 gives these probabilities for $c = 20, 21,$ and 22 and for all initial depths. For $c = 20$ and initial depth 31, this is 12.67%, a figure reached after 55 moves in the 2,000 game experiment.

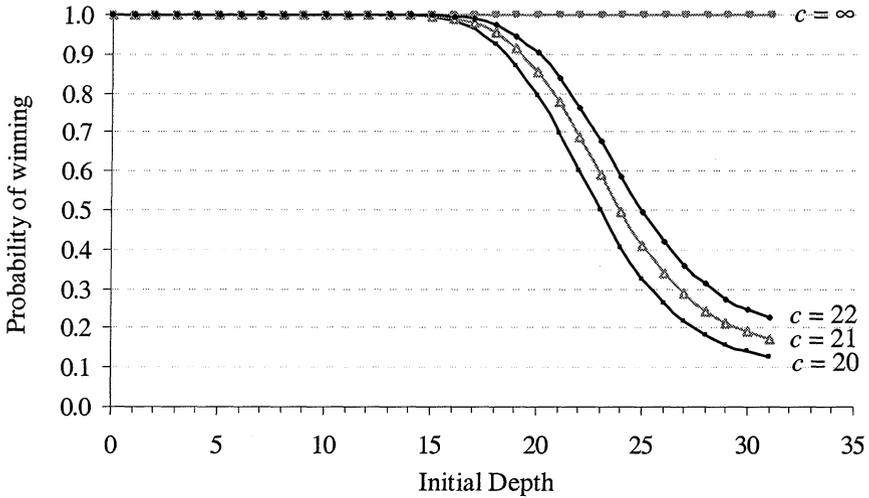


Figure 6. Probability[R_c wins an R_c - R_∞ KQKR game in 50 moves].

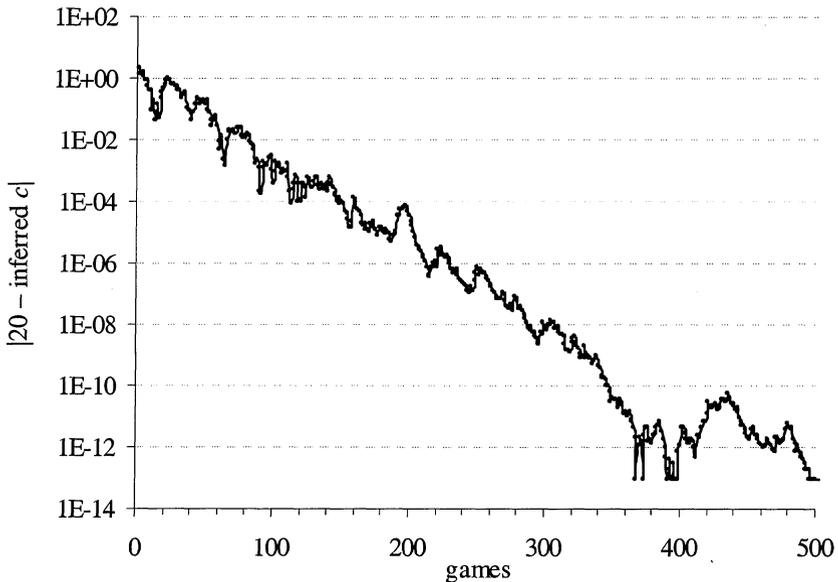


Figure 7. Analyser error in c -estimate versus number of games analysed.

6.4 Analysing R_c 's competence c

The mean of the 2,000 end-of-game apparent c values is not actually the best estimate of R_c 's underlying c .

The reason is that the 2,000 games may be seen as some 191,200 independent move-choices by the R_c . There is no need to associate the *know nothing* uniform distribution probabilities with the possible c_j more than once. In fact, to do so is to interrupt the Bayesian inference processor of the *Analyser* and to negate what the *Analyser* has learned from previous games about the non-uniform distribution of probabilities of the candidate c_j .

Figure 7 shows the *Analyser*'s perception of c approaching the correct value of 20 as it works through the 2,000 games. Even starting with an estimate of $c = 25$, it is accurate to 0.1 after examining 6,000 moves.

7. Apparent Competence of Players

The apparent competence of both carbon and silicon players has been calculated for some published games. The initial assumptions differed slightly from those of Haworth (2003): here, WILHELM was set to analyse with candidate $c = 0$ (0.01) 50 and $\kappa = 1$ and the results are listed in Table 2, showing depth conceded by both sides, net progress and apparent c .

Some background to the games may help put the figures in context. The two Browne-BELLE games (Fenner, 1979; Haworth, 2003; Jansen, 1992a; Levy and Newborn, 1991) are the famous demonstration that the 'easy' KQKR endgame is not so easy to win. Gelfand-Svidler was a tie-breaker rapid-play game played under extreme time pressure. Pinter-Bronstein has been extensively analysed by Roycroft (1988). Timman consulted extensively in a prior adjournment (Breuker et al, 1992). FRITZ (Heise, 2002) played itself with only 3-to-4-man EGTs in an Intel-AMD duel. Lengyel lost the draw three times before our analysis begins (Levy, 1972a,b, 1992).

#	Profile	White	Black	Res.	Year	#m	depth lost		depth gain	Final c	
							Wh.	Bl.		Wh.	Bl.
1	KQKR	Browne	BELLE 1	=	1978	45	27	0	18	19.5	∞
2	KQKR	Browne	BELLE 2	1-0	1978	50	19	0	31	18.4	∞
3	KRKQ	Gelfand	Svidler	=	2001	50	37	79	8	3.5	4.1
4	KNKBB	Pinter	Bronstein	=	1977	50	60	95	14	5.9	15.4
5	KBBKN	Popovich	Korchnoi	=	1984	31	74	45	1	8.1	6.6
6	KBBKN	Timman	Speelman	1-0	1992	25	36	44	33	15.0	7.6
7	KNKBB	FRITZ	FRITZ	=	2002	49	169	210	8	2.9	3.2
8	KQKQN	Lengyel	Levy	0-1	1972	14	7	16	5	4.2	2.8

Table 2. Apparent Competence of Players.

Even after noting that the values c are not necessarily the player's true equivalent c , and are meaningful only in relative rather than absolute terms, the performances of Browne and Timman stand out. FRITZ trades major depth with its clone-opponent and clearly misses the withheld perfect information. The time constraints of Rapid Play, and even third-phase $30'/game$ play in classical chess, mitigate against quality endgame play – arguably a loss to the world of chess.

8. Summary

We have examined the utility of a reference model of Fallible Endgame Play by both experiment and theory, using both a comprehensive REP implementation in WILHELM and Markov methods. Various demonstrations have shown opportunities for exploiting the model, and the robustness of decisions based on it. Experimental results have also been compared with the Markov predictions, with which they agree closely.

Experiments which remain to be carried out include:

- infallible White attacking fallible Black in a drawn position
e.g., in KBBKN, KNNKP, KNPKN, KQNKQ, KQPKP, or KRBKR,
- infallible Black pressing for a draw in a lost position
this requires additional EGT data on draws forced in d moves,
- a more insightful *Predator* searching more than $2p$ plies ahead, and
- use of the *Emulator* as a training partner for human players.

The REP model may be extended to other games where EGTs may be computed – to convergent games such as Chinese Chess, 8×8 checkers, International Draughts, and in principle if not in practice, to divergent *placement* games such as Hex and Othello.

If a search method can propose what it considers the best few moves in a position, each evaluated on an identical basis and therefore comparable, the concept of a stochastic player may be applied more generally than to just endgames for which perfect information is available.

Acknowledgements

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Appendix A: Acronyms, Notation and Terms

<i>Analyser</i>	an agent identifying a fallible opponent as an R_c player
c	the competence index of an REP
$c\delta$	the difference between adjacent c_i assumed by the Analyser
c_{max}	the maximum c assumed possible by the Analyser
c_{min}	the minimum c assumed possible by the Analyser
d	the depth (of win or loss) of a position in the chosen metric, e.g. DTC
DTC	Depth to Conversion, i.e. to change of material and/or mate
DTM	Depth to Mate
<i>Emulator</i>	an agent, E_c , choosing moves to best exhibit apparent competence c
Horizon	a search limit, within which R_c will win or not lose if possible
κ	$\kappa > 0$ ensures that $(d + \kappa)^c$ is finite
λ	a scaling factor, matching the probability of loss to that of a draw
L_i	expected length of win (to conversion in winner's moves) from depth i
maxDTC	maximum DTC (depths)
M_c	a Markov matrix $[m_{i,j}]$
$m_{i,j}$	the probability, averaged over the endgame, that R_c in state s_i moves to s_j
metric	a measure of the depth of a position, usually in winner's moves
n	the number of different c_i assumed by an Analyser
n_1	$n_1 > n_W$, ensures that draws are less preferable than wins
n_2	$n_2 > n_B$, ensures that draws are more preferable than losses
n_B	the number of 'Black win' states
n_W	the number of 'White win' states
ns	the number of states for a chosen endgame and depth metric
$p(x)\cdot\delta x$	the probability that R_c 's $c \in [x, x + \delta x]$

$pc_{0,j}$	the <i>a priori</i> (before a move) probability that the unknown c is c_j
$pc_{i,j}$	the probability, inferred after the i th move, that the unknown c is c_j
Player	an R_c , choosing its moves stochastically with Preference Function S_c
Predator	an agent, choosing the best move possible on the basis of an opponent-model
Predictor	an agent predicting the longer term prospects of a result from Markov theory
REP	Reference Endgame Player
R_0	the REP which prefers no move to any other
R_c	an REP of competence c
R_∞	the player which plays metric-optimal moves infallibly
s	endgame state
s_i	(endgame) state i
$S_c(val_s, d_s)$	the Preference Function for REP R_c , a function of destination value and depth
$S_c(val, d)$	a convenient contraction of $S_c(val_s, d_s)$
$S_c(s)$	a more convenient contraction of $S_c(val_s, d_s)$
$T_c(i)$	the probability that R_c moves to state i , s_i
val	the theoretical value of a position, i.e., <i>win, draw or loss</i>
v_m	a weighting that may be given to a move on chessic grounds

Appendix B: Preference Functions

We require that the set $\{R_c\}$ is in fact a linear, ordered spectrum of R_c players such that:

- for R_0 , all moves are equally likely,
- ' R_∞ ' $\equiv \lim_{c \rightarrow \infty} R_c$ exists and is the infallible player choosing metric-optimal moves,
- ' $R_{-\infty}$ ' $\equiv \lim_{c \rightarrow -\infty} R_c$ exists and is the anti-infallible player choosing anti-optimal moves,
- $c_2 > c_1 \Rightarrow R_{c_2}$'s expectations of successor state, i.e. $E[s]$, are no worse than R_{c_1} 's,
- $c_2 > c_1 \Rightarrow R_{c_2}$'s expectations of theoretical value, i.e. $E[val_d]$, are no worse than R_{c_1} 's.

The following requirements on $S_c(val, d) \equiv S_c(s)$ are natural ones and sufficient to ensure the above, as proved in Haworth (2003):

- $S_c(s)$ is finite and positive: no move has zero or infinite preference for finite c ,⁹
- $S_0(s)$ is a constant,
- for some $n_1 > n_W$ and $n_2 > n_B$, $S_c(draw) = S_c(win, n_1) = S_c(loss, n_2)$,
- $F_j(c) \equiv S_c(s_{i+1})/S_c(s_i)$ decreases as c increases: $\lim_{c \rightarrow \infty} F_j(c) = 0$ and $\lim_{c \rightarrow -\infty} 1/F_j(c) = 0$,
- for $c \neq 0$, $\text{sign}(c) \cdot S_c(s_j)$ decreases (\downarrow) as j increases (\uparrow),
- for $c > (<) 0$, $W_c(d) = S_c(win, d)/S_c(win, d+1) \downarrow (\uparrow)$ as $d \uparrow$ and $\lim_{d \rightarrow \infty} W_c(d) = 1$,
- for $c > (<) 0$, $L_c(d) = S_c(loss, d+1)/S_c(loss, d) \downarrow (\uparrow)$ as $d \uparrow$ and $\lim_{d \rightarrow \infty} L_c(d) = 1$.

The net effect is that:

- the spectrum of R_c is centred as required on the random player, R_0 ,
- the R_c with $c > 0$ prefer better moves to worse moves,
- the R_c demonstrate increasing apparent skill as $c \rightarrow \infty$,
- R_c can be arbitrarily close to being the metric-infallible player for finite c
- as $d \rightarrow \infty$, R_c discriminates less between a win (or loss) of depth d and one of depth $d+1$.

⁹ Hence the requirement that $\kappa > 0$, to accommodate the case of $d = 0$ in $(d + \kappa)^c$.