STABILITY AND LOCAL GROWTH NEAR BOUNDED-STRONG OPTIMAL CONTROLS

Ursula Felgenhauer

Institute of Mathematics

Technical University Cottbus, Germany

felgenh@math.tu-cottbus.de

Abstract Nonlinear constrained optimal control problems as a rule suffer from the so-called two-norm discrepancy, which in particular says that under stable optimality conditions the objective functionals satisfy a quadratic local growth estimate in terms of the L_2 norms but in L_∞ neighborhoods of the solution only. Furthermore, in the case of weak local optima with continuous control functions, stability w.r.t. parameter changes usually can be expected to hold in L_∞ sense rather than in L_p .

Whenever we consider problems with discontinuous optimal control behavior, these results are too restrictive to discuss general variations of the solution including changes in the break points or switches in the active sets. In the paper we show how the use of certain integrated optimality criteria obtained via a duality approach allows for estimates also in the case of discontinuous controls. We consider L_2 and L_1 quadratic growth estimates and discuss consequences for the behavior of minimizing sequences.

Keywords: Constrained control problems, sufficient optimality conditions, stability.

1. Local optimality criteria in integrated form

Consider first a general nonlinear constrained optimal control problem (*primal* problem formulation):

$$(P) \quad \min J(x, u) = k(x(0), x(T)) + \int_0^T r(t, x(t), u(t)) dt$$

$$s.t. \quad \dot{x} = f(t, x(t), u(t)) \quad a.e. \text{ in } [0, T], \quad (1)$$

$$\beta(x(0), x(T)) = 0, \quad (2)$$

$$g(t, x(t), u(t)) \leq 0 \quad a.e. \text{ in } [0, T]. \quad (3)$$

The pair $(x, u) \in W_{\infty}^1(0, 1; \mathbb{R}^n) \times L_{\infty}(0, 1; \mathbb{R}^k)$ is called *admissible* for **(P)** if the state equation (1) (including the boundary conditions (2)) together with the inequality constraints (3) (where $g : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^m$, $\beta : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^s$) is fulfilled. All data functions are assumed to be sufficiently smooth.

Denote by H the HAMILTONIAN, and by $\hat{H}$ the *augmented* HAMILTONIAN related to the problem **(P)**:

$$H(t, x, u, p) = r(t, x, u) + p^T f(t, x, u) ,$$

$$\hat{H}(t, x, u, p, \mu) = H(t, x, u, p) + \mu^T g(t, x, u) , \quad \mu \geq 0 .$$

Further, let W stand for the set

$$W = \{ (t, x, u) : t \in [0, 1], g(t, x, u) \leq 0 \} .$$

Consider the dual variable S given by a function $S : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ and the auxiliary functional θ for $\xi_{1,2} \in \mathbb{R}^n$:

$$\theta(\xi_1, \xi_2, S) = k(\xi_1, \xi_2) + S(0, \xi_1) - S(T, \xi_2) .$$

We assume that S is at least LIPSCHITZ continuous w.r.t. (t, x) whenever $(t, x, u) \in W$. Define

$$T(S) = \inf \{ \theta(\xi_1, \xi_2, S) : \beta(\xi_1, \xi_2) = 0 \}$$

Then the following problem is dual to the original control problem:

$$\begin{aligned} \textbf{(D)} \quad & \max \quad T(S) \\ & \text{s.t.} \quad H(t, x, u, S_x(t, x)) + S_t(t, x) \geq 0 \\ & \text{a.e. on } W = \{ (t, x, u) : t \in [0, 1], g(t, x, u) \leq 0 \} \end{aligned}$$

If (x, u) is an admissible pair for problem **(P)** and S is feasible for **(D)** then the duality relation ([9], also [21] or [4]) holds, i.e.

$$J(x, u) \geq T(S) .$$

The relation turns into an equality if and only if for some admissible (x_0, u_0) and feasible dual S

$$\Psi(x_0, u_0, S) = \int_0^T [H(t, x(t), u(t), S_x(t, x(t))) + S_t(t, x(t))] dt = 0, \quad (4)$$

$$\psi(x_0(0), x_0(T), S) = \theta(x_0(0), x_0(T), S) - T(S) = 0 .$$

In this case, the pair (x_0, u_0) is a solution of **(P)**.

The analysis of the behavior of Ψ and ψ can be further used to characterize local minima of **(P)** in detail including estimates for local growth terms if available (cf. e.g. [7]). It can be also distinguished between *weak* and *strong* local optima in dependence of the reference sets for which optimality holds. To this aim, let us introduce the sets

$$W_\epsilon = W \cap ([0, 1] \times B_\epsilon(x_0, u_0)), \quad \hat{W}_\epsilon = W \cap ([0, 1] \times B_\epsilon(x_0) \times L_\infty)$$

or $\tilde{W}_\epsilon = W \cap ([0, 1] \times B_\epsilon(x_0) \times B_M(0))$

where $\tilde{W}_\epsilon$ with a given constant $M > 0$ is used to check for so-called *bounded-strong* local optima (see e.g. [18] or [20]).

Abstract optimality criteria have been given in [17], [21] or (in slightly generalized formulation) in [7], [6]. The results are summed up in the following Theorem:

Theorem 1 *Let (x_0, u_0) be admissible for **(P)**. Suppose that a function $S : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ exists which is Lipschitz continuous w.r.t. x and piecewise continuously differentiable w.r.t. t such that for a suitably chosen positive constant ϵ the following relations hold with $\gamma = 1$, $D(t) = W_\epsilon(t)$ and $D_\pi = B_\epsilon(x_0(0), x_0(T))$:*

- (R1) $\Psi(x, u; S) > 0$
 $\forall (x, u) \neq (x_0, u_0)$ with $(x(t), u(t)) \in D(t)$ a.e. in $[0, T]$;
- (R2) $\Psi(x_0, u_0; S) = 0$;
- (R3) $\psi(\xi_1, \xi_2, S) > 0 \quad \forall \xi \in D_\pi, \xi \neq (x_0(0), x_0(T))$.

*Then (x_0, u_0) is a strict weak local minimizer of **(P)**.*

If (R1) - (R3) hold true for a certain constant $\epsilon > 0$ with $\gamma = 0$, $D(t) = \hat{W}_\epsilon(t)$ and $D_\pi = B_\epsilon(x_0(0), x_0(T))$, then (x_0, u_0) is a (strict) strong local minimizer.

If the conditions are satisfied with $D(t) = \tilde{W}_\epsilon(t)$ for some $M > 0$, the point (x_0, u_0) is a (strict) bounded-strong local optimum.

The above Theorem differs from former results ([21], [17]) mainly by the consequent usage of the HAMILTON-JACOBI inequality (cf. **(D)**) in integrated form. This fact corresponds to some relaxation in the characterization of local minima by duality means and was first used in [4] for the theoretical convergence analysis of certain discretization methods.

Notice that in some cases it is possible to find estimates for $\Psi(x, u; S) - \Psi(x_0, u_0; S)$ in terms of $\|x - x_0\|_2^2 + \|u - u_0\|_2^2$, i.e. a local quadratic growth condition w.r.t. L_2 topology ([17], [7]) although the reference sets are neighborhoods w.r.t. L_∞ (at least with respect to x). This fact

illustrates once more the effects of the so-called *two-norm discrepancy* appearing in optimal control problems (cf. [10] for details).

The optimality criteria of Theorem 1 in their original parametric formulation have been used in [17] to derive sufficient optimality conditions for general control problems. The approach leads to conditions which in a different way had been obtained via the investigation of so-called *stable* weak optima ([12] or [15], see also [14]).

In particular, it was shown that the following criteria ensure the strict *weak local* optimality of a solution pair (x_0, u_0) :

(PMP) Pontryagin's maximum principle:

$$\begin{aligned} \dot{x} &= \hat{H}_p = f, \quad \beta(x(0), x(T)) = 0; \\ \dot{p} &= -\hat{H}_x, \quad p(0) = -\nabla_1 k - \nabla_1 \beta \rho, \\ &\quad p(T) = \nabla_2 k + \nabla_2 \beta \rho, \quad \rho \in \mathbb{R}^s; \\ \hat{H}_u &= 0; \\ \mu^T g &= 0, \quad \mu \geq 0, \quad g \leq 0. \end{aligned}$$

Notice that in the transversality condition given above the subscripts 1, 2 stand for the gradient components corresponding to the initial and to the final state vectors respectively.

For a given admissible pair (x_0, u_0) , let $g, f, \dots$ be evaluated along the state-control trajectory. We will denote by g^σ , $\sigma > 0$, the set of σ -active constraints at t , e.g. the set of g_i such that $0 \geq g_i(t, x_0(t), u_0(t)) \geq -\sigma$.

(C1) Invertibility: For some positive constant σ , the gradients w.r.t. u of the σ -active constraints, $\nabla_u g^\sigma$, are uniformly linearly independent a.e. on $[0, T]$.

(C2) Controllability: There are functions $y \in W_\infty^1$, $v \in L_\infty$ satisfying

$$\nabla_x g^\sigma y + \nabla_u g^\sigma v = 0; \quad \dot{y} - \nabla_x f y - \nabla_u f v = 0 \quad a.e.$$

together with the boundary condition $\nabla_1 \beta^T y(0) + \nabla_2 \beta^T y(T) = 0$.

In order to formulate second-order conditions, the index set $I_\delta = \{i : \mu_i \geq \delta\}$ and the related tangent spaces

$$\begin{aligned} T_\delta &= \{\zeta : (\nabla_{(x,u)} g^i)^T \zeta = 0 \quad \forall i \in I_\delta\}, \\ T'_\delta &= \{v : (\nabla_u g^i)^T v = 0 \quad \forall i \in I_\delta\} \end{aligned}$$

are useful. The conditions then can be given as follows:

(C3) Legendre-Clebsch Condition: For some positive δ , a constant $\alpha > 0$ exists such that the estimate $v^T \hat{H}_{uu}(t)v \geq \alpha |v|^2$ holds $\forall v \in T'_\delta$ uniformly a.e. on the interval $[0, T]$.

(C4) Riccati Inequality: With the abbreviations $R = -(\nabla_u g^\delta)^+ \nabla_x g^\delta$ and $P = (\nabla_u g^\delta)^\perp$ where $(\cdot)^+$ stands for the *pseudoinverse* and $(\cdot)^\perp$ for the *null-space projector* of a matrix $(\cdot)$, introduce

$$\begin{aligned} a^\delta &= f_x + f_u R, \\ h_{xx}^\delta &= \hat{H}_{xx} + \hat{H}_{xu} R + R^T \hat{H}_{ux} + R^T \hat{H}_{uu} R, \\ (h_{uu}^\delta)^{(-1)} &= P \left(P \hat{H}_{uu} P \right)^{-1} P, \quad h_{xu}^\delta = \hat{H}_{xu} + R^T \hat{H}_{uu}. \end{aligned}$$

For some $\gamma > 0$, the matrix differential inequality

$$\begin{aligned} \dot{Q} \succeq & (h_{xu}^\delta + Q f_u) (h_{uu}^\delta)^{(-1)} (h_{xu}^\delta + Q f_u)^T \\ & - (a^\delta)^T Q - Q a^\delta - h_{xx}^\delta + \gamma I \quad \text{a.e. on } [0, T] \end{aligned} \quad (5)$$

has a bounded in $[0, 1]$ solution satisfying the boundary restrictions

$$\begin{aligned} \xi^T \left(\nabla_1^2 k + \nabla_1^2 \beta \rho + Q(0) \right) \xi &\geq \gamma |\xi|^2 \quad \forall \xi \in \mathbb{R}^n \\ \xi^T \left(\nabla_2^2 k + \nabla_2^2 \beta \rho - Q(1) \right) \xi &\geq \gamma |\xi|^2 \quad \forall \xi \in \mathbb{R}^n. \end{aligned}$$

It is known that **(C1) – (C4)** not only guarantee **(R1) – (R3)** (see [7]), but also a) the LIPSCHITZ stability of the solution in L_∞ w.r.t. small data perturbations ([11], [15], [3]), b) (in the case of a continuous control u_0) the convergence of the EULER and related discretization methods ([14], [4], [2]).

2. Bounded-strong minima. Piecewise conditions

are formulated as integral conditions in terms of Ψ . When we are interested in local growth estimations, this fact gives us the chance to combine piecewise changing growth characterizations as they are typical for switches in the optimal control or for the junction of free arcs and arcs where certain constraints are active.

We begin our optimality analysis with the auxiliary functional $\Psi = \Psi(x, u, S)$ from (4), where the integrand has the form

$$R[t] = (H(t, x, u, \nabla_x S) + S_t) [t]. \quad (6)$$

As it has been shown in [7], using $\Psi(x_0, u_0, S) = 0$ together with an expansion

$$S(t, x) = S_0(t) + p(t)^T (x - x_0(t)) + 0.5 (x - x_0(t))^T Q(t) (x - x_0(t))$$

and the optimal data $\nabla_x S = p_0 + Q(x - x_0)$ and $\dot{S}_0 = -r_0$, one can express R in the following way:

$$\begin{aligned} R[t] &= \hat{H}(x, u, p_0, \mu_0) - \hat{H}(x_0, u_0, p_0, \mu_0) - \hat{H}_x(x_0, u_0, p_0, \mu_0)^T (x - x_0) \\ &\quad + 0.5 (x - x_0)^T \dot{Q}(x - x_0) + (x - x_0)^T Q(f(x, u) - f(x_0, u_0)) \\ &\quad + \mu_0^T (g(x, u) - g(x_0, u_0)) \end{aligned}$$

By rearranging the terms related to variations w.r.t x or u , one can separate two terms $R_{1,2}$ such that $R[t] = R_1[t] + R_2[t]$ and (with $p = p_0 + Q(x - x_0)$)

$$\begin{aligned} R_1[t] &= \hat{H}(x, u_0, p, \mu_0) - \hat{H}(x_0, u_0, p, \mu_0) \\ &\quad - \hat{H}_x(x_0, u_0, p_0, \mu_0)^T (x - x_0) + 0.5 (x - x_0)^T \dot{Q}(x - x_0) \\ &\approx 0.5 (x - x_0)^T \left(\hat{H}_{xx} + Q f_x + f_x^T Q + \dot{Q} \right) (x - x_0); \end{aligned}$$

$$R_2[t] = \hat{H}(x, u, p, \mu_0) - \hat{H}(x, u_0, p, \mu_0) - \mu_0^T g(x, u)$$

Under condition **(C4)**, in particular $R_1[t]$ will be uniformly positive if $\|x - x_0\|_\infty$ is sufficiently small.

In order to estimate Ψ resp. $R[t]$, in [6] the following general result has been proved (Theor. 5 in the cited paper):

Theorem 2 *Suppose (x_0, u_0) to be a weak local minimizer satisfying together with some matrix function $Q \in W_\infty^1$ the conditions **(C1)** - **(C4)**. Let*

$$R_2[t] \geq c_1 |u - u_0(t)|^\nu - c_2 |x - x_0(t)| \quad \forall u : |u| \leq M \quad (7)$$

hold almost everywhere on $[0, 1]$ with $\nu(t) \in \{1, 2\}$ and constants not depending on t . Then (x_0, u_0) is a bounded-strong local minimizer, and positive constants c, ϵ exist such that

$$J(x, u) - J(x_0, u_0) \geq c \|x - x_0\|_2^2 \quad (8)$$

for all admissible (x, u) with $\|x - x_0\|_\infty \leq \epsilon$, $\|u\|_\infty \leq M$.

Important special cases are the following:

Case 1: $\hat{H}(x_0, v, p_0, \mu_0)$ is strongly convex w.r.t. v .

In this case, $R_2 \geq c_1 |u - u_0|^2 - O(|x - x_0|)$, – a situation which can be often observed when $\hat{H}$ takes its minimum in an inner point of the control set.

Case 2: $u_0 = \arg \min \hat{H}(x_0, v, p_0, \mu_0)$, and $-\mu_0^T g(x_0, u) \geq c'_1 |u - u_0|$. Here one can conclude that $R_2 \geq c'_1 |u - u_0| - O(|x - x_0|)$. The situation

occurs when a certain strict complementarity condition holds together with the invertibility assumption **(C1)**.

The criteria given above have been discussed in detail in [6] and tested on a nonlinear example with generically discontinuous optimal control regime in [7]. As a further illustration, we consider here an example from [14]:

Example: The tunnel diode oscillator.

$$\begin{aligned} \min J(x, u) &= \int_0^T (u^2(t) + x^2(t)) dt \\ \text{s.t. } \dot{x}_1 &= x_2, \quad x_1(0) = x_2(0) = -5, \quad x_1(T) = x_2(T) = 0, \\ \dot{x}_2 &= -x_1 + x_2(1.4 - 0.14x_2^2) + 4u, \\ \text{for } |u| &\leq 1 \quad \text{a.e. on } [0, T]. \end{aligned}$$

In the paper [14], the conditions **(C1)** - **(C4)** were checked numerically. In particular, by using a multiple shooting method a bounded matrix function Q was constructed so that the RICCATI condition was satisfied. Thus it is reasonable to assume that a solution to **(C4)** exists with $\|Q\|_\infty < \infty$, and that **(C1)** - **(C3)** hold true for some positive σ and δ . In the example situation for $T = 4.5$ the structure of the optimal control was obtained as $u = 1$ on $[0, \tau_1)$, $u = -1$ on (τ_2, τ_3) and $u = -2p_2$ elsewhere (where $0 < \tau_1 < \tau_2 < \tau_3 < T$ are found approximately from the numerical solution). The points τ_i are the so-called junction points. Consider the term R_2 from our above growth analysis for Ψ :

Using the example data, we get

$$\begin{aligned} R_2 &= u^2 - u_0^2 + 4p_2(u - u_0) + \lambda_{0,1}(1 - u_0) + \lambda_{0,2}(u_0 + 1) \\ &= (2u_0 + 4p_{0,2})(u - u_0) + (u - u_0)^2 + 4(u - u_0)[Q(x - x_0)]_2 \\ &\geq \min \left\{ (\lambda_1 + \lambda_2) |u - u_0|, 0.5 |u - u_0|^2 \right\} - c |x - x_0| \end{aligned}$$

whenever $|x - x_0| \leq \epsilon < 1$. Therefore, the conditions of Theorem 2 are applicable in the example, and the considered solution turns out to be a bounded-strong local minimum satisfying a local L_2 quadratic growth estimate.

3. Bounded-strong optimality in bang-bang case

Consider a dynamical system with linear in state and control equation and known initial position. We will ask for a control regime which in given time gains the system to a final state as close as possible to the origin.

In general, in an arbitrarily given time the system not always can be terminated. Since a (small) deviation from zero in the final position is allowed, this problem class is also called *soft termination control*. As a model case, consider a problem with box control constraints:

$$(\mathbf{P}_S) \quad \min J(x, u) = \frac{1}{2} \|x(T)\|^2$$

$$s.t. \quad \dot{x}(t) = A(t)x(t) + B(t)u(t) \quad a.e. \text{ in } [0, T]; \quad (9)$$

$$x(0) = a; \quad (10)$$

$$|u_i(t)| \leq 1, \quad i = 1, \dots, k, \quad a.e. \text{ in } [0, T]. \quad (11)$$

The HAMILTON function related to $(\mathbf{P}_S)$ has the form

$$H(t, x, u, p) = p^T A(t)x + p^T B(t)u,$$

whereas the *augmented* HAMILTONian reads as

$$\hat{H}(t, x, u, p, \mu) = H + \mu_1^T(u - e) - \mu_2^T(u + e)$$

with $\mu_{1,2} \geq 0$, $e = (1, 1, \dots, 1)^T$.

From PONTYAGIN's maximum principle, we obtain the *switching function*

$$\sigma(t) = B(t)^T p(t), \quad (12)$$

where the costate p satisfies the adjoint equation

$$\dot{p}(t) = -A(t)^T p(t), \quad p(T) = x(T),$$

and the optimal control is given by

$$u_{0,i} \in \begin{cases} \{+1\} & \text{if } \sigma_i(t) < 0, \\ \{-1\} & \text{if } \sigma_i(t) > 0, \\ [-1, +1] & \text{if } \sigma_i(t) = 0, \end{cases} \quad i = 1, \dots, k. \quad (13)$$

Further, the multiplier functions μ_j suffice the relations

$$\mu_1(t) = \left(B(t)^T p(t) \right)_-, \quad \mu_2(t) = \left(B(t)^T p(t) \right)_+$$

where the right-hand sides denote the positive resp. negative part of the related vector components.

In the case that $\sigma_i(t) \equiv 0$ on a certain interval $I \subseteq [0, T]$, we say the control u_0 has a *singular arc* (on I). In our optimality analysis, we will restrict ourselves to the case of piecewise constant u_0 without singular arcs:

Assumption 1 *The optimal control has no singular arcs. In addition, the set of switching points $\Sigma = \{t \in [0, T] : \exists i \in \{1, \dots, m\} \text{ with } \sigma_i(t) = 0\}$ is finite, i.e. $\Sigma = \{t_s : 1 \leq s \leq l\}$ for some $l \in \mathbb{N}$.*

Remark: It is well known, that the above assumption holds true e.g. in the case that A is time-independent and has exclusively real eigenvalues.

Under the Assumption 1, the solution obviously satisfies **(C1)**, **(C2)**: Indeed, $\nabla_u g^0 = \text{diag}(\gamma_j)$ with $\gamma_j \in \{+1, -1\}$, and the controllability assumption **(C2)** in the case of a (linear) initial value problem always trivially holds.

When we try, however, to apply the second order optimality criteria **(C3)** and **(C4)** to our problem, serious difficulties occur:

First of all, the LEGENDRE - CLEBSCH condition **(C3)** becomes singular since $\hat{H}_{uu} = 0$ a.e. on $[0, T]$. It can be fulfilled formally only in the limit sense with $\delta = 0$ where $T_0 = \{0\}$ for all $t \in [0, T] \setminus \Sigma$.

The limit case of the corresponding RICCATI inequality for $\delta \rightarrow 0$ gives

$$\begin{aligned} \dot{Q} + A Q + Q A^T &\succeq \gamma I & \text{a.e.} \\ I - Q(T) &\succeq \gamma I \end{aligned} \quad (14)$$

which obviously can be fulfilled with arbitrary positive γ . In particular, one can choose Q_1 such that in both parts of (14) equality holds with $\gamma = 0.5$. Then for $\gamma < 0.5$, the matrix function $Q_\gamma = 2\gamma Q_1$ solves (14). Moreover, it belongs to $L_\infty(0, T; \mathbb{R}^{n \times n})$ with $\|Q_\gamma\|_\infty = 2\gamma \|Q_1\|_\infty \leq c(A)\gamma$.

It has to be mentioned, however, that in the case of a singular matrix $\hat{H}_{uu}$ the conditions **(C3)**, **(C4)** with $\delta = 0$ in general are *not* sufficient to show the optimality of the solution even in the weak local sense. For our problem class e.g. the linearization in the (nearly-)active constraints will allow only for zero control variation, which together with the admissibility assumption for the linear state equation case reduces the state variation to zero, too. The local technique of deriving estimates for Ψ (resp. for $J - J^*$) from its TAYLOR expansion (see [21] and [7], [6]) therefore fails in the given situation.

Having in mind these arguments, let us restart the optimality analysis for $\Psi = \Psi(x, u, S)$ (cf. (4)) with the integrand $R = R_1 + R_2$ from (6). Notice that from $f(x, u) = Ax + Bu$ we obtain in particular the following estimate for R_1 near x_0 :

$$R_1[t] \approx 0.5 (x - x_0)^T \left(\dot{Q} + Q A + A^T Q + \hat{H}_{xx} \right) (x - x_0) .$$

In the case when **(P_S)** is considered, we have $\hat{H}_{xx} = 0$. Choosing $Q = Q_\gamma$ and $y = x - x_0$ such that $\|y\|_\infty \leq \epsilon_1$, with sufficiently small $\epsilon_1 > 0$ we get

$$R_1[t] \geq 0.25 y^T \left(\dot{Q} + Q A + A^T Q \right) y \geq \frac{\gamma}{4} |y|^2 , \quad (15)$$

We will further derive appropriate estimates for R_2 and the integrals Ψ under the following regularity assumption on the zeros of the switching function σ :

Assumption 2 *There exist positive constants $c_0, \bar{\delta}$ with the following property: For $\delta \in (0, \bar{\delta})$, denote $\omega_\delta = \bigcup_{1 \leq s \leq l} (t_s - \delta, t_s + \delta)$. Then,*

$$\min_i \left| \left(B^T p(t) \right)_i \right| \geq c_0 \delta \quad \forall t \in [0, T] \setminus \omega_\delta .$$

In the case of problem $(\mathbf{P}_S)$, the term R_2 connected with variations w.r.t. u may be expressed by

$$\begin{aligned} R_2[t] &= \hat{H}(x, u, p_0, \mu_0) - \hat{H}(x, u_0, p_0, \mu_0) + (x - x_0)^T Q B (u - u_0) \\ &\quad - \mu_{0,1}^T (u - u_0) + \mu_{0,2}^T (u - u_0). \end{aligned}$$

Denoting $v = u - u_0$ with an arbitrary feasible control u , we have:

$$\begin{aligned} \left(B^T p_0 \right)_i > 0 &\Rightarrow (u_0)_i = -1 \Rightarrow v_i \geq 0 \\ \left(B^T p_0 \right)_i < 0 &\Rightarrow (u_0)_i = +1 \Rightarrow v_i \leq 0 . \end{aligned}$$

Therefore,

$$R_2[t] = \left(B^T p_0 \right)^T v + y^T Q B v \geq \sum_{i=1}^m \left| B^T p_0 \right|_i \cdot |v_i| - \left| y^T Q B v \right| .$$

In order to estimate the integral over R_2 , each part of the right-hand side will be integrated separately and estimated now:

$$J_1 = \int_0^T \left| y^T Q B v \right| dt \leq \|y\|_\infty \|Q\|_\infty \|B\|_\infty \int_0^T |v(t)| dt .$$

From the state equation it follows that $\|y\|_\infty \leq c(A, B) \|v\|_1$, thus

$$J_1 \leq \|Q\|_\infty \|B\|_\infty c(A, B) \|v\|_1^2 =: \gamma c_2 \|v\|_1^2$$

when $Q = Q_\gamma$, and for a certain constant $c_2 = c_2(A, B) > 0$.

The estimation of

$$J_2 = \int_0^T \sum_{i=1}^m \left| B^T p_0 \right|_i \cdot |v_i| dt$$

will make essential use of the Assumption 2 on the switching function behavior near its zeros. Rewriting

$$J_2 = \int_0^T \dots dt = \int_{[0, T] \setminus \omega_\delta} \dots dt + \int_{\omega_\delta} \dots dt ,$$

it is easy to see that the second part is nonnegative. Neglecting this integral (which for small δ is of small size), from Assumption 2 we conclude

$$J_2 \geq c_0 \delta \int_{[0,T] \setminus \omega_\delta} \sum_{i=1}^m |v_i(t)| dt \geq c_0 \delta \int_{[0,T] \setminus \omega_\delta} |v(t)| dt$$

(where $|v|$ stands for the EUKLIDEAN vector norm of $v = v(t)$ in $\mathbb{R}^m$). On the other hand side,

$$\begin{aligned} \|v\|_1 &= \int_0^T |v(t)| dt = \int_{[0,T] \setminus \omega_\delta} |v(t)| dt + \int_{\omega_\delta} |v(t)| dt \\ &\leq \int_{[0,T] \setminus \omega_\delta} |v(t)| dt + c_1 \delta \end{aligned}$$

for $c_1 = 4kM$ if only $\|u\|_\infty$ and $\|u_0\|_\infty$ are bounded by M ($M = 1$ in $(\mathbf{P}_S)$ e.g.). Inserting this estimate into our relation for J_2 , we obtain

$$J_2 \geq c_0 \delta (\|v\|_1 - c_1 \delta) .$$

Combining now the estimates for J_1 and J_2 , an estimate for $\int R_2 dt$ results:

$$\int_0^T R_2[t] dt \geq c_0 \delta (\|v\|_1 - c_1 \delta) - c_2 \gamma \|v\|_1^2 . \quad (16)$$

LEMMA 1 (Weak local optimality.)

Let the Assumptions 1, 2 hold for the optimal data. Then positive constants ϵ , c_w exist such that

$$\int_0^T R[t] dt \geq c_w \left(\|x - x_0\|_2^2 + \|u - u_0\|_1^2 \right) \quad (17)$$

for all admissible $(x, u) \in W_\epsilon$. Therefore, (x_0, u_0) is a strict weak local minimizer (with $J(x, u) - J(x_0, u_0)$ satisfying a local quadratic growth condition similar to (17)).

Proof: Our previous analysis allows to discuss the estimate (16) for various δ . In particular, choose

$$\delta = \frac{1}{2c_1} \|v\|_1 .$$

Then, for $\|v\|_\infty = \|u - u_0\|_\infty \leq \epsilon < \epsilon_2 = (2c_1\bar{\delta})/T$,

$$\delta \leq \frac{1}{2c_1} T \|v\|_\infty < \bar{\delta} ,$$

and we obtain

$$\int_0^T R_2[t] dt \geq \left(\frac{c_0}{4c_1} - c_2\gamma \right) \|v\|_1^2 =: \tilde{c}(\gamma) \|v\|_1^2 .$$

If γ is taken from $\left(0, \frac{c_0}{4c_1c_2} \right)$ then $\tilde{c}(\gamma)$ is positive, so that together with (15) the last estimate leads to

$$\int_0^T R[t] dt \geq \frac{\gamma}{4} \|x - x_0\|_2^2 + \tilde{c}(\gamma) \|u - u_0\|_1^2 .$$

Thus, the desired conclusion follows by setting $c_w = \max_{\gamma>0} \min\{\frac{\gamma}{4}, \tilde{c}(\gamma)\}$ and $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. $\square$

LEMMA 2 (Strong local optimality.)

Let the Assumptions 1, 2 hold true. Then positive constants ϵ, c_s exist such that

$$\int_0^T R[t] dt \geq c_s \|x - x_0\|_\infty^2 \quad \text{for all admissible } (x, u) \in \tilde{W}_\epsilon . \quad (18)$$

Therefore, (x_0, u_0) is a strict strong local minimizer (with $J(x, u) - J(x_0, u_0)$ satisfying a local quadratic growth condition similar to (18)).

Proof: Consider (16) with

$$\delta = \min \left\{ \frac{1}{2c_1}, \frac{\bar{\delta}}{2MT} \right\} \|v\|_1 =: c_3 \|v\|_1 .$$

Since $\|v\|_\infty \leq 2M$, we have $\delta \leq \frac{\bar{\delta}}{2MT} T \|v\|_\infty \leq \bar{\delta}$ so that, for $\gamma < (c_0c_3)/(4c_2)$ and $\|y\|_\infty \leq \epsilon_1$,

$$\int_0^T R[t] dt \geq \frac{c_0c_3}{4} \|v\|_1^2 . \quad (19)$$

follows. Notice that due to the state equation we have

$$\|y\|_\infty \leq c(A, B) \|v\|_1 ,$$

and consequently (18) holds for $c_s \leq c_0c_3/(4c^2(A, B))$. $\square$

Remark: Notice that in the case of a compact control set as in the model problem $(\mathbf{P}_S)$ the definitions of strong and of bounded-strong local optimality coincide.

As a conclusion from the last two Lemmas we easily obtain:

Theorem 3 *Let the Assumptions 1, 2 be satisfied for the solution (x_0, u_0) of problem (P_S) . Then, (x_0, u_0) is a (bounded-)strong local minimizer, and positive constants c, ϵ exist such that*

$$J(x, u) - J(x_0, u_0) \geq c \left(\|x - x_0\|_2^2 + \|u - u_0\|_1^2 \right) \quad (20)$$

for all admissible pairs with $\|x - x_0\|_\infty \leq \epsilon$.

4. Minimizing sequence stabilization

In this final section, it will be shown how the results of Theorems 2 and 3 can be used to obtain certain preliminary convergence results for minimizing sequences of (P) resp. (P_S) . The results are orientated on [6] (Propos. 6).

For convenience, the following assumption is made on the system dynamics:

Assumption 3 *There exists a constant $M > 0$ such that for any piecewise continuous u with $\|u\|_\infty \leq M$ the state boundary value problem (1), (2) has a bounded solution on $[0, T]$. If solutions corresponding to $u_{1,2}$ are denoted by $x_{1,2}$ resp., then $\|x_1 - x_2\|_\infty \leq c_0 \|u_1 - u_2\|_1$ holds true for some constant $c_0 > 0$*

Consider first the situation of section 3 where the LEGENDRE-CLEBSCH condition is fulfilled in the sense of **(C3)**, and where Theorem 2 allows for a local quadratic growth estimation of the objective functional.

LEMMA 3 *Let (x_0, u_0) be a bounded-strong local minimizer for (P) and suppose the Assumptions 1-3 to hold true. Further, assume the estimate (8), i.e.*

$$J(x, u) - J(x_0, u_0) \geq c \|x - x_0\|_2^2$$

to be valid for $\|x - x_0\|_\infty \leq \epsilon$. If (x_k, u_k) is a minimizing sequence with uniformly bounded and piecewise continuous controls u_k , and if for all k , $\|u_k - u_0\|_1$ is sufficiently small, then $x_k \rightarrow x_0$ in L_2 sense.

The proof is similiar to that of Proposition 6 in [6]. Notice, that in practice a minimizing sequence can be obtained e.g. by the EULER discretization approach ([1], also [2]). The error estimates in the cited papers are given in terms of the maximal deviation in the discretization grid points, a condition which for piecewise continuous controls u_0 is sufficient for their L_1 closeness to the solution.

We add an analogous result for the bang-bang situation, although in this case (without an appropriate coercivity assumption like **(C3)**) the convergence of discretization schemes and their benefit for constructing minimizing sequences theoretically is yet an open question:

LEMMA 4 *Let (x_0, u_0) be a bounded-strong local minimizer for $(\mathbf{P}_S)$ and suppose the Assumptions 1-3 to hold true. Further, assume the estimate (20), i.e.*

$$J(x, u) - J(x_0, u_0) \geq c \left(\|x - x_0\|_2^2 + \|u - u_0\|_1^2 \right)$$

to be valid for $\|x - x_0\|_\infty \leq \epsilon$. If (x_k, u_k) is a minimizing sequence with uniformly bounded and piecewise continuous controls u_k , and if for all k , $\|u_k - u_0\|_1$ is sufficiently small, then $x_k \rightarrow x_0$ in L_2 , and moreover, $u_k \rightarrow u_0$ w.r.t. L_1 topology.

References

- [1] Dontchev, A. L., and Hager, W. W. Lipschitzian stability in nonlinear control and optimization, *SIAM J. Control and Optimization*, 31:569-603, 1993.
- [2] Dontchev, A. L., and Hager, W. W. The Euler approximation in state constrained optimal control, *Math. Comp.*, 70(233):173-203, 2001.
- [3] Dontchev, A. L., and Malanowski, K. A characterization of Lipschitzian stability in optimal control, In Ioffe et al. [8], pages 62-76.
- [4] Felgenhauer, U. *Diskretisierung von Steuerungsproblemen unter stabilen Optimalitätsbedingungen*, Habilitationsschrift, Brandenburgische Technische Universität Cottbus, 1999. (122 p., in German)
- [5] Felgenhauer, U. On smoothness properties and approximability of optimal control functions. In D. Klatte, J. Rückmann, and D. Ward, editors, *Optimization with Data Perturbation II*, vol. 101 of *Ann. Oper. Res.* Baltzer Sc. Publ., The Netherlands, vol. 101:23-42, 2001.
- [6] Felgenhauer, U. Structural properties and approximation of optimal controls, *Nonlinear Analysis*, (TMA), 47(3):1869-1880, 2001.
- [7] Felgenhauer, U. Weak and strong optimality conditions for constrained control problems with discontinuous control, *J. Optim. Theor. Appl.*, 110(2):361-387, 2001.
- [8] A. Ioffe, S. Reich, and I. Shafrir, editors. *Calculus of variations and optimal control. Haifa 1998*, vol. 411 of *Chapman & Hall/CRC Res. Notes Math.*, Boca Raton, FL, 2000. Chapman & Hall/CRC.
- [9] Klötzler, R. On a general conception of duality in optimal control, In: vol. 703 of *Lect. Notes Math.*, 189-196. Springer-Verlag, New York, Heidelberg, Berlin, 1979.
- [10] Malanowski, K. Two-norm approach in stability and sensitivity analysis of optimization and optimal control problems, *Adv. in Math. Sc. and Applic.*, 2:397-443, 1993.

- [11] Malanowski, K. Stability and sensitivity analysis of solutions to infinite-dimensional optimization problems, In J. Henry nad J.-P. Yvon, editors, *Proc. 16th IFIP-TC7 Conference on System Modelling and Optimization*, vol. 197 of *Lect. Notes in Control and Inf. Sci.*, 109-127, London, 1994. Springer-Verlag.
- [12] Malanowski, K. Stability and sensitivity analysis of solutions to nonlinear optimal control problems, *Appl. Math. and Optim.*, 32:111-141, 1995.
- [13] Malanowski, K. Stability analysis of solutions to parametric optimal control problems, In J. Guddat, H. T. Jongen, F.Nožička, G. Still, and F.Twilt, editors, *Proc. IV. Conference on "Parametric Optimization and Related Topics"* Enschede 1995, *Ser. Approximation and Optimization*, 227-244, Frankfurt, 1996. Peter Lang Publ. House.
- [14] Malanowski, K., Büskens, C., and Maurer, H. Convergence of approximations to nonlinear control problems, In A. V. Fiacco, editor, *Mathematical Programming with Data Perturbation*, vol. 195 of *Lect. Notes Pure Appl. Mathem.*, 253-284. Marcel Dekker, Inc., New York, 1997.
- [15] Malanowski, K., and Maurer, H. Sensitivity analysis for parametric optimal control problems with control-state constraints, *Comput. Optim. Appl.*, 5:253-283, 1996.
- [16] Malanowski, K., and Maurer, H. Sensitivity analysis for state constrained optimal control problems, *Discrete Contin. Dynam. Systems*, 4:241 - 272, 1998.
- [17] Maurer, H., and Pickenhain, S. Second order sufficient conditions for optimal control problems with mixed control-state constraints, *J. Optim. Theor. Appl.*, 86:649-667, 1995.
- [18] Milyutin, A. A., and Osmolovskii, N. P. *Calculus of variations and optimal control*, Amer. Mathem. Soc., Providence, Rhode Island, 1998.
- [19] Osmolovskii, N. P. Quadratic conditions for nonsingular extremals in optimal control (A theoretical treatment). *Russian J. of Mathem. Physics*, 2:487-512, 1995.
- [20] Osmolovskii, N. P. Second-order conditions for broken extremals. In Ioffe et al. [8], pages 198-216.
- [21] Pickenhain, S. Sufficiency conditions for weak local minima in multidimensional optimal control problems with mixed control-state restrictions, *Zeitschr. f. Analysis u. Anwend. (ZAA)*, 11:559-568, 1992.