RICCATI EQUATION OF STOCHASTIC CONTROL AND STOCHASTIC UNIFORM OBSERVABILITY IN INFINITE DIMENSIONS

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Abstract We establish that under stabilizability and observability conditions the Riccati equation arising in the stochastic quadratic control problem has a unique uniformly positive bounded solution.

Keywords: Riccati equation, stochastic uniform observability, stabilizability, detectability, uniform controllability

Introduction

G.Da Prato and I. Ichikawa proved in [3] that under stabilizability and detectability conditions the Riccati equation (3.1) arising in the stochastic quadratic control problem has a unique nonnegative bounded solution. We replace the detectability condition with the uniform observability property and we obtain the same conclusion and, moreover, we show that this solution is uniformly positive. So, we generalize the result obtained by T.Morozan in [7] for finite dimensional case. We also prove that our result is distinct from the one of G.Da Prato and I. Ichikawa.

1. Stabilizability, detectability, uniform observability and uniform controllability for linear differential stochastic equations

Let H, U, V be separable real Hilbert spaces. $L(H, V)$ is the Banach space of all bounded linear operators from H into V (if $H = V$ we put $L(H, V) = L(H)$). We denote by $\mathcal{H}$ the subspace of $L(H)$ formed by

The original version of this chapter was revised: The copyright line was incorrect. This has been corrected. The Erratum to this chapter is available at DOI: [10.1007/978-0-387-35690-7_44](https://doi.org/10.1007/978-0-387-35690-7_44)

V. Barbu et al. (eds.), *Analysis and Optimization of Differential Systems*

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all self-adjoint operators. We write $\langle \cdot, \cdot \rangle$ for the inner product and $\|\cdot\|$ for norms of elements and operators. The operator $S \in L(H)$ is called nonnegative ($S \geq 0$) if S is self-adjoint and $\langle Sx, x \rangle \geq 0$ for all $x \in H$. We set $L^+(H) = \{S \in L(H), S \geq 0\}$. For each interval $J \subset \mathbf{R}_+$, we denote by $C_s(J, L(H))$ the space of all strongly continuous mappings $G(t) : J \subset \mathbf{R}_+ \rightarrow L(H)$ and by $C_b(J, L(H))$ the subspace of $C_s(J, L(H))$, which consist of all mappings $G(t)$ such that $\sup_{t \in J} \|G(t)\| < \infty$. If E is a Banach space we also denote by $C(J, E)$ the space of all continuous mappings $G(t) : J \subset \mathbf{R}_+ \rightarrow E$. We need the following assumption:

- P_1 : a) $A(t), t \in [0, \infty)$ is a closed linear operator on H with constant domain D dense in H .
 b) there exist $M > 0$, $\eta \in (\frac{1}{2}\pi, \pi)$ and $\delta \in (-\infty, 0)$ such that $S_{\delta, \eta} = \{\lambda \in C; |\arg(\lambda - \delta)| < \eta\} \subset \rho(A(t))$, for all $t \geq 0$ and $\|R(\lambda, A(t))\| \leq \frac{M}{|\lambda - \delta|}$ for all $\lambda \in S_{\delta, \eta}$.
 c) there exist numbers $\alpha \in (0, 1)$ and $\tilde{N} > 0$ such that $\|A(t)A^{-1}(s) - I\| \leq \tilde{N}|t - s|^\alpha$, $t \geq s \geq 0$, where we denote by $\rho(A)$, $R(\lambda, A)$ the resolvent set of A and respectively the resolvent of A .

It is known (see [4]) that if P_1 holds, then the family $A(t), t \geq 0$ generates an evolution operator $U(t, s)$. For any $n \in N$ we have $n \in \rho(A(t))$. The operators $A_n(t) = n^2 R(n, A(t)) - nI$ are called the Yosida approximations of $A(t)$. If we denote by $U_n(t, s)$ the evolution operator relative to $A_n(t)$ for each $x \in H$ one has $\lim_{n \rightarrow \infty} U_n(t, s)x = U(t, s)x$ uniformly on any bounded subset of $\{(t, s); t \geq s \geq 0\}$.

Let $(\Omega, F, F_t, t \in [0, \infty), P)$ be a stochastic basis. We consider the stochastic equation

$$dy(t) = A(t)y(t)dt + \sum_{i=1}^m G_i(t)y(t)dw_i(t), \quad y(s) = x \in H,$$

denoted by $\{A; G_i\}$. The family $A(t)$ satisfies the hypothesis P_1 , $G_i \in C_s([0, \infty), L(H))$, $i = 1, \dots, m$ and w_i 's are independent real Wiener processes relative to F_t .

It is known (see [3]) that $\{A; G_i\}$ has a unique mild solution in the space $C([s, T], L^2(\Omega; H))$ that is adapted to F_t ; namely the solution of

$$y(t) = U(t, s)x + \sum_{i=1}^m \int_s^t U(t, r)G_i(r)y(r)dw_i(r). \quad (1.1)$$

Let $y(t, s; x)$ be the mild solution of $\{A; G_i\}$. We have the following definition (see [5] for the autonomous case):

Definition 1 The equation $\{A; G_i\}$ is uniformly exponentially stable if there exist constants $M \geq 1, \omega > 0$ such that $E\|y(t, s; x)\|^2 \leq M e^{-\omega(t-s)} \|x\|^2$ for all $t \geq s \geq 0$ and $x \in H$.

If $C \in C_s([0, \infty), L(H, V))$, we consider the system $\{A, G_i; C\}$ formed by equation $\{A; G_i\}$ and the observation relation $z(t) = C(t)y(t, s, x)$.

Definition 2 [7] The system $\{A, C; G_i\}$ is uniformly observable if there exist $\tau > 0$ and $\gamma > 0$ such that $E \int_s^{s+\tau} \|C(t)y(t, s; x)\|^2 dt \geq \gamma \|x\|^2$ for all $s \in \mathbf{R}_+$ and $x \in H$.

Definition 3 [3] Let $D \in C_b([0, \infty), L(H))$. The system $\{A, D; G_i\}$ is detectable if there exists $L \in C_b([0, \infty), L(H))$ such that $\{A + LD; G_i\}$ is uniformly exponentially stable.

If $B \in C_b([0, \infty), L(U, H))$ and $u \in L^2(\mathbf{R}_+, U)$, we associate to $\{A; G_i\}$ the following stochastic linear control equation, denoted by $\{A, B; G_i\}$:

$$dy(t) = A(t)y(t)dt + B(t)u(t)dt + \sum_{i=1}^m G_i(t)y(t)dw_i(t), \quad y(s) = x.$$

Definition 4 [3] The equation $\{A, B; G_i\}$ is stabilizable if there exists $F \in C_b([0, \infty), L(H, U))$ such that $\{A + BF; G_i\}$ is uniformly exponentially stable.

In the deterministic case it is known (see [6] for the autonomous case) that uniform observability implies detectability. We will prove in the following section that this assertion is not true in the stochastic case.

2. A counter example for the implication "uniform controllability implies stabilizability"

Assume that $H = \mathbf{R}^n, V = \mathbf{R}^p$ and $U = \mathbf{R}^d$ (where $\mathbf{R}^n$ is the real n -dimensional space) and functions A, B, C and G_i are defined on the whole real axis $\mathbf{R}$ and are assumed to be continuous and bounded. By $Y(t, s), t \geq s$ we denote the fundamental random matrix solution associated with the equation $\{A; G_i\}$ (see [8]).

Definition 5 The equation $\{A, B; G_i\}$ is uniformly controllable if there exist $\tau > 0$ and $\gamma > 0$ such that $E \int_{s-\tau}^s Y(s, t)B(t)B^*(t)Y^*(s, t)dt \geq \gamma I$ for all $s \in \mathbf{R}$, where I is the identity operator on H .

Remark 1 If $G_i = 0$ and $Y(s, t)$ is replaced with the evolution operator $U(s, t)$ associated to family $A(t), t \in \mathbf{R}$ we obtain the definition for the uniform controllability of the deterministic equation $\{A, B\}$.

Similarly, the Definition 2 can be extended for all $s \in \mathbf{R}$. So, by corollaries C.1 and C.2 of [8] we deduce that uniform controllability implies (or does not imply) stabilizability if and only if uniform observability implies (or does not imply) detectability. Suppose all the operators are constants, $m=1$ and $G_1=G$. In the time invariant case uniform controllability is called controllability. From Proposition 4.1 in [3], it follows:

Remark 2 If $\{A, B; G\}$ is stabilizable, then there exists a solution in $L^+(H)$ of the Riccati equation

$$A^*K + KA + G^*KG + I - KBB^*K = 0. \quad (2.1)$$

Lemma 1 ¹ (see [9]) *If the deterministic equation $\{A, B\}$ is controllable (see the Remark 1), then the stochastic equation $\{A, B; G\}$ is controllable.*

Proof. Let us consider the operators $\widetilde{M}, \widetilde{G}: \mathcal{H} \rightarrow \mathcal{H}$, $\widetilde{M}(P) = AP + PA^*$, $\widetilde{G}(K) = GKG^*$. From proposition P.1 in [8], we see that $\{A, B; G\}$ (resp. $\{A, B\}$) is controllable, if and only if there exist $\tau, \gamma > 0$ such that $\int_{s-\tau}^s e^{(s-p)(\widetilde{M}+\widetilde{G})} BB^* dp \geq \gamma I$ (resp. $\int_{s-\tau}^s e^{(s-p)\widetilde{M}} BB^* dp \geq \gamma I$) for all $s \in \mathbf{R}$. If we consider the Cauchy problem

$$\frac{dR(t)}{dt} = (\widetilde{M} + \widetilde{G})(R(t)), R(s) = BB^*, t \geq s \geq 0$$

we have (by variation constants formula)

$$R(t) = e^{(t-s)\widetilde{M}}(BB^*) + \int_s^t e^{(t-r)\widetilde{M}} \widetilde{G}(R(r)) dr.$$

On the other hand $R(t) = e^{(t-s)(\widetilde{M}+\widetilde{G})}(BB^*) \geq 0$.

Since $\widetilde{G}(K) \geq 0$ and $e^{(t-s)\widetilde{M}}(K) \geq 0$ for all $K \in L^+(H)$, we deduce $e^{(t-s)(\widetilde{M}+\widetilde{G})}(BB^*) \geq e^{(t-s)\widetilde{M}}(BB^*)$ for all $t \geq s \geq 0$. From the hypothesis and the last inequality the conclusion follows. ■

Counter-example. (see[9]) Let us consider the stochastic equation $\{A, B; G\}$, where $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $G = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Since $\text{rank}(B, AB) = 2$, $\{A, B\}$ is controllable and, from the previous lemma, we deduce that $\{A, B; G\}$ is controllable. We will prove that $\{A, B; G\}$

¹This result was proved in [9] in the hypothesis that the operators A and G commute, but prof.T.Morozan noticed that this assumption can be removed.

is not a stabilizable equation. Assume by contradiction that $\{A, B; G\}$ is stabilizable. Then, from Remark 2, follows that there exists a solution $K = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$ in $L^+(H)$ of the Riccati equation (2.1), which satisfies the conditions

$$x_1 x_3 \geq x_2^2, x_1 \geq 0. \quad (2.2)$$

For this solution, equation (2.1) is equivalent with the following system:

$$\begin{aligned} (x_1 + x_2)^2 &= 3x_1 + 1, (x_1 + x_2)(x_2 + x_3) = 3x_2, \\ (x_2 + x_3)^2 &= 4x_3 + 1. \end{aligned} \quad (2.3)$$

In order to solve (2.3) the following situations arise : a) if $x_1 + x_2 = 0$ (respectively $x_2 + x_3 = 0$), then we have $x_1 = -1/3$ (respectively $x_3 = -1/4$) and the condition (2.2) does not hold;

b) if $x_1 + x_2 \neq 0$ and $x_2 + x_3 \neq 0$, we obtain $\frac{x_1+x_2}{x_2+x_3} = \frac{3x_1+1}{3x_2} = \frac{3x_2}{4x_3+1}$. Hence it follows successively $9x_2^2 = 12x_1x_3 + 3x_1 + 4x_3 + 1 > 12x_1x_3$ and $9/12x_2^2 > x_1x_3$. From (2.2) we deduce $9/12x_2^2 > x_2^2$ and we get a contradiction. Thus, we proved that there exists a stochastic system, which is controllable and is not stabilizable. Consequently, an uniform observable system it not necessarily a detectable one.

3. Bounded solutions of Riccati equation of stochastic control

In this section assume that P_1 holds, $B \in C_b([0, \infty), L(U, H)), B^* \in C_b([0, \infty), L(H, U)), C \in C_b([0, \infty), L(H, V)), C^*C, G_i \in C_b([0, \infty), L(H)), K(t) \in C_b([0, \infty), L^+(U))$ and there exists $\delta_0 > 0$ such that $K(t) \geq \delta_0 I$ for all $t \in [0, \infty)$. Consider the Riccati equation

$$\begin{aligned} P'(s) + A^*(s)P(s) + P(s)A(s) + \sum_{i=1}^m G_i^*(s)P(s)G_i(s) + \\ + C^*(r)C(r) - P(s)B(s)(K(s))^{-1}B^*(s)P(s) = 0 \end{aligned} \quad (3.1)$$

We say that P is a mild solution on an interval J of (3.1)(see [3]), if $P \in C_s(J, L^+(H))$ and if it satisfies

$$\begin{aligned} P(s)x = U^*(t, s)P(t)U(t, s)x + \sum_{i=1}^m \int_s^t U^*(r, s)[G_i^*(r)P(r)G_i(r) + \\ + C^*(r)C(r) - P(r)B(r)(K(r))^{-1}B^*(r)P(r)]U(r, s)xd r \end{aligned} \quad (3.2)$$

for all $s \leq t, s, t \in J$. Moreover, if P is a mild solution on $\mathbf{R}_+$ of (3.1) and $\sup_{s \in \mathbf{R}_+} \|P(s)\| < \infty$, then P is said to be a *bounded solution*. If

$$\begin{aligned} P'_n(s) + A_n^*(s)P_n(s) + P_n(s)A_n(s) + \sum_{i=1}^m G_i^*(s)P_n(s)G_i(s) + \\ + C^*(r)C(r) - P_n(s)B(s)(K(s))^{-1}B^*(s)P_n(s) = 0 \end{aligned} \quad (3.3)$$

is the approximating equation of (3.1), then we have the following lemma:

Lemma 2 [3] *Assume P_1 holds. Let $0 < T < \infty$ and let $R \in L^+(H)$. Then there exists a unique mild (resp. classical) solution P (resp. P_n) of (3.1) (resp. (3.3)) on $[0, T]$ such that $P(T) = R$ (resp. $P_n(T) = R$) and for each $x \in H$, $P_n(s)x \rightarrow P(s)x$ uniformly on $[0, T]$.*

Assume that (3.1) has a bounded solution $P(s)$ and consider $S(s) = -(K(s))^{-1}B^*(s)P(s)$, $s \geq 0$. We denote $L^s(H) = L^2(\Omega, F_s, P, H)$ and if $\xi \in L^s(H)$ we consider the equation

$$dz(t) = [A(t) + B(t)S(t)]z(t)dt + \sum_{i=1}^m G_i(t)z(t)dw_i(t), z(s) = \xi \quad (3.4)$$

We associate to (3.4) the integral equation:

$$\begin{aligned} z(t) = & U(t, s)\xi + \int_s^t U(t, r)B(r)S(r)z(r)dr + \\ & + \sum_{i=1}^m \int_s^t U(t, r)G_i(r)z(r)dw_i(r) \end{aligned} \quad (3.5)$$

If $A_n(t)$, $n \in N$ are the Yosida approximations of $A(t)$, then we have the following approximating system of (3.4)

$$\begin{aligned} dz_n(t) = & [A_n(t) + B(t)S(t)]z_n(t)dt + \sum_{i=1}^m G_i(t)z_n(t)dw_i(t), \\ z_n(s) = & \xi \end{aligned} \quad (3.6)$$

Lemma 3 *Let $0 \leq s \leq T$. If $P(t), t \in \mathbf{R}_+$ is a bounded solution of (3.1), then (3.5) has a unique solution denoted by $z(t, s; x)$, which belongs to $C([s, T], L^2(\Omega; H))$ and is adapted to F_t and the system (3.6) has a unique classical solution z_n . Moreover, $z_n \rightarrow z$ in mean square uniformly on $[s, T]$.*

The unique solution of (3.5) which belongs to $C([s, T], L^2(\Omega; H))$ is called the mild solution of (3.4).

Proof. From Theorem 2.3.1 in [1] it follows that (3.4) and respectively (3.6) have unique mild solutions in $C([s, T], L^2(\Omega; H))$. It is a simple exercise to verify that the mild solution of (3.6) coincides with the strong (classical) solution. We only have to prove that $z_n \rightarrow z$ in mean square uniformly on $[s, T]$, $s \geq 0$. Since $U_n(t, r)x - U(t, r)x \xrightarrow{n \rightarrow \infty} 0$ for every $x \in H$ uniformly on $\{(t, r), T \geq t \geq r \geq 0\}$, we deduce, from the uniform boundedness theorem, that there exists $\widetilde{M}_T > 0$ such as $\|U_n(t, r) - U(t, r)\| \leq \widetilde{M}_T$ for all $0 \leq r \leq t \leq T$ and $n \in N$. If $M_T = \sup_{0 \leq r \leq t \leq T} \|U(t, r)\|$, $\widetilde{G}_i = \sup_{r \in [0, T]} \|G_i(r)\|$, $i = 1, \dots, m$ and $\widetilde{D} = \sup_{r \in [0, T]} \|B(r)S(r)\|$, then it is not very difficult to see that we have

$$\begin{aligned}
E \|z_n(t) - z(t)\|^2 &\leq (2m+3)\{\|U_n(t,s)x - U(t,s)x\|^2 + \\
&+ \widehat{N} \int_s^t E \|z_n(r) - z(r)\|^2 dr + \int_s^t E \|[U_n(t,r) - U(t,r)]D(r)z(r)\|^2 dr + \\
&+ \sum_{i=1}^m \int_s^t E \|[U_n(t,r) - U(t,r)]G_i(r)z(r)\|^2 dr\},
\end{aligned} \quad (3.7)$$

where $\widehat{N}$ is a constant depending on $\widetilde{M}_T, M_T, \widetilde{D}$ and $\widetilde{G}_i, i = 1, \dots, m$. Denote the last two integrals of the previous inequality by $I_{1,n}$ and $I_{2,n}$, respectively. It is easy to see that if we prove that $I_{1,n} \xrightarrow{n \rightarrow \infty} 0$ (resp. $I_{2,n} \xrightarrow{n \rightarrow \infty} 0$) uniformly with respect to t , then for every $\varepsilon > 0$ there exists $n_\varepsilon \in N$ such as $\|U_n(t,s)x - U(t,s)x\|^2 + I_{1,n} + I_{2,n} < \varepsilon$ for all $n \in N, n \geq n_\varepsilon$. Using this relation in (3.7) and Gronwall's lemma it follows the conclusion. We prove only the statement for $I_{1,n}$, because the proof for $I_{2,n}$ goes on similarly. Since $\phi : [s, T] \rightarrow L^2(\Omega, H)$ $\phi(r) = D(r)z(r)$ is continuous, then $\phi([s, T]) = \Delta \subset L^2(\Omega, H)$ is compact. Let fix $r \in [s, T]$ and $\alpha_n(r) = \sup_{(t, \xi) \in [r, T] \times \Delta} E \|[U_n(t, r) - U(t, r)]\xi\|^2$. It is easy to see that

the map $(t, \xi) \xrightarrow{\Psi} E \|[U_n(t, r) - U(t, r)]\xi\|^2$ is continuous on $[r, T] \times \Delta$ for every $n \in N$. Thus there exists $(t_{n,r}, \xi_{n,r}) \in [r, T] \times \Delta$ such that $\alpha_n(r) = \Psi(t_{n,r}, \xi_{n,r})$. Since r is fixed, we write, by convenience, (t_n, ξ_n) instead of $(t_{n,r}, \xi_{n,r})$ and so, $\alpha_n(r) = E \|[U_n(t_n, r) - U(t_n, r)]\xi_n\|^2$. We will prove that $\alpha_n(r) \xrightarrow{n \rightarrow \infty} 0$. Assume by contradiction that $\alpha_n(r) \not\xrightarrow{n \rightarrow \infty} 0$. Let $\delta > 0$. Then there exists a subsequence of $\alpha_n(r)$ such as $\alpha_n(r) \geq \delta$. Since $\{t_n\}_{n \in N}, \{\xi_n\}_{n \in N}$ belongs to compact sets, they have convergent subsequences. Consider a subsequence of $\{\alpha_n(r)\}_{n \in N}$ (still denoted $\alpha_n(r)$) such as the corresponding $\{t_n\}$ and $\{\xi_n\}$ converge. We have $\lim_{n \rightarrow \infty} E \|[U_n(t_n, r) - U(t_n, r)]\xi_n\|^2 \leq 5 \lim_{n \rightarrow \infty} \{E \|U_n(t_n, r)(\xi_n - \xi)\|^2 + E \|[U_n(t_n, r) - U(t_n, r)]\xi\|^2 + 2E \|[U(t_n, r) - U(t, r)]\xi\|^2 + E \|U(t_n, r)(\xi_n - \xi)\|^2\} = 0$.

We obtain, $\lim_{n \rightarrow \infty} \alpha_n(r) = 0 \geq \delta$ and we deny the hypothesis; hence $\lim_{n \rightarrow \infty} \alpha_n(r) = 0$. Consider the map $\Phi_n(t, r, \xi) : [s, T] \times [s, T] \times \Delta \rightarrow \mathbf{R}_+$,

$$\Phi_n(t, r, \xi) = E \|[U_n(t, r) - U(t, r)]\xi\|^2 \chi_{\{(t, r), s \leq r \leq t \leq T\}}(t, r).$$

Since $(t, \xi) \rightarrow \Phi_n(t, r, \xi)$ (resp. $r \rightarrow \Phi_n(t, r, \xi)$) is continuous on $[s, T] \times \Delta$ (resp. on $[s, T]$), $\sup_{(t, \xi) \in [s, T] \times \Delta} \Phi_n(t, r, \xi) \leq \alpha_n(r)$ and $[s, T] \times \Delta$ is separable, it follows that $r \rightarrow \sup_{(t, \xi) \in [s, T] \times \Delta} \Phi_n(t, r, \xi)$ is Borel measurable. Then

$$I_{1,n} = \int_s^T \Phi_n(t, r, \xi) dr \leq \int_s^T \sup_{(t, \xi) \in [s, T] \times \Delta} \Phi_n(t, r, \xi) dr \xrightarrow{n \rightarrow \infty} 0 \text{ by bounded}$$

convergence theorem ($\Phi_n(t, r, \xi) \leq \widetilde{M}_T \sup_{\xi \in \Delta} E \|\xi\|^2 < \infty$). $\blacksquare$

We introduce the following hypothesis:

P_2 : The evolution operator $U(t, s)$ has an exponentially growth that is, there exist M_0 and ω positive constants such that $\|U(t, s)\| \leq M_0 e^{\omega(t-s)}$.

Taking mean square in (3.5) and using P_2 and Gronwall's inequality we obtain:

Lemma 4 Assume P_2 holds. If $z(t, s; x)$ is the mild solution of (3.4), then there exists a continuous function $\phi : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ such that

$$E \|z(t, s; x)\|^2 \leq \phi(t-s) \|x\|^2 \text{ for all } x \in H \text{ and } 0 \leq s \leq t.$$

Definition 6 A self-adjoint solution of (3.1) is called stabilizing for $\{A; G_i\}$ if $\{A + BS; G_i\}$ is uniformly exponentially stable, where $S(t) = -(K(t))^{-1}B^*(t)P(t)$.

Theorem 1 Assume that $\{A, G_i; C\}$ is uniformly observable and P_2 holds. If $P(t)$ is a nonnegative bounded solution of (3.1) then

a) there exists $\delta > 0$ such that $P(t) \geq \delta I$ for all $t \in [0, \infty)$ (P is uniformly positive on $\mathbf{R}_+$);

b) P is a stabilizing solution (for $\{A; G_i\}$).

Proof. The main idea is the one in [7]. Let $P(t)$ be a nonnegative bounded solution of the Riccati equation (3.1) and γ and τ as in Definition 2. If $z(t, s; x)$ is the mild solution of (3.4) then we introduce the linear nonnegative operator $Q(s)$, which satisfies the relation

$$\begin{aligned} \langle Q(s)x, x \rangle &= E \int_s^{s+\tau} \|C(t)z(t, s; x)\|^2 + \\ &+ \langle S^*(t)K(t)S(t)z(t, s; x), z(t, s; x) \rangle dt. \end{aligned} \quad (3.8)$$

The above relation defines a unique linear nonnegative operator $Q(s)$.

We will prove that $\mathcal{I} = \inf \{ \langle Q(s)x, x \rangle, s \geq 0, x \in H, \|x\| = 1 \} > 0$.

Assume by contradiction that $\mathcal{I} = 0$. Then, for every $\varepsilon > 0$ there exist $s_\varepsilon \in [0, \infty)$, $\hat{x}_\varepsilon \in H$, $\|\hat{x}_\varepsilon\| = 1$, such that $\langle Q(s_\varepsilon)\hat{x}_\varepsilon, \hat{x}_\varepsilon \rangle < \varepsilon$.

Let $z_\varepsilon(t) = z(t, s_\varepsilon; \hat{x}_\varepsilon)$, $u_\varepsilon(t) = S(t)z_\varepsilon(t)$ for all $t \geq s_\varepsilon$. We get

$$\begin{aligned} \varepsilon > \langle Q(s_\varepsilon)\hat{x}_\varepsilon, \hat{x}_\varepsilon \rangle &\geq E \int_{s_\varepsilon}^{s_\varepsilon+\tau} \langle K(t)S(t)z(t, s_\varepsilon; \hat{x}_\varepsilon), S(t)z(t, s_\varepsilon; \hat{x}_\varepsilon) \rangle dt \geq \\ &\geq \delta_0 E \int_{s_\varepsilon}^{s_\varepsilon+\tau} \|u_\varepsilon(t)\|^2 dt. \end{aligned}$$

On the other hand we have

$$\begin{aligned}
\varepsilon &> \langle Q(s_\varepsilon)\widehat{x}_\varepsilon, \widehat{x}_\varepsilon \rangle \geq E \int_{s_\varepsilon}^{s_\varepsilon+\tau} \|C(t)z_\varepsilon(t)\|^2 dt \text{ and} \\
\varepsilon &\geq 1/2E \int_{s_\varepsilon}^{s_\varepsilon+\tau} \|C(t)y(t, s_\varepsilon; \widehat{x}_\varepsilon)\|^2 dt - \\
&\quad - \widetilde{C}^2 \int_{s_\varepsilon}^{s_\varepsilon+\tau} E \|y(t, s_\varepsilon; \widehat{x}_\varepsilon) - z(t, s_\varepsilon; \widehat{x}_\varepsilon)\|^2 dt,
\end{aligned} \tag{3.9}$$

where $y(t, s_\varepsilon; \widehat{x}_\varepsilon)$ is the mild solution of $\{A; G_i\}$ with the initial condition $y(s_\varepsilon) = \widehat{x}_\varepsilon$ and $\widetilde{C} = \sup_{0 \leq r < \infty} \|C(r)\|$. We need an upper estimation for

$$\begin{aligned}
&\int_{s_\varepsilon}^{s_\varepsilon+\tau} E \|y(t, s_\varepsilon; \widehat{x}_\varepsilon) - z(t, s_\varepsilon; \widehat{x}_\varepsilon)\|^2 dt. \text{ From Lemma 3, (1.1) and (3.5) we} \\
&\text{get } z(t, s_\varepsilon; \widehat{x}_\varepsilon) - y(t, s_\varepsilon; \widehat{x}_\varepsilon) = \int_{s_\varepsilon}^t U(t, r)B(r)S(r)z(r, s_\varepsilon; \widehat{x}_\varepsilon)dr + \\
&\quad + \sum_{i=1}^m \int_{s_\varepsilon}^t U(t, r)G_i(r)[z(r, s_\varepsilon; \widehat{x}_\varepsilon) - y(r, s_\varepsilon; \widehat{x}_\varepsilon)]dw_i(r).
\end{aligned}$$

Consider $\widetilde{B} = \sup_{0 \leq r < \infty} \|B(r)\|$. Taking mean square, using P_2 and Fubini's theorem it follows $E \|y(t, s_\varepsilon; \widehat{x}_\varepsilon) - z(t, s_\varepsilon; \widehat{x}_\varepsilon)\|^2 \leq (m+1)M_0^2 e^{2\omega\tau} [\widetilde{B}^2 \varepsilon / \delta_0 + \sum_{i=1}^m \widetilde{G}_i^2 \int_{s_\varepsilon}^t E \|z(r, s_\varepsilon; \widehat{x}_\varepsilon) - y(r, s_\varepsilon; \widehat{x}_\varepsilon)\|^2 dr]$ for all $t \in [s_\varepsilon, s_\varepsilon + \tau]$, because $\varepsilon \geq \delta_0 E \int_{s_\varepsilon}^{s_\varepsilon+\tau} \|u_\varepsilon(t)\|^2 dt$. By Gronwall's inequality and since

$$\begin{aligned}
&t \in [s_\varepsilon, s_\varepsilon + \tau] \text{ we obtain } E \|y(t, s_\varepsilon; \widehat{x}_\varepsilon) - z(t, s_\varepsilon; \widehat{x}_\varepsilon)\|^2 \leq \varepsilon r_1, \text{ where} \\
&r_1 = (m+1)(M_0^2 e^{2\omega\tau} \widetilde{B}^2 1 / \delta_0) \exp(\tau M_0^2 e^{2\omega\tau} (m+1) \sum_{i=1}^m \widetilde{G}_i^2).
\end{aligned}$$

Since $\{A, G_i; C\}$ is uniformly observable, (3.9) becomes $\varepsilon \geq (1/2)\gamma - r_1 \widetilde{C}^2 \varepsilon$. We get a contradiction since $\varepsilon > 0$ is arbitrary. Hence there exists $r_2 > 0$ such that

$$\langle Q(t)x, x \rangle \geq r_2 \|x\|^2 \tag{3.10}$$

for all $t \in \mathbf{R}_+$ and $x \in H$.

If $P_n(t) = P_n(s+\tau, t, P(s+\tau))$ is the classical solution of (3.3) with the initial condition $P_n(s+\tau) = P(s+\tau)$ and $P(s)$ is the bounded solution

of (3.1), then we apply the Ito's formula for the classical solution of (3.6) and the function $v_n(t, x) = \langle P_n(t)x, x \rangle$. Taking expectations, we get

$$E \langle P_n(s + \tau)z_n(s + \tau, s; x), z_n(s + \tau, s; x) \rangle - \langle P_n(s)x, x \rangle = \\ = -E \int_s^{s+\tau} \|C(t)z_n(t, s; x)\|^2 + \langle K(t)S_n(t)z_n(t, s; x), S_n(t)z_n(t, s; x) \rangle dt,$$

where $S_n(t) = K(t)B^*(t)P_n(t)$. As $n \rightarrow \infty$ in the last equality and using Lemma 2 and 3 we get $E \langle P(s + \tau)z(s + \tau, s; x), z(s + \tau, s; x) \rangle - \langle P(s)x, x \rangle = -E \int_s^{s+\tau} \|C(t)z(t, s; x)\|^2 + \langle K(t)S(t)z(t, s; x), S(t)z(t, s; x) \rangle dt$.

Therefore we get $\langle P(s)x, x \rangle - E \langle P(s + \tau)z(s + \tau, s; x), z(s + \tau, s; x) \rangle = \langle Q(s)x, x \rangle$ for all $s \geq 0$ and $x \in H$. From (3.10) and since $P(\cdot)$ is bounded on $\mathbf{R}_+$ and nonnegative, we deduce that there exists $r_3 > 0$ such as

$$r_3 \|x\|^2 \geq \langle P(s)x, x \rangle \geq \langle Q(s)x, x \rangle \geq r_2 \|x\|^2 \quad (3.11)$$

for all $s \geq 0$ and $x \in H$. If we take $\delta = r_2$ it follows a).

Let us prove b). From (3.11) we get $\frac{r_2}{r_3} \langle P(s)x, x \rangle \leq r_2 \|x\|^2 \leq$

$$\leq \langle P(s)x, x \rangle - E \langle P(s + \tau)z(s + \tau, s; x), z(s + \tau, s; x) \rangle$$

and $(1 - \frac{r_2}{r_3}) \langle P(s)x, x \rangle \geq E \langle P(s + \tau)z(s + \tau, s; x), z(s + \tau, s; x) \rangle$.

Let $q = (1 - \frac{r_2}{r_3}) < 1$, $v(s, x) = \langle P(s)x, x \rangle$ and $g(s, x) = Ev(s + \tau, z(s + \tau, s; x))$, $s \in \mathbf{R}_+$, $x \in H$. Then

$$g(s, x) \leq qv(s, x) \quad (3.12)$$

for any $s \in \mathbf{R}_+$, $x \in H$. We will prove that for all $s, p \in \mathbf{R}_+$ and $u \in H$

$$Eg(s, z(s, p, u)) = Ev(s + \tau, z(s + \tau, p; x)). \quad (3.13)$$

Let $\xi \in L^s(H)$ be a simple random variable (it takes on only a finite number of values) then $\xi = \sum_{i=1}^n \chi_{A_i} x_i$, where $\{A_i\}_{i=1, \dots, m}$, $A_i \in F_s$ is a partition of Ω , χ_{A_i} is the characteristic function of A_i and $x_i \in H$. Then $g(s, \xi) = \sum_{i=1}^n \chi_{A_i} g(s, x_i)$. By the linear dependence of $z(s + \tau, s; x)$ to the initial conditions and since $z(s + \tau, s; x)$ is F_s -independent we obtain

$$E \langle P(s + \tau)z(s + \tau, s; \xi), z(s + \tau, s; \xi) \rangle = \\ = E \left\langle P(s + \tau)z(s + \tau, s; \sum_{i=1}^n \chi_{A_i} x_i), z(s + \tau, s; \sum_{i=1}^n \chi_{A_i} x_i) \right\rangle = \\ = E \left(\sum_{i=1}^n \chi_{A_i} \langle P(s + \tau)z(s + \tau, s; x_i), z(s + \tau, s; x_i) \rangle \right) = \\ = \sum_{i=1}^n P(A_i) g(s, x_i) = Eg(s, \xi).$$

Consequently

$$Eg(s, \xi) = E(\langle P(s + \tau)z(s + \tau, s; \xi), z(s + \tau, s; \xi) \rangle). \quad (3.14)$$

Since $z(s, p, u) \in L^s(H)$ for all $s \geq p \geq 0$ and $u \in H$, there exists a sequence $\{\xi_n\}_{n \in \mathbb{N}} \subset L^s(H)$ of simple random variables, which converges to $z(s, p, u)$ and satisfies (3.14).

Thus $g(s, \xi_n) = E(\langle P(s + \tau)z(s + \tau, s; \xi_n), z(s + \tau, s; \xi_n) \rangle)$.

As $n \rightarrow \infty$ in the last equality we get

$$Eg(s, z(s, p, u)) = E \langle P(s + \tau)z(s + \tau, s; z(s, p, u)), z(s + \tau, s; z(s, p, u)) \rangle$$

From the uniqueness of the mild solution of (3.4) we deduce

$$Eg(s, z(s, p, u)) = E(\langle P(s + \tau)z(s + \tau, p, u), z(s + \tau, p, u) \rangle).$$

We obtain (3.13). Then from (3.12) we deduce

$$E[g(s, z(s, p; x))] \leq qEv(s, z(s, p; x)) \text{ or}$$

$$Ev(s + \tau, z(s + \tau, p; x)) \leq qEv(s, z(s, p; x))$$

From the last inequality, Lemma 4 and (3.11) it follows that there exist $\beta \geq 1$ and $\alpha > 0$ ($\alpha = -\frac{\ln q}{\tau}$) such as $E \|z(t, p; x)\|^2 \leq \beta e^{-\alpha(t-p)} \|x\|^2$ for all $t \geq p \geq 0$ and $x \in H$. Hence P is a stabilizing solution. ■

The following results are the infinite dimensional versions of Proposition 5 and Theorem 1 from [7].

Proposition 1 *Under the assumptions of the above theorem the Riccati equation (3.1) has at most one nonnegative bounded solution.*

Proof. Since it is not very difficult to prove (see Corollary 3.2 in [3]) that any bounded and stabilizing solution of (3.1) is maximal (see [3]) in the class of bounded solutions, the conclusion follows from the above theorem. ■

The following theorem is the main result of this section.

Theorem 2 *Assume $\{A, G_i; B\}$ is stabilizable and $\{A, G_i; C\}$ is uniformly observable. Then the Riccati equation (3.1) has a unique nonnegative bounded on $\mathbf{R}_+$ solution $P(t)$, which is a stabilizing solution and there exists $\delta > 0$ such that $P(t) \geq \delta I$ for all $t \in [0, \infty)$.*

Proof. From Theorem 4.1 in [3] it follows that under stabilizability conditions the equation (3.1) has a bounded solution on $\mathbf{R}_+$. From Theorem 1 and the above proposition we deduce the conclusion. ■

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