OPTIMIZATION OF STEADY-STATE FLOW OF INCOMPRESSIBLE FLUIDS *

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Abstract An optimal boundary-control problem for the steady-state buoyancy-driven flow of an incompressible, heat conductive fluid is investigated as far as optimality conditions and an increment formula concerns. The controlled system covers, in particular, the Oberbeck-Boussinesq model. Regularity of the Navier-Stokes system coupled with heat equation and its adjoint system is substantially used.

Keywords: Buoyancy-driven flow, optimality conditions, increment formula.

1. Introduction

Optimal control problem of steady-state incompressible flow coupled with heat equation was already studied in [2,4,8,9,19,20]. The isothermal case (i.e. without heat equation) was studied, e.g., in [1,3,5,10,15,16,17,27] and [22; Section III.11]. Due to generality of the controlled system (1) below, this contribution covers a lot of these results as a special case.

Besides, optimal control of evolutionary Navier-Stokes system was treated in a lot of other works, see e.g. monographs [11,12,22] and references therein.

2. The controlled system

Our a bit academical problem can be interpreted, after necessary simplifications, as a flow of, say, water heated up from outside to prepare a

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morning tee in a completely closed container Ω . Anyhow, one can think, e.g., about transport in a high-pressure vapor reactor [19,20] or flow a melted steel in a mold during a casting process, too.

We consider Ω a bounded smooth domain in $\mathbb{R}^d$, $d \leq 3$. To cover various possibilities, we consider the following fairly general system:

$$(u \cdot \nabla)u - \nu \Delta u + \nabla p = g(1 - \alpha_0 \theta), \quad (1a)$$

$$\operatorname{div} u = 0 \quad (1b)$$

$$u \cdot \nabla \theta - \kappa \Delta \theta = \alpha_1 |\nabla u|^2 + \alpha_2 \theta g \cdot u, \quad (1c)$$

where u is the *velocity* vector field, p the *pressure*, θ *temperature*, and g an external (e.g. gravity) force. The material parameters are: ν the *viscosity*, κ *heat conductivity*, α_0 linearized relative mass density variation with respect to temperature, α_1 reflects *dissipation* effects, while α_2 expresses *adiabatic heat* effects. For a rigorous derivation of a system like (1) we refer to Kagei, Růžička and Thäter [21; System (16)] or Rajagopal, Růžička and Srinivasa [25]: it is shown that the coefficient α_1 depends on Ostrach's dissipation number, while the coefficient α_2 depends also on the Reynolds and the Prandtl numbers. The conventional *Oberbeck-Boussinesq model* uses $\alpha_1 = \alpha_2 = 0$. For derivation of the model with $\alpha_1 > 0$ but $\alpha_2 = 0$ see [24; Chap.I].

The system should be completed by boundary conditions. For simplicity, we will consider no-slip boundary condition for velocity and Newton's boundary condition for temperature, i.e.

$$u = 0, \quad \kappa \frac{\partial \theta}{\partial n} + b\theta = h \quad \text{on } \Gamma, \quad (2)$$

with n being the unit outward normal to the boundary $\Gamma := \partial\Omega$ of Ω .

We call the pair $(u, \theta) \in W_{0,\operatorname{DIV}}^{1,2}(\Omega; \mathbb{R}^d) \times W^{1,2}(\Omega)$ a weak solution to the boundary-value problem (1)–(2) if

$$\begin{aligned} & ((u \cdot \nabla)u, v) + \nu(\nabla u, \nabla v) + \kappa(\nabla \theta, \nabla \tilde{v}) + (u \cdot \nabla \theta, \tilde{v}) \\ & + (b\theta, \tilde{v})_\Gamma - ((1 - \alpha_0 \theta)g, v) - (\alpha_1 |\nabla u|^2 + \alpha_2 \theta g \cdot u, \tilde{v}) \\ & - (h, \tilde{v})_\Gamma = 0 \quad \forall v \in W_{0,\operatorname{DIV}}^{1,2}(\Omega; \mathbb{R}^d), \quad \tilde{v} \in W^{1,2}(\Omega), \end{aligned} \quad (3)$$

where $W_{0,\operatorname{DIV}}^{1,2}(\Omega; \mathbb{R}^d) := \{v \in W_0^{1,2}(\Omega; \mathbb{R}^d); \operatorname{div} v = 0\}$, $(\cdot, \cdot)$ denotes the scalar product in $L^2(\Omega)$ or $L^2(\Omega; \mathbb{R}^d)$ or $L^2(\Omega; \mathbb{R}^{d \times d})$ and $(\cdot, \cdot)_\Gamma$ is the scalar product in $L^2(\Gamma)$. Moreover, in what follows we will denote by $\|\cdot\|_p$ the norm of $L^p(\Omega)$ or $L^p(\Omega; \mathbb{R}^d)$ or $L^p(\Omega; \mathbb{R}^{d \times d})$ and by $\|\cdot\|_{p,\Gamma}$ the norm of $L^p(\Gamma)$. Besides, N_Γ will be the norm of the trace operator $W^{1,2}(\Omega) \rightarrow L^2(\Gamma)$, and N_p and $N_{p,q}$ will denote the norm of the embedding $W_0^{1,2}(\Omega) \subset L^p(\Omega)$ and $L^p(\Omega) \subset L^q(\Omega)$, respectively.

The existence of a weak solution to the whole problem (1)-(2) is not automatic unless $\alpha_1 = \alpha_2 = 0$. Its uniqueness is even more delicate because, due to the quadratic terms on the right-hand side of (1c), we lack any global a-priori estimate if $\alpha_1 \neq 0 \neq \alpha_2$ and therefore we can get the uniqueness only of those solutions whose energy does not exceed certain limits.

For some $h_{\min} < h_{\max}$, we denote the set of admissible h 's in (2) by

$$\mathfrak{H}_{\text{ad}} = \{h \in L^\infty(\Gamma); h_{\min} \leq h(x) \leq h_{\max} \text{ for a.a. } x \in \Gamma\}. \quad (4)$$

Proposition 1 *Let $\varrho > N_2 N_\Gamma c_P^{-1} \max(|h_{\min}|, |h_{\max}|) \sqrt{\text{meas}_{d-1}(\Gamma)}$ and $\alpha_0, h_{\min}, h_{\max} \in \mathbb{R}$ be arbitrary, where $c_P = c_P(\kappa, b_{\min}, \Omega)$ is the constant from the Poincaré inequality $c_P \|\theta\|_{W^{1,2}(\Omega)}^2 \leq \kappa \|\nabla \theta\|_2^2 + b_{\min} \|\theta\|_{2,\Gamma}^2$. Let the assumptions*

$$\alpha_0 N_2 N_4^2 \|g\|_\infty \varrho < \nu, \quad (5a)$$

$$g \in L^\infty(\Omega; \mathbb{R}^d) \text{ has a potential, i.e. } g = \nabla \varphi, \quad (5b)$$

$$\alpha_1 \geq 0, \alpha_2 \geq 0 \text{ sufficiently small,} \quad (5c)$$

$$b_{\max} \geq b(x) \geq b_{\min} > 0 \text{ for a.a. } x \in \Gamma \quad (5d)$$

be satisfied. Then, for each $h \in \mathfrak{H}_{\text{ad}}$, there exists a weak solution (u, θ) to (1)-(2) satisfying $\|\theta\|_2 \leq \varrho$. Moreover, if

$$\nu, \kappa, b_{\min} \text{ are sufficiently large,} \quad (6)$$

this solution is determined uniquely and the mapping $h \mapsto (u, \theta) : L^2(\Omega) \rightarrow W^{1,2}(\Omega; \mathbb{R}^{d+1})$, restricted on $\mathfrak{H}_{\text{ad}}$, is Lipschitz continuous and (weak, norm)-continuous.

Proof. For the existence, we use a Schauder fixed-point argument similarly (but not entirely the same) as in [26]: we take $\vartheta \in L^2(\Omega)$, $\|\vartheta\|_2 \leq \varrho$, find a unique $u \in W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^d)$ solving (in the weak sense) the Navier-Stokes system

$$(u \cdot \nabla)u - \nu \Delta u + \nabla p = g(1 - \alpha_0 \vartheta), \quad \text{div } u = 0, \quad u|_\Gamma = 0, \quad (7)$$

which is possible thanks to the assumption (5a). Testing (3) by $(v, \tilde{v}) := (u, 0)$ (note that the term $(g, v) = (\nabla \varphi, v) = -(\varphi, \text{div } v) = 0$ in (3) vanishes due to (5b)), the basic a-priori estimate

$$\|\nabla u\|_2 \leq \frac{N_2}{\nu} \|g\|_\infty \alpha_0 \varrho =: C_1 \quad (8)$$

is easily obtained. Then we take the unique $\theta \in W^{1,2}(\Omega)$ solving (in the weak sense) the heat equation

$$\begin{aligned} u \cdot \nabla \theta - \kappa \Delta \theta &= f, \quad \kappa \frac{\partial \theta}{\partial n} + b\theta|_{\Gamma} = h, \\ \text{with } f &\equiv f(u, \vartheta) := \alpha_1 |\nabla u|^2 + \alpha_2 \vartheta g \cdot u, \end{aligned} \quad (9)$$

The (nowadays standard) regularity result

$$\|u\|_{\infty} \leq c \|u\|_{W^{2,2}(\Omega; \mathbb{R}^d)} \leq C_2 \equiv C_2(\Omega, \varrho) \quad (10)$$

is known, see e.g. [7] or [14; Chap.VIII, Thm.5.2]. This shows, in particular, that $f \in L^2(\Omega)$ so that the unique weak solution to (9) certainly does exist. This regularity together with the compact embedding $W^{2,2}(\Omega; \mathbb{R}^d) \subset W^{1,6-\varepsilon}(\Omega; \mathbb{R}^d)$ shows that the mapping $\vartheta \mapsto f(u(\vartheta), \vartheta) : L^2(\Omega) \rightarrow L^2(\Omega)$ is (weak,weak)-continuous; the limit passage in the term $\alpha_2 \vartheta g \cdot u$ is easy because, due to (10) and compactness of the embedding $W^{2,2}(\Omega) \subset L^{\infty}(\Omega)$, the mapping $\vartheta \mapsto u : L^2(\Omega) \rightarrow L^{\infty}(\Omega; \mathbb{R}^d)$ is (weak,norm)-continuous. By standard arguments, $f \mapsto \theta : L^2(\Omega) \rightarrow W^{1,2}(\Omega)$ is (weak,norm)-continuous. Altogether, $\vartheta \mapsto \theta : L^2(\Omega) \rightarrow L^2(\Omega)$ is (weak,norm)-continuous. Note that, if $\alpha_1 = \alpha_2 = 0$, then

$$c_P \|\theta\|_2 \leq N_2 N_{\Gamma} \max(|h_{\min}|, |h_{\max}|) \sqrt{\text{meas}_{d-1}(\Gamma)}. \quad (11)$$

If ϱ is taken as assumed and (5c) holds, the mapping $\vartheta \mapsto \theta$ maps the ball $\{\vartheta \in L^2(\Omega); \|\vartheta\|_2 \leq \varrho\}$ into itself and, by Schauder's theorem, it has a fixed point θ . Then $\theta = \vartheta$ together with the corresponding $u = u(\vartheta)$ from (7) solves (1)-(2).

Now, we will show the claimed Lipschitz-continuous dependence on h of this fixed point and, as a by-product, we get its uniqueness as well as its (weak,norm)-continuity. Write the integral identity (3) for two right-hand sides, say h_1 and h_2 . Then (u_1, θ_1) and (u_2, θ_2) will denote (some of) the corresponding solutions. Now we subtract these identities (3) and test it by $v := u_1 - u_2$ and $\tilde{v} = \theta_1 - \theta_2$. Then, abbreviating shortly $u_{12} \equiv u_1 - u_2$, $\theta_{12} \equiv \theta_1 - \theta_2$, and $h_{12} \equiv h_1 - h_2$ and using the identities

$$\begin{aligned} ((u_1 \cdot \nabla)u_1 - (u_2 \cdot \nabla)u_2, u_{12}) &= ((u_1 \cdot \nabla)u_{12}, u_{12}) \\ &+ ((u_{12} \cdot \nabla)u_2, u_{12}) = ((u_{12} \cdot \nabla)u_2, u_{12}) \end{aligned} \quad (12)$$

and

$$\begin{aligned} (u_1 \cdot \nabla \theta_1 - u_2 \cdot \nabla \theta_2, \theta_{12}) &= (u_1 \cdot \nabla \theta_{12}, \theta_{12}) \\ &+ (u_{12} \cdot \nabla \theta_2, \theta_{12}) = (u_{12} \cdot \nabla \theta_2, \theta_{12}) \end{aligned} \quad (13)$$

(which hold thanks to $\operatorname{div} u_1 = 0$ and $u_1|_\Gamma = 0$) and also

$$(|\nabla u_1|^2 - |\nabla u_2|^2, \theta_{12}) = (\nabla(u_1 + u_2), (\nabla u_{12})\theta_{12}) \quad \text{and} \quad (14)$$

$$(\theta_1 g \cdot u_1 - \theta_2 g \cdot u_2, \theta_{12}) = (\theta_1 g \cdot u_{12}, \theta_{12}) + (g \cdot u_2, \theta_{12}^2), \quad (15)$$

we get, by testing (3) by $(v, \tilde{v}) := (u_{12}, \theta_{12})$,

$$\begin{aligned} & \nu \|\nabla u_{12}\|_2^2 + \kappa \|\nabla \theta_{12}\|_2^2 + (b\theta_{12}, \theta_{12})_\Gamma \\ &= (h_{12}, \theta_{12})_\Gamma - ((u_{12} \cdot \nabla)u_2, u_{12}) - (u_{12} \cdot \nabla \theta_2, \theta_{12}) \\ & \quad - \alpha_0(\theta_{12}, g \cdot u_{12}) + \alpha_1(\nabla(u_1 + u_2), (\nabla u_{12})\theta_{12}) \\ & \quad + \alpha_2(\theta_1 g \cdot u_{12}, \theta_{12}) + \alpha_2(g \cdot u_2, \theta_{12}^2) \\ & \leq N_\Gamma \|h_{12}\|_2 \|\theta_{12}\|_{W^{1,2}(\Omega)} + \|\nabla u_2\|_2 \|u_{12}\|_4^2 + \|u_{12}\|_4 \|\nabla \theta_2\|_2 \|\theta_{12}\|_4 \\ & \quad + \alpha_0 \|\theta_{12}\|_2 \|g\|_\infty \|u_{12}\|_2 + \alpha_1 N_{6,4}^2 \|\nabla(u_1 + u_2)\|_6 \|\nabla u_{12}\|_2 \|\theta_{12}\|_6 \\ & \quad + \alpha_2 \|\theta_1\|_6 \|g\|_\infty \|u_{12}\|_6 \|\theta_{12}\|_6 + \alpha_2 N_{6,3}^2 \|g\|_\infty \|u_2\|_\infty \|\theta_{12}\|_2^2. \end{aligned} \quad (16)$$

Using Young and Poincaré inequalities, all right-hand-side terms but $\|h_{12}\|_2^2$ can be absorbed in the left-hand side as (6) is assumed. ■

Remark 2 The above uniqueness proof suggests a modification of the existence proof by showing a Lipschitz continuity of the mapping $\vartheta \mapsto \theta$ and then to use Banach's fixed-point theorem instead of the Schauder one. The requirement of contractivity of $\vartheta \mapsto \theta$ would, however, require (6) quantitatively stronger.

Remark 3 Usually, we are given by ν , κ , and $b_{\min}$. Then the choice of a suitable ϱ needs further analysis and (6) needs, in particular, a very small Reynolds number, cf. [2] for quantitative estimates.

3. The optimal control problem

We will consider the velocity/temperature tracking problem. For the purpose of Proposition 10 below, we consider two different norms for velocity, distinguished by $\xi = 0, 1$, and then the optimal-control problem:

$$(\mathfrak{P}) \left\{ \begin{array}{ll} \text{Minimize} & J(u, \theta, h) := \int_\Omega \frac{\xi}{2} |\nabla u - \nabla u_d|^2 + \frac{\xi - 1}{2} |u - u_d|^2 \\ & + \frac{1}{2} |\theta - \theta_d|^2 dx + \int_\Gamma h^2 dS \quad (\text{cost functional}) \\ \text{subject to} & (u, \theta, h) \in W_{0,\operatorname{DIV}}^{1,2}(\Omega; \mathbb{R}^d) \times W^{1,2}(\Omega) \times L^\infty(\Gamma) \\ & \text{solves (1)–(2) weakly,} \quad (\text{state system}) \\ & \|\theta\|_2 \leq \varrho, \quad (\text{"selection" criterion}) \\ & h_{\min} \leq h(x) \leq h_{\max} \text{ for } x \in \Gamma, \quad (\text{control constraints}) \end{array} \right.$$

where u_d and θ_d are the desired velocity and temperature profiles. Let us emphasize that $(\mathfrak{P})$ is not a linear/quadratic optimization problem because the controlled system (1), containing 4 bilinear/quadratic terms, is obviously nonlinear.

Remark 4 (*The selection criterion.*) The criterion $\|\theta\|_2 \leq \varrho$ uses ϱ from Proposition 1 and is not a state constraint in the usual sense because it does not constrain implicitly the controls but it only wants to avoid a possible nonuniqueness in the response. This selection criterion can be omitted for the Oberbeck-Boussinesq system (i.e. if $\alpha_1 = \alpha_2 = 0$).

Proposition 5 *Let u_d belong to $L^2(\Omega; \mathbb{R}^d)$ (if $\xi = 0$) or to $W^{1,2}(\Omega; \mathbb{R}^d)$ (if $\xi = 1$), $\theta_d \in L^2(\Omega)$, and the assumptions (5)–(6) be satisfied. Then $(\mathfrak{P})$ has a solution.*

Proof. The set $\mathfrak{H}_{ad}$ of admissible controls is compact in weak L^2 -topology in which the state mapping $h \mapsto (u(h), \theta(h))$, $\|\theta\|_2 \leq \varrho$, is continuous and the cost functional J lower-semicontinuous. Hence the minimum of $\Phi(h) = J(u(h), \theta(h), h)$ does exist by Bolzano-Weierstraß' theorem. ■

4. Optimality conditions

Of course, since the controlled system is nonlinear, there may exist (beside the globally optimal control whose existence has been claimed in Proposition 5) also locally optimal controls. Let us consider a fixed locally optimal reference pair $(\bar{u}, \bar{\theta}, \bar{h})$. We begin with the first-order optimality conditions. Formally, they can be found by applying the well-known Lagrange principle, where the state-equations are eliminated by the *Lagrange function*

$$\begin{aligned} L(u, \theta, h, w, \vartheta) &= J(u, \theta, h) - ((u \cdot \nabla)u, w) - \nu(\nabla u, \nabla w) \\ &\quad - \kappa(\nabla \theta, \nabla \vartheta) - (u \cdot \nabla \theta, \vartheta) + (h - b\theta, \vartheta)_\Gamma \\ &\quad + ((1 - \alpha_0 \theta)g, w) + (\alpha_1 |\nabla u|^2 + \alpha_2 \theta g \cdot u, \vartheta), \end{aligned} \quad (17)$$

cf. (3). Obviously, for fixed *multipliers* $w \in W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^d)$ and $\vartheta \in W^{1,2}(\Omega)$, the function $L(\cdot, \cdot, \cdot, w, \vartheta) : W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^d) \times W^{1,2}(\Omega) \times L^2(\Gamma) \rightarrow \mathbb{R}$ is quadratic and continuous, hence it is a C^2 -function, too. According to the Lagrange principle, $(\bar{u}, \bar{\theta}, \bar{h})$ should satisfy the necessary optimality conditions for minimizers of L with respect to $h \in \mathfrak{H}_{ad}$, i.e.

$$[L'_u(\bar{u}, \bar{\theta}, \bar{h}, w, \vartheta)](u) = 0 \quad \forall u \in W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^d), \quad (18a)$$

$$[L'_\theta(\bar{u}, \bar{\theta}, \bar{h}, w, \vartheta)](\theta) = 0 \quad \forall \theta \in W^{1,2}(\Omega), \quad (18b)$$

$$[L'_h(\bar{u}, \bar{\theta}, \bar{h}, w, \vartheta)](h - \bar{h}) \geq 0 \quad \forall h \in \mathfrak{H}_{ad}. \quad (18c)$$

The identities (18a,b) lead to the *adjoint system* which has (in the classical formulation) the form:

$$(\nabla \bar{u})^\top w - (\bar{u} \cdot \nabla)w - \nu \Delta w + \nabla \pi = 2\alpha_1 \operatorname{div}(\vartheta \nabla \bar{u}) + \alpha_2 \bar{\theta} g \vartheta \quad (19a)$$

$$+ \bar{\theta} \nabla \vartheta + \begin{cases} \bar{u} - u_d & \text{for } \xi = 0 \\ \Delta(u_d - \bar{u}) & \text{for } \xi = 1 \end{cases}$$

$$\operatorname{div} w = 0, \quad (19b)$$

$$\kappa \Delta \vartheta + \bar{u} \nabla \vartheta + \alpha_2 g \cdot \bar{u} \vartheta = \alpha_0 g \cdot w + \theta_d - \bar{\theta} \quad (19c)$$

with the boundary conditions

$$w = 0, \quad \kappa \frac{\partial \vartheta}{\partial n} + b \vartheta = 0 \quad \text{on } \Gamma, \quad (20)$$

for the so-called *adjoint velocity*, *pressure*, and *temperature* $(w, \pi, \vartheta) \in W_{0,\operatorname{DIV}}^{1,2}(\Omega; \mathbb{R}^d) \times L^2(\Omega) \times W^{1,2}(\Omega)$ associated with a given state $(\bar{u}, \bar{\theta})$. Note that $(\nabla u)^\top w - (u \cdot \nabla)w$ means $(\sum_{k=1}^n (\frac{\partial u_k}{\partial x_i} w_k - u_k \frac{\partial w_i}{\partial x_k}))_{\xi=1, \dots, n}$.

Proposition 6 (1st-order necessary conditions.) *Let (5)–(6) hold, and let $\bar{h}$ be a locally optimal control for $(\mathfrak{P})$ with associated state $(\bar{u}, \bar{\theta})$. Then the variational inequality*

$$\bar{h}(x) = \begin{cases} h_{\max} & \text{if } -\vartheta(x) > 2h_{\max}, \\ -\vartheta(x)/2 & \text{if } -\vartheta(x) \in [2h_{\min}, 2h_{\max}], \\ h_{\min} & \text{if } -\vartheta(x) < 2h_{\min}, \end{cases} \quad (21)$$

is satisfied for $\vartheta = \vartheta(\bar{u}, \bar{\theta}) \in W^{1,2}(\Omega)$ being, together with $w = w(\bar{u}, \bar{\theta})$, the unique weak solution to the adjoint system (19)–(20).

Proof. (Sketched.) In view of the special form (4) of $\mathfrak{H}_{\text{ad}}$, (18c) is equivalent to (21) and, as already told, (18a,b) is equivalent to the adjoint system (19)–(20) whose solution can be shown to exist under the assumption (5)–(6). ■

To perform 2nd-order analysis (as, in the context of fluid control used in [10,18,27]), we need 2nd-order derivative of the Lagrange function. In view of (17), the 2nd differential of $L(\cdot, \cdot, \cdot, w, \vartheta)$ at a point (u, θ, h) is

$$\begin{aligned} L''_{(u, \theta, h)}(u, \theta, h, w, \vartheta)[(u_1, \theta_1, h_1), (u_2, \theta_2, h_2)] = & -((u_1 \cdot \nabla)u_2, w) \quad (22) \\ & -((u_2 \cdot \nabla)u_1, w) - (u_1 \cdot \nabla \theta_2, \vartheta) - (u_2 \cdot \nabla \theta_1, \vartheta) \\ & + 2\alpha_1(\vartheta \nabla u_1, \nabla u_2) + \alpha_2(\theta_1 g \cdot u_2, \vartheta) + \alpha_2(\theta_2 g \cdot u_1, \vartheta) \\ & + (\theta_1, \theta_2) + (h_1, h_2)_\Gamma + \begin{cases} (u_1, u_2) & \text{if } \xi = 0, \\ (\nabla u_1, \nabla u_2) & \text{if } \xi = 1. \end{cases} \end{aligned}$$

This quadratic form is therefore independent of (u, θ, f) and bounded provided the multipliers w and ϑ are bounded in L^∞ -norm, as indeed shown in the proof of Proposition 10 below under the assumption b smooth and, for $\gamma = 2$,

$$\theta_d \in L^\gamma(\Omega), \quad u_d \in \begin{cases} L^\gamma(\Omega; \mathbb{R}^d) & \text{if } \xi = 0, \\ W^{2,\gamma}(\Omega; \mathbb{R}^d) & \text{if } \xi = 1. \end{cases} \quad (23)$$

By using Green formula and $\operatorname{div} \tilde{u} = 0$ and $\tilde{u}|_\Gamma = 0$, (22) restricted on the diagonal $(u_1, \theta_1, h_1) = (u_2, \theta_2, h_2)$ becomes symmetric and takes the form

$$\begin{aligned} L''_{(u,\theta,h)}(u, \theta, h, w, \vartheta)(\tilde{u}, \tilde{\theta}, \tilde{h})^2 &= 2((\tilde{u} \cdot \nabla)w, \tilde{u}) + 2(\tilde{u} \cdot \nabla \vartheta, \tilde{\theta}) \\ &+ 2\alpha_1(\vartheta \nabla \tilde{u}, \nabla \tilde{u}) + 2\alpha_2(\tilde{\theta} g, \tilde{u} \vartheta) + \|\tilde{h}\|_{2,\Gamma}^2 + \|\tilde{\theta}\|_2^2 + \begin{cases} \|\tilde{u}\|_2^2 & \text{if } \xi = 0, \\ \|\nabla \tilde{u}\|_2^2 & \text{if } \xi = 1. \end{cases} \end{aligned} \quad (24)$$

Proposition 7 (2nd-order sufficient conditions.) *Let (5)–(6) be valid, b smooth, (23) hold with $\gamma = 2$, and let $(\bar{u}, \bar{\theta}, \bar{h}, w, \vartheta)$ satisfy (1)–(2), $\bar{h} \in \mathfrak{H}_{\text{ad}}$, the first-order necessary conditions (19)–(20), together with the second-order sufficient condition:*

$$L''_{(u,\theta,h)}(\bar{u}, \bar{\theta}, \bar{h}, w, \vartheta)(u, \theta, h)^2 \geq \delta \|h\|_{2,\Gamma}^2 \quad (25)$$

for $L''_{(u,\theta,h)}$ from (24), some $\delta > 0$, and for all (u, θ, h) solving the system (1)–(2) linearized at $(\bar{u}, \bar{\theta}, \bar{h})$, i.e. for (u, θ, h) satisfying (1b), (2), and

$$(u \cdot \nabla)\bar{u} + (\bar{u} \cdot \nabla)u - \nu \Delta u + \nabla p = -g\alpha_0 \theta, \quad (26a)$$

$$u \cdot \nabla \bar{\theta} + \bar{u} \cdot \nabla \theta - \kappa \Delta \theta = -2\alpha_1(\nabla \bar{u}, \nabla u) + \alpha_2 g \cdot (\theta \bar{u} + \bar{\theta} u) \quad (26b)$$

in the weak sense. Then $(\bar{u}, \bar{\theta}, \bar{h})$ is locally optimal with respect to the topology of $W^{1,2}(\Omega; \mathbb{R}^{d+1}) \times L^2(\Gamma)$.

Proof. (Sketched.) By [6], (25)–(26) yields $\Phi''(\bar{h})(h, h) \geq \delta_1 \|h\|_{2,\Gamma}^2$ for some $\delta_1 > 0$ and for all $h \in \mathfrak{H}_{\text{ad}}$. Moreover, one can prove that $(u, \theta, h, w, \vartheta) \mapsto L''_{(u,\theta,h)}(u, \theta, h, w, \vartheta)$ is continuous, and this continuity is inherited also by $\Phi''(\cdot)(h, h)$, so that one can conclude that $\bar{h}$ is locally optimal for Φ with respect to the norm of $L^2(\Gamma)$. By the continuity of the state mapping $h \mapsto (u, \theta) : L^2(\Gamma) \rightarrow W^{1,2}(\Omega; \mathbb{R}^{d+1})$, see Proposition 1, this gives the claimed local optimality of $(\bar{u}, \bar{\theta}, \bar{h})$. ■

5. The increment formula

For $h \in \mathfrak{H}_{\text{ad}}$, as in the proof of Proposition 5, we denote $\Phi(h) = J(u(h), \theta(h), h)$ with $\|\theta(h)\|_2 \leq \varrho$. Recall that such $\theta = \theta(h)$ and $u = u(h)$ are unique under the assumption (5)–(6).

For the following lemma, let us abbreviate $z := (u, \theta) \in Z := W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^d) \times W^{1,2}(\Omega)$ and $H = L^2(\Gamma)$, and define $\Pi : Z \times H \rightarrow Z^*$ by $\langle \Pi(z, h), \tilde{z} \rangle =$ the left-hand side of (3); thus $\Pi(z, h) = 0$ is just (3).

Lemma 8 (Increment formula for bi-quadratic problems.) *Let $J : Z \times H \rightarrow \mathbb{R}$ and $\Pi : Z \times H \rightarrow Z^*$ be quadratic mappings and $\lambda \in Z^{**} \cong Z$ satisfy $L'_z(z, h, \lambda) = 0$ with $L(z, h, \lambda) = J(z, h) - \lambda \circ \Pi(z, h)$, and let $\Pi(z, h) = 0$ and $\Pi(\tilde{z}, \tilde{h}) = 0$. Then*

$$\Phi(\tilde{h}) - \Phi(h) - \Phi'(h)(\tilde{h} - h) = \frac{1}{2} L''_{(z,h)}(z, h, \lambda)(\tilde{z} - z, \tilde{h} - h)^2. \quad (27)$$

Proof. As $L(\cdot, \cdot, \lambda)$ is quadratic, the Taylor expansion up to 2nd-order term is exact, i.e.

$$\begin{aligned} L(\tilde{z}, \tilde{h}, \lambda) &= L(z, h, \lambda) + L'_{(z,h)}(z, h, \lambda)(\tilde{z} - z, \tilde{h} - h) \\ &\quad + \frac{1}{2} L''_{(z,h)}(z, h, \lambda)(\tilde{z} - z, \tilde{h} - h)^2. \end{aligned} \quad (28)$$

As $\Pi(z, h) = 0$, $L(z, h, \lambda) = J(z, h) - \lambda \circ \Pi(z, h) = \Phi(h)$. Similarly, $\Pi(\tilde{z}, \tilde{h}) = 0$ implies $L(\tilde{z}, \tilde{h}, \lambda) = \Phi(\tilde{h})$. Moreover, using the adjoint equation $L'_z(z, h, \lambda) = 0$, we get

$$\begin{aligned} L'_{(z,h)}(z, h, \lambda)(\tilde{z} - z, \tilde{h} - h) &= L'_z(z, h, \lambda)(\tilde{z} - z) + L'_h(z, h, \lambda)(\tilde{h} - h) \\ &= J'_h(z, h)(\tilde{h} - h) - [\Pi'_h(z, h)(\tilde{h} - h)]^* \lambda = \Phi'(h)(\tilde{h} - h). \quad \blacksquare \end{aligned} \quad (29)$$

As the controlled system (1) as well as the cost functional in $(\mathfrak{P})$ are quadratic, we can use Lemma 8 for our problem:

Corollary 9 *The following increment formula for $(\mathfrak{P})$ holds:*

$$\begin{aligned} J(\tilde{u}, \tilde{\theta}, \tilde{h}) - J(u, \theta, h) - (\vartheta + 2h, \tilde{h} - h)_\Gamma \\ &= (((\tilde{u} - u) \cdot \nabla)w, \tilde{u} - u) + ((\tilde{u} - u) \cdot \nabla \vartheta, \tilde{\theta} - \theta) \\ &\quad + \alpha_1(\vartheta(\nabla(\tilde{u} - u), \nabla(\tilde{u} - u)) + \alpha_2((\tilde{\theta} - \theta)g, (\tilde{u} - u)\vartheta) \\ &\quad + \frac{1}{2} \|\tilde{h} - h\|_{2,\Gamma}^2 + \frac{1}{2} \|\tilde{\theta} - \theta\|_2^2 + \begin{cases} \frac{1}{2} \|\tilde{u} - u\|_2^2 & \text{if } \xi = 0, \\ \frac{1}{2} \|\nabla(\tilde{u} - u)\|_2^2 & \text{if } \xi = 1. \end{cases} \end{aligned} \quad (30)$$

where $\tilde{u} = u(\tilde{f})$, $u = u(f)$, $w = w(u, \theta)$, and $\vartheta = \vartheta(u, \theta)$.

Proof. Use (27) with L'' from (24) and $\lambda = (w, \vartheta)$, and realize that $\Phi'(h) = L'_h(u, \theta, h, w, \vartheta) = \vartheta + 2h$. $\blacksquare$

The formula (30) can be used for global analysis of $(\mathfrak{P})$ provided one shows the multipliers w and ϑ sufficiently small in the $W^{1,\infty}$ -norm.

Proposition 10 *Let (5)–(6) be valid, b smooth, (23) hold for $\gamma > d$, let $\xi = 1$ or $\alpha_1 = 0$, and, for any $h \in \mathfrak{H}_{\text{ad}}$, the corresponding adjoint variables (w, ϑ) satisfy*

$$\frac{1}{2} \geq A_{w, \vartheta} := \begin{cases} \|\nabla w\|_\infty + \frac{1}{2}\|\nabla \vartheta\|_\infty + \frac{1}{2}\alpha_2\|g\|_\infty\|\vartheta\|_\infty & \text{if } \xi = 0, \\ N_2^2(\|\nabla w\|_\infty + \frac{1}{2}\|\nabla \vartheta\|_\infty \\ \quad + \frac{1}{2}\alpha_2\|g\|_\infty\|\vartheta\|_\infty) + \alpha_1\|\vartheta\|_\infty & \text{if } \xi = 1, \end{cases} \quad (31a)$$

$$\frac{1}{2} \geq A_\vartheta := \frac{1}{2}\|\nabla \vartheta\|_\infty + \frac{1}{2}\alpha_2\|g\|_\infty\|\vartheta\|_\infty. \quad (31b)$$

Then Φ is strictly convex on $\mathfrak{H}_{\text{ad}}$.

Proof. First, let us prove $W^{1,\infty}$ -estimates of the multipliers w and ϑ . The adjoint equation (19) can equally be viewed as the Stokes system and the Poisson equation with the right-hand sides

$$f_1 = -(\bar{u} \cdot \nabla)\bar{w} + (\nabla \bar{u})^\top \bar{w} + 2\alpha_1 \nabla \vartheta \cdot \nabla \bar{u} + 2\alpha_1 \vartheta \Delta \bar{u} \quad (32)$$

$$+ \alpha_2 \bar{\theta} g \cdot \vartheta - \bar{\theta} \nabla \vartheta + \begin{cases} \bar{u} - u_d & \text{for } \xi = 0 \\ \Delta(u_d - \bar{u}) & \text{for } \xi = 1 \end{cases}$$

$$f_2 = \bar{u} \nabla \vartheta + \alpha_2 g \cdot \bar{u} \vartheta + \bar{\theta} - \theta_d. \quad (33)$$

By the proof of Proposition 1, we know that $\bar{u} \in W^{2,2}(\Omega; \mathbb{R}^d)$ and $\bar{\theta} \in W^{1,2}(\Omega)$, so that $f := (\bar{u} \cdot \nabla)\bar{u} + \alpha_0 g \bar{\theta} \in L^6(\Omega; \mathbb{R}^d)$ and, by the $W^{2,p}$ -regularity [14] of Stokes' system with the right-hand side f , we get $\bar{u} \in W^{2,\gamma}(\Omega; \mathbb{R}^d)$; as $n \leq 3$ we may assume $\gamma \leq 6$. Besides, certainly $w \in W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^d)$ and $\vartheta \in W^{1,2}(\Omega)$. As $W^{1,2}(\Omega) \subset L^6(\Omega)$ and $W^{2,\gamma}(\Omega) \subset W^{1,\infty}(\Omega)$, we can see that $f_2 \in L^2(\Omega)$ due to the estimate

$$\|f_2\|_2 \leq \|\bar{u}\|_\infty \|\nabla \vartheta\|_2 + \alpha_2 N_{6,2} \|g\|_\infty \|\bar{u}\|_\infty \|\vartheta\|_6 + \|\bar{\theta} - \theta_d\|_2. \quad (34)$$

By the $W^{2,2}$ -regularity of the equation $\kappa \Delta \vartheta + f_2 = 0$ with the boundary conditions $\kappa \partial \vartheta / \partial n + b \vartheta = 0$ (here we need b smooth), we can see that $\vartheta \in W^{2,2}(\Omega)$. Then we have also $f_1 \in L^2(\Omega; \mathbb{R}^d)$ due to the estimate

$$\begin{aligned} \|f_1\|_2 &\leq \|\bar{u}\|_\infty \|\nabla \bar{w}\|_2 + N_{6,4}^2 \|\nabla \bar{u}\|_6 \|\bar{w}\|_6 + 2\alpha_1 N_{6,4}^2 \|\nabla \vartheta\|_6 \|\nabla \bar{u}\|_6 \\ &\quad + 2\alpha_1 \|\vartheta\|_\infty \|\Delta \bar{u}\|_2 + \alpha_2 N_{\infty,2} \|\bar{\theta}\|_\infty \|g\|_\infty \|\vartheta\|_\infty \\ &\quad + N_{6,2} \|\bar{\theta}\|_\infty \|\nabla \vartheta\|_6 + \begin{cases} N_{\infty,2} \|\bar{u} - u_d\|_\infty & \text{for } \xi = 0 \\ \|\Delta(u_d - \bar{u})\|_2 & \text{for } \xi = 1 \end{cases} \end{aligned} \quad (35)$$

Again by regularity for Stokes' systems [14; Chap. IV, Thm.6.1], we have $w \in W^{2,2}(\Omega; \mathbb{R}^d)$. By usual bootstrap argument, we can see that f_2 then belongs even to $L^\gamma(\Omega)$ (recall that $\gamma \leq 6$) due to the estimate

$$\begin{aligned} \|f_2\|_\gamma &\leq N_{6,\gamma} \|\bar{u}\|_\infty \|\nabla \vartheta\|_6 \\ &\quad + \alpha_2 N_{\infty,\gamma} \|g\|_\infty \|\bar{u}\|_\infty \|\vartheta\|_\infty + \|\bar{\theta} - \theta_d\|_\gamma. \end{aligned} \quad (36)$$

Hence, we get one of the desired estimates: $\vartheta \in W^{2,\gamma}(\Omega) \subset W^{1,\infty}(\Omega)$. Then, by the obvious “bootstrap” modification of (35), we have also $f_1 \in L^\gamma(\Omega; \mathbb{R}^d)$. Again by $W^{2,\gamma}$ -regularity for Stokes’ system [14], we have the remaining desired estimate $w \in W^{2,\gamma}(\Omega; \mathbb{R}^d) \subset W^{1,\infty}(\Omega; \mathbb{R}^d)$.

Then, in the case $\xi = 1$, we can estimate the terms with indefinite signs in (30) by the Hölder and Young inequalities as follows:

$$\begin{aligned} & (((\tilde{u} - u) \cdot \nabla)w, \tilde{u} - u) + ((\tilde{u} - u) \cdot \nabla \vartheta, \tilde{\theta} - \theta) \\ & + \alpha_1 (\vartheta \nabla(\tilde{u} - u), \nabla(\tilde{u} - u)) + \alpha_2 ((\tilde{\theta} - \theta)g, (\tilde{u} - u)\vartheta) \\ & \geq -\|\tilde{u} - u\|_2^2 \|\nabla w\|_\infty - \|\tilde{u} - u\|_2 \|\nabla \vartheta\|_\infty \|\tilde{\theta} - \theta\|_2 \\ & \quad - \alpha_1 \|\vartheta\|_\infty \|\nabla(\tilde{u} - u)\|_2^2 - \alpha_2 \|\tilde{u} - u\|_2 \|g\|_\infty \|\vartheta\|_\infty \|\tilde{\theta} - \theta\|_2 \\ & \geq -A_{w,\vartheta} \|\nabla(\tilde{u} - u)\|_2^2 - A_\vartheta \|\tilde{\theta} - \theta\|_2^2. \end{aligned} \tag{37}$$

Then (31) ensures that these terms are dominated by last two terms in (30) so that $\Phi(\tilde{h}) - \Phi(h) - \Phi'(h)(\tilde{h} - h) \geq 0$ by (27). Replacing the role of h and $\tilde{h}$, we get $\Phi(h) - \Phi(\tilde{h}) - \Phi'(\tilde{h})(h - \tilde{h}) \geq 0$, and by adding these inequalities we get monotonicity of Φ' , hence convexity of Φ . The term $\frac{1}{2} \|\tilde{h} - h\|_{2,\Gamma}^2$ in (30) then ensures strict convexity of Φ .

For $\alpha_1 = 0$, we can handle the case $\xi = 0$ by simplified estimation as suggested in (31a). ■

Corollary 11 (1st-order sufficient condition.) *Under the conditions in Proposition 10, (21) ensures that $\tilde{h}$ is the unique optimal control to $(\mathfrak{P})$.*

Remark 12 The increment-formula technique has been developed in a general context in [13]. Increment formula (30) has been derived in [23] for mere Navier-Stokes system (in particular, the $W^{1,\infty}$ -regularity of w has been proved by Málek [23] for this case) and in [2] for Oberbeck-Boussinesq system (i.e. $\alpha_1 = \alpha_2 = 0$). Severe restrictions of conditions like (31) have been analyzed by Bubák [2].

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