GENERAL CONNECTIONS BETWEEN STRONG OPTIMIZATION AND PARETO EFFICIENCY

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Abstract In this research paper we present some connections between the Strong Optimization and the Approximate Pareto type efficiency in the usual Vector Optimization, first in Ordered Vector Spaces by the natural convex cones and, then, in Hausdorff Locally Convex Spaces. This fact may be very useful for possible scalarization methods.

Keywords: Pareto efficiency, nuclear (supernormal) cone, full nuclear cone.

1. Introduction

One of the direction concerning the scientific research in Vector Optimization is the study of the existence for Pareto type efficient points (see, for instance, the recent results, comments and examples in [1], [3]–[7], [10]–[14], [16]– [18]). In this context, it can be immediately remarked very clearly that an important mathematical tool, imposed especially by the implications and applications in the field of this kind of optimality dedicated to the efficiency, remains the concept of nuclear (supernormal) cone introduced by Isac G. [4], published in [5] and developed with significant applications until now in [5]–[8], [10]–[13], [17], [18].

The still unpublished research work [8] inspired us in this paper, having in mind the connections with the approximate Pareto efficiency, concept introduced here by us, that is, with its particular case represented by the usual Pareto efficiency. All the results, comments and examples are original.

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2. The main results and related topics

Let X be a vector space ordered by a convex cone K , K_1 a non-void subset of K and A a non-empty subset of X . The following definition introduces a new concept of (approximate) Pareto type efficient points which leads to the well know notion of Pareto efficiency (in fact, the generalization in abstract spaces of the finite dimensional notion as we shall see in the next considerations).

Definition 2.1. We say that $a_0 \in A$ is a K_1 -Pareto (minimal) efficient point of A , in notation, $a_0 \in \text{eff}(A, K, K_1)$ (or $a_0 \in \text{MIN}_{K+K_1}(A)$) if it satisfies one of the following equivalent conditions:

- (i) $A \cap (a_0 - K - K_1) \subseteq a_0 + K + K_1$;
- (ii) $(K + K_1) \cap (a_0 - A) \subseteq -K - K_1$;

In a similar manner one defines the Pareto (maximal) efficient points by replacing $K + K_1$ with $-(K + K_1)$.

Remark 2.1. $a_0 \in \text{eff}(A, K, K_1)$ iff it is a fixed point for one of the following multifunctions $F_i : A \rightarrow A, i = \overline{1, 4}$ defined by

$$F_1(t) = \{a \in A : A \cap (a - K - K_1) \subseteq t + K + K_1\};$$

$$F_2(t) = \{a \in A : A \cap (t - K - K_1) \subseteq a + K + K_1\}.$$

Consequently, for the existence of the Pareto type efficient points one can apply appropriate fixed point theorems concerning the multifunctions (see, for instance, [1]).

Remark 2.2. in [2] it is shown that whenever $K_1 \subset K \setminus \{0\}$, the existence of this new type of efficient points for lower bounded sets characterizes the semi Archimedian ordered vector spaces and the regular ordered locally convex spaces.

Remark 2.3. When K is pointed, that is, $K \cap (-K) = \{0\}$ and $K_1 = \{0\}$, then, from Definition 2.1, we obtain the well known usual notion of Pareto (minimal, efficient, optimal or admissible) point, abbreviated $a_0 \in \text{eff}(A, K)$ (or $a_0 \in \text{MIN}_K(A)$), that is, satisfying the next equivalent properties:

- (i) $A \cap (a_0 - K) = \{a_0\}$;
- (ii) $A \cap (a_0 - K \setminus \{0\}) = \emptyset$;
- (iii) $K \cap (a_0 - A) = \{0\}$;
- (iv) $(K \setminus \{0\}) \cap (a_0 - A) = \emptyset$

and it is clear that for any $\varepsilon \in K \setminus \{0\}$, taking $K_1 = \{\varepsilon\}$, it follows that $a_0 \in \text{eff}(A, K, K_1)$ if and only if $A \cap (a_0 - \varepsilon - K) = \emptyset$. In all these cases, the set $\text{eff}((A, K, K_1))$ was denoted by $\varepsilon - \text{eff}(A, K)$ (or $\varepsilon - \text{MIN}_K(A)$ as in [2],[12] and [13]) and it is obvious that $\text{eff}(A, K) =$

$$\bigcap_{\varepsilon \in K \setminus \{0\}} [\varepsilon - \text{eff}(A, K)].$$

Remark 2.4. The following theorem offers the first important connection between the strong optimization and the (approximate) Pareto efficiency in the environment of ordered vector spaces, described initially on the previous Definition 2.1.

Theorem 2.1. *If we denote by $S(A, K, K_1) = \{a_1 \in A : A \subseteq a_1 + K + K_1\}$ and $S(A, K, K_1) \neq \emptyset$, then $S(A, K, K_1) = \text{eff}(A, K, K_1)$.*

Proof. Clearly, $S(A, K, K_1) \subseteq \text{eff}(A, K, K_1)$.

Indeed, if $a_0 \in S(A, K, K_1)$ and $a \in A \cap (a_0 - K - K_1)$ are arbitrary elements, then $a \in a_0 + K + K_1$, that is, $a_0 \in \text{eff}(A, K, K_1)$, by virtue of (i) in Definition 2.1. Suppose now that $\bar{a} \in S(A, K, K_1) \neq \emptyset$ and there exists $a_0 \in \text{eff}(A, K, K_1) \setminus S(A, K, K_1)$. From $\bar{a} \in S(A, K, K_1)$ it follows that $a_0 \in \bar{a} + K + K_1$, that is, $\bar{a} \in a_0 - K - K_1$, from which, since $\bar{a} \in A$ and $a_0 \in \text{eff}(A, K, K_1)$ we conclude that $\bar{a} \in a_0 + K + K_1$. Therefore, $A \subseteq \bar{a} + K + K_1 \subseteq a_0 + K + K_1$, in contradiction with $a_0 \notin S(A, K, K_1)$ as claimed.

Remark 2.5. The above theorem shows that, for any non-empty subset of an arbitrary vector space, the set of all strong minimal elements with respect to any convex cone through the agency of every non-void subset of it coincides with the corresponding set of Pareto (minimal) efficient points whenever there exists at least a strong minimal element, the result remaining obviously valid for the strong maximal elements and the Pareto maximal efficient points, respectively.

Using this result and our abstract construction given in [9] for the H -locally convex spaces introduced by Th. Precupanu in [15] as separated locally convex spaces with any seminorm satisfying the parallelogram law, we established in [12] that the only best simultaneous and vectorial approximation for each element in the direct sum of a (closed) linear subspace and its orthogonal with respect to a linear (continuous) operator between two H -locally convex spaces is its spline function. We also note that it is possible to have $S(A, K, K_1) = \emptyset$ and $\text{eff}(A, K, K_1) = A$. Thus, for example, if one considers $X = R^2$ endowed with the separated locally convex topology generated by the seminorms $p_1, p_2 : X \rightarrow R_+$, $p_1(x, y) = |x|$, $p_2(x, y) = |y|$, $K = R_+^2 = \{(x, y) \in R^2 : x, y \geq 0\}$, $K_1 = \{(0, 0)\}$ and $A = \{(\lambda, 1 - \lambda) : 0 \leq \lambda \leq 1\}$, then it is clear that $S(A, K, K_1) = \emptyset$ and $\text{eff}(A, K, K_1) = A$.

In all our further considerations we suppose that X is a Hausdorff locally convex space having the topology induced by family $P = \{p_\alpha : \alpha \in I\}$ of seminorms, ordered by a convex cone K and its topological dual space X^* . In this framework, the next theorem contains a significant criterion for the existence of the approximative Pareto (minimal) efficient points, in particular, for the usual Pareto (minimal) efficient points, taking into

account that the dual cone of K is defined by $K^* = \{x^* \in X^* : x^*(x) \geq 0, \forall x \in K\}$ and its attached polar cone is $K^0 = -K^*$. The version for the approximate Pareto (maximal) efficient points is straightforward.

Theorem 2.2. *If A is any non-empty subset of X and K_1 is every non-void subset of K , then $a_0 \in \text{eff}(A, K, K_1)$ whenever for each $p_\alpha \in P$ and $\eta \in (0, 1)$ there exists u^* in the polar cone K^0 of K such that $p_\alpha(a_0 - a) \leq x^*(a_0 - a) + \eta, \forall a \in A$.*

Proof. We follow the general lines of the proof for Theorem 2.5 in [12].

Let us suppose that, under the above hypotheses,

$$(K + K_1) \cap (a_0 - A) \not\subseteq -(K + K_1),$$

that is, there exists $a \in A$ so that $a_0 - a \in K + K_1 \setminus (-(K + K_1))$. Then, $a_0 - a \neq 0$ and, because X is separated in Hausdorff's sense, there exists $p_\alpha \in P$ such that $p_\alpha(a_0 - a) > 0$. On the other hand, there exists $n \in \mathbb{N}^*$ sufficiently large with $p_\alpha(a_0 - a)/n \in (0, 1)$ and the relation given by the hypothesis of theorem leads to $p_\alpha(a_0 - a) \leq x^*(a_0 - a) + p_\alpha(a_0 - a)/n$ with $x^* \in K^0$ and $n \rightarrow \infty$, which implies that $p_\alpha(a_0 - a) \leq 0$, a contradiction and the proof is completed.

Remark 2.6. The above theorem represents an immediate extension of Precupanu's result given in Proposition 1.2 of [15]. In general, the converse of this theorem is not valid at least in (partially) ordered separated locally convex spaces as we can see from the example considered in Remark 2.5. Indeed, if one assumes the contrary in the corresponding, mathematical background, then, taking $\eta = \frac{1}{4}$ it follows that for each $\lambda_0 \in [0, 1]$ there exists $c_1, c_2 \leq 0$ such that $|\lambda_0 - \lambda| \leq (c_1 - c_2)(\lambda_0 - \lambda) + \frac{1}{4}, \forall \lambda \in [0, 1]$. Taking $\lambda_0 = \frac{1}{4}$ one obtains $|1 - 4\lambda| \leq (c_1 - c_2)(1 - 4\lambda) + 1, \forall \lambda \in [0, 1]$ which for $\lambda = 0$ implies that $c_2 \leq c_1$ and for $\lambda = \frac{1}{2}$ leads to $c_1 \leq c_2$, that is, $|1 - 4\lambda| \leq 1, \forall \lambda \in [0, 1]$, a contradiction.

The beginning and the considerations of Section 4 in [8] suggested us to consider for each function $\varphi : P \rightarrow K^* \setminus \{0\}$ the convex cone $K_\varphi = \{x \in X : p(x) \leq \varphi(p)(x), \forall p \in P\}$ and to extend Theorem 7 [8] in a more general context, which represents also a new link between strong optimization and the approximative vector optimization together with its usual particular variant, respectively.

Theorem 2.3. *If there exists $\varphi : P \rightarrow K^* \setminus \{0\}$ with $K \subseteq K_\varphi$, then*

$$\bigcup_{\substack{a \in A \\ \varphi \in P \rightarrow K^* \setminus \{0\}}} S(A \cap (a - K - K_1), K_\varphi, \{0\}) \subseteq \text{eff}(A, K, K_1)$$

for any non-empty subset K_1 of K .

Proof. Let $a_1 \in S(A \cap (a_0 - K - K_1), K_\varphi, \{0\})$ for at least one elements $a_0 \in A$ and $\varphi : P \rightarrow K^* \setminus \{0\}$. Then, $a_1 \in A \cap (a_0 - K - K_1)$ and $A \cap (a_0 - K - K_1) - a_1 \subseteq K_\varphi$, that is, $p(a - a_1) \leq \varphi(p)(a - a_1), \forall a \in$

$A \cap (a_0 - K - K_1), p \in P$ which implies immediately that $p(a_1 - a) \leq -\varphi(p)(a_1 - a) + \eta, \forall a \in A \cap (a_0 - K - K_1), p \in P, \eta \in (0, 1)$ and, by virtue of Theorem 2.2 one obtains $a_1 \in \text{eff}(A \cap (a_0 - K - K_1), K, K_1)$. But $\text{eff}(A \cap (a_0 - K - K_1), K, K_1) \subseteq \text{eff}(A, K, K_1)$.

Indeed, for any $t \in \text{eff}(A \cap (a_0 - K - K_1), K, K_1)$ and $h \in A \cap (t - K - K_1)$ we have $h \in A \cap (a_0 - K - K_1) \cap (t - K - K_1) \subseteq t + K + K_1$ that is, $A \cap (t - K - K_1) \subseteq t + K + K_1$ and by point (i) of Definition 2.1 one obtains $t \in \text{eff}(A, K, K_1)$. This completes the proof.

Remark 2.7. The hypothesis $K \subseteq K_\varphi$ imposed upon the convex cone K is automatically satisfied whenever K is a supernormal (nuclear) cone. When K is any pointed convex cone, A is a nonempty subset of X and $a_0 \in \text{eff}(A, K)$, then, by virtue of (i) in Remark 2.3, it follows that $A \cap (a_0 - K) = \{a_0\}$, that is, $A \cap (a_0 - K) - a_0 = \{0\} \subset K_\varphi$. Hence, $a_0 \in S(A \cap (a_0 - K), K_\varphi, \{0\})$ for every mapping $\varphi : P : K^* \setminus \{0\}$ and the next corollary is valid.

Corollary 2.1. *For every non-empty subset A of any Hausdorff locally convex space ordered by an arbitrary, pointed convex cone K with its dual cone K^* we have*

$$\text{eff}(A, K) = \bigcup_{\substack{a \in A \\ \varphi: P \rightarrow K^* \setminus \{0\}}} S(A \cap (a - K), K_\varphi)$$

Remark 2.8. Clearly, the result concerns the possibilities of scalarization for the study of Pareto efficiency in separated locally convex spaces, as we can see also in the final comments of [8] for the particular cases of Hausdorff locally convex spaces ordered by closed, pointed and normal cones.

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