THE CONTROL OF SAFFMAN-TAYLOR INSTABILITY

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1. The Saffman–Taylor formula

The Secondary Oil Recovery process is considered: the oil contained in a two dimensional homogeneous porous medium is obtained by injection of an immiscible second (less viscous) fluid. The Hele-Shaw approximation is considered. The instability of the sharp interface between the fluids appears, first studied by Saffman and Taylor, [3]. The (constant) water and oil viscosities are denoted by μ_1 and μ_2 . The velocity components are (u, v) . The perturbations of the horizontal velocity are

$$u'(x, y, t) = f(x) \cdot \exp(iky + \sigma t),$$

where σ is the growth constant and k is the wave number in the Oy direction. The Saffman - Taylor value for the growth constant is

$$\sigma = \frac{(\mu_2 - \mu_1)Uk - Tk^3}{\mu_2 + \mu_1},$$

where T is the surface tension on $x = 0$ (the water - oil interface). A maximal value σ_m (in terms of k) is obtained for the wave number k_m :

$$\sigma_m = \sigma_m(k_m) = \frac{2(\mu_2 - \mu_1)U}{3(\mu_2 + \mu_1)\sqrt{3}} \cdot \sqrt{\frac{(\mu_2 - \mu_1)U}{T}}, \quad k_m = \frac{1}{\sqrt{3}} \sqrt{\frac{(\mu_2 - \mu_1)U}{T}}.$$

We use in the sequel the dimensionless quantities given in [3]:

$$\left\{ \begin{array}{l} \sigma^* = \frac{2\sigma}{3\sqrt{3}\sigma_m}, \quad \mu^*(x) = \mu_0(x)/\mu_1, \quad k^* = k/(k_m\sqrt{3}), \quad \alpha = \mu_2/\mu_1, \\ x^* = k_mx\sqrt{3}, \quad L = k_ml\sqrt{3}, \quad f^*(x) = f(x)/U, \quad \lambda = 1/\sigma. \end{array} \right.$$

The maximum dimensionless value of the growth constant in the Saffman-Taylor case is $\sigma_{ST} = 2/(3\sqrt{3}) \approx 0.38$.

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2. The model of Gorell and Homsy

An intermediate region PS, containing a polymer solute is considered in a model given by Gorell and Homsy, [2]. A steady basic solution, with straight initial interfaces exists. The three regions are moving by the water velocity far upstream. The linear stability of interfaces is governed by a Sturm-Liouville problem, with eigenvalues (the growth constant of perturbations) in the boundary conditions. The unknown viscosity in PS, denoted by μ , is a parameter used to improve the stability of the PS - oil interface.

On the interfaces we consider the Laplace's law; moreover, on the interface water-PS a continuous viscosity is considered.

The Sturm-Liouville problem governing the stability is (written without *):

$$\begin{cases} -(\mu f_x)_x + k^2 \mu f = \lambda k^2 \beta \mu_x \cdot f, & x \in (-L, 0) \\ f_x(0) = (\lambda a + b) \cdot f(0) \\ f_x(-L) = k \cdot f(-L), \end{cases}$$

$$\begin{cases} a = k^2 \beta \{\alpha - \mu(o) - k^2(\alpha - 1)\} / \mu(0), & \alpha = \mu_2 / \mu_1 \\ b = -k \cdot \alpha / \mu(0), & \beta = (\alpha + 1) / (\alpha - 1), \quad \lambda = 1 / \sigma. \end{cases}$$

where L is the length of PS. In a mobil system of coordinates, moving with the velocity U of water far upstream, the intermediate region is contained in the domain $x \in [-L, 0]$ - see [1] and [3]. The total amount C of polymer is:

$$C = \int_{-L}^0 \mu(x) dx.$$

The problem is to find the optimal value of μ d PS, which gives us the smallest growth constant σ , and to obtain an improvement of stability, compared with the Saffman-Taylor case. In [3] a numerical algorithm is used to give an "optimal" exponential viscosity profile in PS, which gives us a growth constant less than the Saffman-Taylor value.

3. The estimation of the growth constant

Carasso and Pasa, [1], obtain formula for an optimal viscosity in PS, and an upper estimation for the growth constant, in terms of $\mu(0)$ - the limit value of the viscosity in PS on the PS - oil interface. The above Sturm-Liouville problem is discretized by using the finite-difference method. We consider the points $x_i = -ih$, $h = L/N$, $i = 0, 1, \dots, N$ and obtain (by using the Greshgorin's localization theorem):

$$\sigma \leq \text{Max}\{H(k), \frac{\beta \mu'_i}{\mu_i}\}, \quad i = 1, 2, \dots, (N-1), \quad \mu_i = \mu(x_i),$$

$$H(k) = \frac{k\beta}{\alpha}[\alpha - \mu(0) - k^2(\alpha - 1)], \quad ' = d/dx.$$

The function $H(k)$ has a maximum value with respect to k :

$$\text{Max}_k\{H(k)\} = F(\mu(0)) = \frac{2\beta}{3\alpha} \frac{(\alpha - \mu(0))^{3/2}}{\sqrt{3(\alpha - 1)}}.$$

We consider the viscosity profile (1) in PS and obtain the estimation (2):

$$(\beta\mu'_i)/\mu_i \leq F(\mu(0)) \quad (1)$$

$$\sigma \leq \text{Max}\{F(\mu(0)), \frac{\beta\mu'_i}{\mu_i}\} = F(\mu(0)) < \frac{2}{3\sqrt{3}} \approx \sigma_{ST}, \quad (2)$$

if the following condition holds:

$$F(\mu(0)) < 2/(3\sqrt{3}). \quad (3)$$

We imposed in [1] some restrictions for $\mu(0)$. The profile (1) gives us:

$$\mu(x) \leq \exp\{(x + L) \cdot \frac{F(\mu(0))}{\beta}\} \quad (4)$$

and for $x = 0$ we obtain the condition:

$$\mu(0) \leq \exp\{L \frac{F(\mu(0))}{\beta}\}. \quad (5)$$

We integrated (4) and obtained a relation involving the total amount of polymer C , the injection length L and the limit value $\mu(0)$:

$$L \geq \frac{\beta}{F(\mu(0))} \cdot \ln\{\frac{CF(\mu(0))}{\beta} + 1\}. \quad (6)$$

We can solve the following problem: for a given C , we compute the corresponding $\mu(0)$, L and the growth constant. Then it is not possible to obtain in a direct manner a "prescribed" improvement in stability, compared with the Saffman-Taylor case.

4. A new optimal profile in terms of $\mu(0)$

In this paper we can solve the inverse of the above problem. The relation (5) is considered as a "restriction" for L , in terms of $\mu(0)$:

$$L \geq \frac{\beta}{F(\mu(0))} \cdot \ln\{\mu(0)\}. \quad (7)$$

Therefore this time we have only one restriction for $\mu(0)$: it lies between water and oil viscosities. The relation (7) is considered as a second restriction for L . The point is to find an estimation of C in terms of $\mu(0)$ such that (6) and (7) would be compatible. For this, we consider the relations:

$$C \geq \beta\{\mu(0) - 1\}/F(\mu(0)) \iff CF(\mu(0))/\beta + 1 \geq \mu(0) \iff \frac{\beta}{F(\mu(0))} \ln\{CF(\mu(0))/\beta + 1\} \geq \frac{\beta}{F(\mu(0))} \cdot \ln(\mu(0)). \quad (8)$$

We use the relations (6) and (8) to obtain

$$L \geq \frac{\beta}{F(\mu(0))} \cdot \ln\left\{\frac{CF(\mu(0))}{\beta} + 1\right\} \geq \frac{\beta}{F(\mu(0))} \cdot \ln\{\mu(0)\}. \quad (9)$$

We consider the viscosity profile (1). Then the relations (2), (6), (8) give us the growth constant σ , the injection length L and the total amount C of polymer C in terms of $\mu(0)$. The limit value $\mu(0)$ verifies only one restriction: it lies between the water and oil viscosities. The condition (3) gives us an improved stability, compared with the Saffman–Taylor case. We emphasize that $\sigma \rightarrow 0$ for $\mu(0) \rightarrow \alpha$. We must avoid the case $\mu(0) = \alpha$, because (1) gives us $\mu' = 0$ and we obtain a constant viscosity in PS.

5. Numerical results

The function $F(\mu(0))$ is plotted in Figure 1, for the viscosity profile (1) and $\alpha = 100$, $\mu(0) \in [1, 90)$. The optimal viscosity profile in PS and the values of L are given in Figure 2, also in terms of $\mu(0)$. The relations (2), (6), (8) give us:

$$\mu(0) = 40 \Rightarrow F(\mu(0)) = \sigma_G = 0.17, \quad L \geq 20.49, \quad C \geq 216;$$

$$\mu(0) = 80 \Rightarrow F(\mu(0)) = \sigma_G = 0.03, \quad L \geq 126, \quad C \geq 2373;$$

$$\mu(0) = 90 \Rightarrow F(\mu(0)) = \sigma_G = 0.01, \quad L \geq 368, \quad C \geq 7442.$$

Figure 1. Maximal value $\sigma_G = F(\mu(0))$,

Figure 2. Optimal viscosity profiles: $\mu(0)=40, 80, 90$

In the Figure 3 are plotted the compared evolutions of the perturbations

$$u'(x, y, t) = f(x) \exp(\sigma \cdot t) \cdot \left\{ \sum_{n=2}^{n=9} [\cos(0.05 \cdot n \cdot y) + \sin(0.05 \cdot n \cdot y)] \right\} \quad (10)$$

up to the time moment $t = 10$, for $\sigma = 0.38$ (without PS) and $\sigma = 0.03$. The initial amplitude is $f(0) \approx 1/10$.

Figure 3. Compared evolution of the perturbations (10), $t_2=10$

References

- [1] Carasso, C., Paşa, G., An optimal viscosity profile in the secondary oil recovery, *Modélisation Mathématique et Analyse Numérique*, **32**, 1998, 211-221.
- [2] Gorell, S.B., Homsy, G.M., A theory of the optimal policy of oil recovery by the secondary displacement process, *SIAM J. Appl. Math.*, **43**, 1983, 79-98.
- [3] Saffman, P.G., Taylor, G., The penetration of a fluid in a porous medium or Hele-Shaw cell containing a more viscous liquid, *Proc.Roy.Soc.*, **A 245**, 1958, 312-329.