METHOD OF EXTREMAL SHIFT IN PROBLEMS OF RECONSTRUCTION OF AN INPUT FOR PARABOLIC VARIATIONAL INEQUALITIES

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Abstract The problem of dynamical reconstruction of a control in a parabolic variational inequality [1–4] through inaccurate measurement of phase state is considered. This problem belongs to the class of dynamical inverse problems which consist in reconstruction of unknown input (a control) of dynamical systems from (inaccurate) measurement of outputs.

Keywords: reconstruction, parabolic inequality

Introduction

Let V and H be real Hilbert spaces, V be a dense subspace of H and $V \subset H = H^* \subset V^*$ algebraically and topologically. We denote by $|\cdot|_V$ and $|\cdot|_H$ the norms in V and H , respectively, by $(\cdot, \cdot)$ a scalar product in H , and by $\langle \cdot, \cdot \rangle$ duality between V and its dual V^* . Let a dynamical system be described by the parabolic variational inequality

$$(x_t(t) - f(t), x(t) - z) + \langle Ax(t) - Bu(t), x(t) - z \rangle + \varphi(x(t)) - \varphi(z) \leq 0 \quad (1)$$

a. e. $t \in T = [t_0, \vartheta] \forall z \in V, x(t_0) = x_0 \in D(\varphi)$.

Here, $A : V \rightarrow V^*$ is a linear continuous symmetrical operator satisfying (with a certain $c > 0$ and $\omega \in R$) the following condition

$$\langle Ay, y \rangle + \omega |y|_H^2 \geq c |y|_V^2; \quad (2)$$

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$f(\cdot) \in W^{1,2}(T; H)$ is a given disturbance; $\varphi : V \rightarrow R^+ = \{r \in R : r \geq 0\} \cup \{+\infty\}$ is a weakly lower semicontinuous convex function, $D(\varphi) = \{x \in V : \varphi(x) < +\infty\}$, U is a uniformly convex Banach space (the space of controls), $W^{1,2}(T; H) = \{x(\cdot) \in L_2(T; H) : x_t(\cdot) \in L_2(T; H)\}$, the derivative $x_t(\cdot)$ is understood in a sense of distribution; B is a linear continuous operator from U to H .

Introduce a function $\varphi_\omega(y) : H \rightarrow R^+$,

$$\varphi_\omega(y) = \begin{cases} 1/2\langle Ay, y \rangle + \omega/2|y|_H^2 + \varphi(y), & \text{if } y \in D(\varphi), \\ +\infty & \text{otherwise.} \end{cases}$$

From Theorem 4.1, 1.13 [2] it follows that for any $u(\cdot) \in L_2(T, U)$, $x_0 \in D(\varphi)$ there exists a unique solution of the inequality (1) with the properties: $x(\cdot) = x(\cdot; t_0, x_0, u(\cdot)) \in W_*(T) = W^{1,2}(T; H) \cap L_2(T; V)$, $x(t) \in D(\varphi_\omega) \quad \forall t \in T, \quad t \rightarrow \varphi_\omega(x(t)) \in AC(T)$. Here $AC(T)$ is a space of absolutely continuous functions.

The problem may be formulated in the following way. Let a uniform net $\Delta = \{\tau_i\}_{i=0}^m$, $\tau_i = \tau_{i-1} + \delta$, $\tau_0 = t_0$, $\tau_m = \vartheta$ be fixed on a given time interval T . Let a motion of system (1) proceed on the interval T . Its trajectory (the solution of (1)) $x_r(\cdot) = x(\cdot; t_0, x_0, u_r(\cdot))$ depends on time-varying unknown control $u(\cdot) = u_r(\cdot) \in U_T = L_2(T; U)$. Here U_T is a set of admissible controls. The trajectory $x_r(\cdot)$ is unknown. At moments τ_i the phase state $x_r(\tau_i)$ is inaccurately measured. The results of measurements $\xi_i^h \in H$, $i \in [0 : m-1]$ satisfy the inequality

$$|\xi_i^h - x_r(\tau_i)|_H \leq h. \quad (3)$$

Here, $h \in (0, 1)$ is the value of the level of informational noise. It is required to design an algorithm allowing to reconstruct (synchro with the process) some unknown input $u_*(\cdot)$ generating solution $x_r(\cdot)$ to inequality (1). This is meaningful statement of the problem.

1. The approach to solving the problem

One of the approaches to solving the problems of similar type was described, for example, in [5–12]. (See more detail on this approach in surveys [13–16]). Briefly remind about the essence of this approach. Let $U(x_r(\cdot))$ be the set of all inputs $u(\cdot) \in U_T$ which are compatible with $x_r(\cdot)$, Ξ_T be the set of all measurements, i. e. the set of all piece-wise functions on T with values in H , $\Xi(x_r(\cdot), h)$ be the set of all h -accurate measurements, i. e. functions $\xi^h(\cdot) \in \Xi_T$ satisfying (3). An auxiliary system M (a model) described by a parabolic variational inequality of the form

$$\begin{aligned} (w_t^h(t) - f(t), w^h(t) - z) + \langle Aw(t) - B_t(w^h(\cdot), \xi^h(\cdot))v^h(t), w^h(t) - z \rangle + \\ \varphi(w^h(t)) - \varphi(z) \leq 0 \text{ a. e. } t \in T \quad \forall z \in V, \quad w^h(t_0) = w_0^h \in D(\varphi) \end{aligned} \quad (4)$$

is considered. Here $B_t(\cdot, \cdot) : U \rightarrow H$ is a family of operators, whose structure will be specify below. The motion of the model is denoted as $w^h(\cdot) = w^h(\cdot; t_0, w_0^h, v^h(\cdot)) \in W_*(T)$. Here $v^h(\cdot)$ is a control in the model. Initial state w_0^h of the model is chosen by using the value ξ_0^h of measurement at initial time moment t_0 in accordance with some rule $\mathcal{W}_h$ fixed in advance:

$$w_0^h = w^h(t_0) = \mathcal{W}_h(\xi_0^h) \in H. \quad (5)$$

Model control rules are identified with pairs $S_h = (\Delta_h, \mathcal{U}_h)$, where $\Delta_h = \{\tau_{h,i}\}_{i=0}^{m_h}$ is a partition of the interval T into half-intervals $[\tau_{h,i}, \tau_{h,i+1})$, $\tau_{h,i+1} = \tau_{h,i} + \delta$, $\delta = \delta(h)$, $\tau_{h,0} = t_0$, $\tau_{h,m_h} = \vartheta$, $\mathcal{U}_h$ is a function relating element

$$v_i^h = \mathcal{U}_h(\tau_i, \xi_i^h, w^h(\tau_i)) \in U \quad (6)$$

to every triple $(\tau_i, \xi_i^h, w^h(\tau_i))$, $i \in [0 : m_h - 1]$, where $\tau_i = \tau_{h,i}$, $w^h(\tau_i) = w^h(\tau_i; t_0, w_0^h, v^h(\cdot))$, $\xi_i^h = \xi^h(\tau_i)$, $\xi^h(\cdot) \in \Xi(x_r(\cdot), h)$. Thus, quadruple $(M, \mathcal{W}_h, \Delta_h, \mathcal{U}_h)$ for every $h \in (0, 1)$ determines some algorithm D_h on space of measurements $\xi(\cdot) \in \Xi(x_r(\cdot), h)$ ($D_h : \Xi_T \rightarrow U_T$) forming output $v^h(\cdot) = D_h \xi(\cdot)$ according to the feedback principle (4)–(6). We identify the algorithm D_h with quadruple $(M, \mathcal{W}_h, \Delta_h, \mathcal{U}_h)$. The result of work of the algorithm on the interval T is a piecewise constant control $v^h(\cdot)$ of the form

$$v^h(t) = v_i^h, \quad t \in [\tau_i, \tau_{i+1}). \quad (7)$$

Let the following condition be fulfilled.

Condition 1 The set $U_*(x_r(\cdot))$ of $L_2(T; U)$ -norm minimal inputs in $U(x_r(\cdot))$ is one-element, i. e. $U_*(x_r(\cdot)) = \{u_*(\cdot; x_r(\cdot))\}$.

A family D_h , $h \in (0, 1)$ of operators from Ξ_T to U_T is called *regularizing* if

$$\lim_{h \rightarrow 0} \sup\{|D_h \xi^h(\cdot) - u_*(\cdot; x_r(\cdot))|_{L_2(T; U)} : \xi^h(\cdot) \in \Xi(x_r(\cdot), h)\} = 0.$$

We consider the problems of construction of regularizing families of algorithms of modeling

$$D_h = (M, \mathcal{W}_h, \Delta_h, \mathcal{U}_h), \quad h \in (0, 1) \quad (8)$$

of the form (4)–(7). We call them *positional algorithms of modeling*.

For parabolic variational inequality in [9–11], some regularizing families of algorithms of modeling D_h of type (8) were indicated for the case when the set of admissible controls U_T has the form: $U_T = \{u(\cdot) \in L_2(T; U) : u(t) \in P \text{ for a.e. } t \in T\}$. Here $P \subset U$ is a convex, bounded, and closed set. The constructions of works [9–11] are based on combination of the theory of control with a model [17] and the known in

the theory of ill-posed problems method of smoothing functional (also often called Tikhonov's method). In the present work, we modify algorithms [5–8, 10–12] for the case when set U_T is unbounded. Note that works [9, 13] indicate some other regularizing algorithms D_h of reconstruction of unbounded controls in parabolic variational inequalities based on a dynamical modification of the discrepancy method. Questions on program control of parabolic variational inequalities have been discussed in monographs [2].

After model (4) and its initial state (5) are chosen the work of the algorithm D_h (for fixed h) corresponds to the following outline. First, before the moment t_0 , a partition $\Delta = \Delta_h = \{\tau_i\}_{i=0}^m$, ($\tau_i = \tau_{h,i}$) of the interval T is chosen and fixed. At the i -th step carried out during the time interval $[\tau_i, \tau_{i+1})$, the following sequence of actions takes place. The output $x_r(\tau_i)$ is inaccurately measured, i. e. the value $\xi_i^h \in H$ with the properties (3) is calculated. Then the model control is determined by (6), (7) and after that we form the new part of the model trajectory $w^h(t)$, $t \in (\tau_i, \tau_{i+1}]$ instead of $w_{t_0, \tau_i}^h(\cdot)$ (memory correction). The procedure stops at the time moment ϑ .

Construction of a family D_h is based on Theorem 1 formulated below. Let us fix a functional $\Lambda^0(\cdot, \cdot)$ on $W_*(T) \times W_*(T)$.

Definition 1 [7, 15] A family D_h , $h \in (0, 1)$, (8) of positional algorithms of modeling is said to be Λ^0 -stable if there exist functions $k_1(\cdot)$, $k_2(\cdot)$, $k_3(\cdot)$: $[0, \infty) \rightarrow [0, \infty)$ such that $k_1(h) \rightarrow 1$, $k_2(h) \rightarrow 0$, $k_3(h) \rightarrow 0$ as $h \rightarrow 0$, and for every observation record $\xi^h(\cdot) \in \Xi(x_r(\cdot), h)$, it holds that

$$|v^h(\cdot)|_{L_2(T; U)} \leq k_1(h)|u_*(\cdot; x_r(\cdot))|_{L_2(T; U)} + k_2(h), \quad (9)$$

$$\Lambda^0(x_r(\cdot), w^h(\cdot)) \leq k_3(h), \quad (10)$$

where $v^h(\cdot) = D_h \xi^h(\cdot)$ and $w^h(\cdot)$ is the model motion generated by D_h under the observation record $\xi^h(\cdot)$.

Then the following theorem is true.

Theorem 1 [15] Let a family D_h of positional algorithms of modeling be a) Λ^0 -stable, and b) for every $h_k > 0$, $h_k \rightarrow 0+$ as $k \rightarrow +\infty$, $\xi^{h_k}(\cdot) \in \Xi(x_r(\cdot), h_k)$, $w^{h_k}(\cdot) = w^{h_k}(\cdot; t_0, w^{h_k}(t_0), v^{h_k}(\cdot))$, $v^{h_k}(\cdot) = D_{h_k} \xi^{h_k}(\cdot)$, the limit correlations $v^{h_k}(\cdot) \rightarrow v(\cdot)$ weakly in $L_2(T; U)$, $\Lambda^0(x_r(\cdot), w^{h_k}(\cdot)) \rightarrow 0$ imply that $v(\cdot) \in U(x_r(\cdot))$. Then the family D_h , $h \in (0, 1)$, is regularizing.

2. Case $u_*(\cdot; x_r(\cdot)) \in L_\infty(T; U)$

First we consider a case where a control $u_*(\cdot; x_r(\cdot))$ is a bounded function, i.e., $u_*(\cdot; x_r(\cdot)) \in L_\infty(T; U)$. Let a number $a > 0$ such that $x_0 \in X_0 = \{x \in D(\varphi) : |x|_H^2 + \varphi_\omega(x) \leq a < +\infty\}$ be chosen.

We take a family of partitions (Δ_h) of the interval T , and functions $\alpha(h):R^+ \rightarrow R^+$ and $d(h):R^+ \rightarrow R^+$ satisfying the following conditions:

$$\begin{aligned} \alpha(h) &\rightarrow 0, \quad \delta(h) \rightarrow 0, \quad d(h) \rightarrow +\infty, \\ d(h)(h + \delta(h))/\alpha(h) &\rightarrow 0, \quad \alpha(h)d^2(h) \rightarrow 0 \text{ as } h \rightarrow 0+. \end{aligned}$$

This condition holds, for example, if $\delta(h) \leq ch$, $\alpha(h) = c_1 h^\gamma$, $\gamma = 0, 5 + \gamma_0$, $\gamma_0 \in (0; 0, 25)$, $d(h) = c_2 h^{-1/4}$, c, c_1, c_2 are positive constants. The family $(\mathcal{W}_h)$ of t_0 -algorithms is defined by rule (5), where

$$w_0^h \in B(\xi_0^h) = \{x \in X_0 : |\xi_0^h - x|_H \leq 2h\}. \quad (11)$$

From (3) and inclusion $x_0 \in X_0$, it follows $B(\xi_0^h) \neq \emptyset$. In its turn, the model control law $S_h = (\Delta_h, \mathcal{U}_h)$ is defined following rule (6), (7), where we assume

$$\mathcal{U}_h(\tau_i, \xi_i^h, w^h(\tau_i)) = \arg \min \{l(\alpha, v, s_i) : v \in S(d(h))\}, \quad (12)$$

$$\begin{aligned} l(\alpha, v, s_i) &= 2(s_i, Bv) + \alpha(h)|v|_U^2, \quad s_i = w^h(\tau_i) - \xi_i^h, \\ S(d(h)) &= \{u \in U : |u|_U \leq d(h)\}. \end{aligned} \quad (13)$$

For a fixed h (and, consequently, a fixed family $\Delta_h = \{\tau_i\}_{i=0}^{m_h}$ with diameter $\delta(h)$) model M is given by the mapping associated with each triple $(w_0^h, \xi^h(\cdot), v^h(\cdot))$, $w_0^h \in X_0$, $\xi^h(\cdot) \in \Xi(x(\cdot), h)$, $v^h(\cdot) \in L_2(T; U)$ with a function $w^h(\cdot) = w^h(\cdot; t_0, w_0^h, \xi^h(\cdot), v^h(\cdot)) \in W_*(T)$ which is a unique solution of variational inequality (4), where

$$B_t(\xi^h(\cdot), w^h(\cdot))v^h(t) = Bv^h(t) - \omega^*(w^h(\tau_i) - \xi_i^h) \quad (14)$$

for $t \in \delta_i = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{h,i}$, $\xi_i^h = \xi^h(\tau_i)$, $\omega^* = 0$, if $\omega \leq 0$, $\omega^* = \omega + \varepsilon_0$, otherwise ($\varepsilon_0 \geq 0$). Let

$$\Lambda^0(x(\cdot), w^h(\cdot)) = |x(\cdot) - w^h(\cdot)|_{C(T; H)}^2 + 2c|x(\cdot) - w^h(\cdot)|_{L_2(T; V)}^2. \quad (15)$$

Theorem 2 *The family of positional modeling algorithms D_h (8) of the form (4)–(7), (11)–(14) satisfies conditions of Theorem 1 and is regularizing.*

Before proving the theorem, we formulate auxiliary statements. Let $\omega_* = \omega$, if $\omega > 0$, $\omega_* = 0$ otherwise. Introduce a mapping $l(\cdot) : W_*(T) \rightarrow R^+$,

$$l(y(\cdot)) = |y(\cdot)|_{C(T; H)} + |y_t(\cdot)|_{L_2(T; H)} + |y(\cdot)|_{L_2(T; V)}.$$

The following lemmas are true.

Lemma 1 *There exists a number $K_* = K_*(\omega_*, c, |B|_{\mathcal{L}(T; U)})$ such that for any $x_0 \in D(\varphi)$, $u(\cdot) \in U_T$, $x(\cdot) = x(\cdot; t_0, x_0, u(\cdot))$ the inequality*

holds.

$$l(x(\cdot)) \leq K_*(1 + \omega_*|x_0|_H + \varphi_\omega^{1/2}(x_0) + |u(\cdot)|_{L_2(T;U)})$$

Lemma 2 *There exists a number $K^* = K^*(\omega_*, c, |B|_{\mathcal{L}(T;U)}, \varepsilon_0)$ such that for any $w_0^h \in D(\varphi)$, $v^h(\cdot) \in U_T$, $\xi^h(\cdot) \in \Xi(x_r(\cdot), h)$, $w^h(\cdot) = w^h(\cdot; t_0, w_0^h, \xi^h(\cdot), v^h(\cdot))$, the inequality*

$$l(w^h(\cdot)) \leq K^*(1 + \omega_*|w_0^h|_H + \varphi_\omega^{1/2}(w_0^h) + |v^h(\cdot)|_{L_2(T;U)})$$

holds.

These lemmas are proved by the standard scheme [1, 2].

Let us emphasize that constants K_* and K^* do not depend on w_0^h , $\xi^h(\cdot)$, $u(\cdot)$ or $v^h(\cdot)$.

Proof of Theorem 2. Let us show that the family D_h (4)–(7), (11)–(14) is Λ^0 -stable. Let $\xi^h(\cdot) \in \Xi(x_r(\cdot), h)$, $v^h(\cdot) = D_h \xi^h(\cdot)$ and $w^h(\cdot)$ is the model motion (4) generated by D_h under the observation record $\xi^h(\cdot)$. Consider the value

$$\varepsilon_h(t) = |\mu^h(t)|_H^2 + 2c|\mu^h(\cdot)|_{L_2([t_0, t]; V)}^2 + \alpha(h) \int_{t_0}^t \{|v^h(\tau)|_U^2 - |u_*(\tau)|_U^2\} d\tau.$$

Here and below $\mu^h(\cdot) = x_r(\cdot) - w^h(\cdot)$. Setting in (1) and (4) $z = w^h(t)$ and $z = x_r(t)$, respectively, and summing the corresponding inequalities, we obtain

$$\begin{aligned} (\mu_t^h(t), \mu^h(t)) + \langle A\mu^h(t), \mu^h(t) \rangle &\leq (B(u_*(t) - v^h(t)), \mu^h(t)) - \\ &\omega^*(w^h(\tau_i) - \xi_i^h, \mu^h(t)) \quad \text{for a.a. } t \in \delta_i = [\tau_i, \tau_{i+1}). \end{aligned}$$

($\tau_i = \tau_{h,i}$, $\xi_i^h = \xi^h(\tau_{h,i})$). Hence, taking into account (2) we derive a.e. in δ_i

$$\begin{aligned} 1/2d|\mu^h(t)|_H^2/dt + c|\mu^h(t)|_V^2 - \omega|\mu^h(t)|_H^2 &\leq \\ (B(u_*(t) - v^h(t)), \mu^h(t)) + \omega^*(w^h(\tau_i) - \xi_i^h, \mu^h(t)). \end{aligned} \quad (16)$$

It is easy to see that for $t \in \delta_i$

$$\leq (h + \int_{\tau_i}^t \{|w_\tau^h(\tau)|_H + |x_{r\tau}(\tau)|_H\} d\tau) |\mu^h(t)|_H - |\mu^h(t)|_H^2. \quad (17)$$

In this case, for a.a. $t \in \delta_i$

$$1/2d|\mu^h(t)|_H^2/dt + c|\mu^h(t)|_V^2 \leq (B(u_*(t) - v^h(t)), \xi_i^h - w^h(\tau_i)) + \rho_i(t, h, \delta), \quad (18)$$

where $k_h = \omega^* + |B|(Q + d(h))$, $Q = |u_*(\cdot)|_{L_\infty(T;U)}$,

$$\rho_i(t, h, \delta) = k_h(h + \int_{\tau_i}^t \{|w_\tau^h(\tau)|_H + |x_{r\tau}(\tau)|_H\} d\tau).$$

Let $h_* > 0$ such that $d(h) > Q$ for $h \in (0, h_*)$. Hence, using the rule of definition of a control $v^h(\cdot)$, we obtain

$$\varepsilon_h(t) \leq \varepsilon(\tau_i) + 2(t - \tau_i)\rho_i(t, h, \delta), \quad t \in \delta_i.$$

Consequently, by Lemmas 1 and 2, the inequality

$$\varepsilon_h(t) \leq \varepsilon_h(t_0) + 2k_h(h(\vartheta - t_0) + \delta(h)) \int_{t_0}^t \{|w_\tau^h(\tau)|_H + |x_{r\tau}(\tau)|_H\} d\tau \leq k_1(1 + d(h))(h + \delta(h)) \quad t \in T$$

is valid. We deduce

$$\Lambda^0(x_r(\cdot), w^h(\cdot)) \leq k_1(1 + d(h))(h + \delta(h)) + \alpha(h)k_2(1 + d^2(h)), \quad (19)$$

$$|v^h(\cdot)|_{L_2(T;U)} \leq |u_*(\cdot)|_{L_2(T;U)} + k_1(\alpha^{-1}(h)(1 + d(h))(h + \delta(h)))^{1/2}. \quad (20)$$

Here, the constant k_1 does not depend on h and can be written explicitly. Thus, the family D_h is Λ^0 -stable. It is easy to see that the condition b) of Theorem 1 holds, if

$$y(\cdot) = x_r(\cdot), \quad (21)$$

where $y(\cdot) = x(\cdot; t_0, x_0, v(\cdot))$. Prove (21). Note, that

$$|x_r(\cdot) - w^{h_k}(\cdot)|_{C(T;H)} \rightarrow 0 \quad \text{as } k \rightarrow \infty, \quad (22)$$

if $h_k \rightarrow 0$. Here $w^{h_k}(\cdot) = w(\cdot; t_0, w^{h_k}(t_0), \xi^{h_k}(\cdot), v^{h_k}(\cdot))$, $v^{h_k}(\cdot) = D_{h_k}(\xi^{h_k}(\cdot))$. The inequality

$$\begin{aligned} & |y(t) - w^{h_k}(t)|_H^2 + 2 \int_{t_0}^t \{c|y(\tau) - w^{h_k}(\tau)|_V^2 - \omega|y(\tau) - w^{h_k}(\tau)|_H^2\} d\tau \leq \\ & 2 \int_{t_0}^t (B(v(\tau) - v^{h_k}(\tau)), y(\tau) - w^{h_k}(\tau))_H d\tau + \\ & |\omega^*| \max_{i \in [0:m_{h_k}]} |\xi_i^{h_k} - w^{h_k}(\tau_i)|_H \int_{t_0}^t |y(\tau) - w^{h_k}(\tau)|_H d\tau \end{aligned}$$

is true. Hence, taking into account (22), inclusion $\xi^{h_k}(\cdot) \in \Xi(x_r(\cdot), h_k)$, convergence h_k to 0, and weak convergence of controls $v^{h_k}(\cdot)$ to $v(\cdot)$ in $L_2(T;U)$ as $k \rightarrow \infty$, in virtue of Gronwall inequality, we obtain

$$|y(\cdot) - w^{h_k}(\cdot)|_{C(T;H)} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Therefore, relation (21) holds. The theorem follows from Theorem 1.

Let the following condition be fulfilled

Condition 2 The function φ is differentiable and operator $Cx = \text{grad}\varphi(x) : V \rightarrow V^*$ is Lipschitz.

Then the estimate of convergence rate of the algorithm is true.

Theorem 3 Let $U = V$, B be the operator of canonical embedding V in H and $u_*(\cdot) = u_*(\cdot; x_r(\cdot))$ be a function with bounded variation. Then the estimate

$$|u_*(\cdot) - v^h(\cdot)|_{L_2(T;H)} \leq K\{[d(h)(h + \delta(h)) + \alpha(h)d^2(h)]^{1/2} + \alpha^{-1}(h)d(h)(h + \delta(h))\}$$

holds.

The theorem is proved analogously to [15].

3. Case $u_*(\cdot; x_r(\cdot)) \in L_2(T; U)$

Consider the case when $u_*(\cdot; x_r(\cdot))$ is a norm-square integrable function, i.e., $u_*(\cdot; x_r(\cdot)) \in L_2(T; U)$. As the model M we take the variational inequality (4) with the family of operators $B_t(\cdot; \cdot) : U \rightarrow H$ of the form (14). The family $(\mathcal{W}_h)$ of t_0 -algorithms is defined as mentioned above, i.e., by rule (5), (11), the model control law $\mathcal{S}_h = (\Delta_h, \mathcal{U}_h)$ being defined by the rule (6), (7), where we assume

$$\mathcal{U}_h(\tau_i, \xi_i^h, w^h(\tau_i)) = \arg \min \{l(\alpha, v, s_i) : v \in U\} = -\alpha^{-1} B' s_i, \quad (23)$$

$s_i = w^h(\tau_i) - \xi_i^h$. Let a family of partitions (Δ_h) of the interval T , and function $\alpha(h) : R^+ \rightarrow R^+$ satisfying the following conditions:

$$\begin{aligned} h\delta^{-1}(h) \leq C, \quad \delta(h)\alpha^{-2}(h) \leq C, \\ \alpha(h) \rightarrow 0, \quad \delta(h) \rightarrow 0, \quad (h + \delta(h))/\alpha(h) \rightarrow 0 \quad \text{as } h \rightarrow 0+. \end{aligned} \quad (24)$$

Here $C = \text{const} > 0$ that does not depend on h .

Theorem 4 *The family of positional modeling algorithms D_h (8) of the form (4)–(7), (11), (14), (23) satisfies conditions of Theorem 1 and is regularizing.*

Proof of theorem is carried out by the scheme of that for Theorem 1. Taking into account (2) we derive a.e. in δ_i (16). It is also easy to see that for $t \in \delta_i$ inequality (17) holds along with

$$\begin{aligned} (B(u_*(t) - v^h(t), \mu^h(t)) \leq (B(u_*(t) - v^h(t)), \xi_i^h - w^h(\tau_i)) + \\ + |B|\{|u_*(t)| + |v^h(t)|\}(h + \int_{\tau_i}^t g^h(\tau) d\tau), \end{aligned}$$

In this case, for a.a. $t \in \delta_i$ estimate (18) holds, provided

$$\begin{aligned} \rho_i(t, h, \delta) = \{\omega^* + |B|[\|u_*(t)\|_U + |v^h(t)|_U]\}(h + \int_{\tau_i}^t g^h(\tau) d\tau), \\ g^h(\tau) = |w_\tau^h(\tau)|_H + |x_{r\tau}(\tau)|_H. \end{aligned}$$

Hence, using the rule of definition of a control $v^h(\cdot)$ (23), we obtain

$$\begin{aligned} \varepsilon_h(t) \leq \varepsilon_h(\tau_i) + \omega^* \delta h + \delta \omega^* \int_{\tau_i}^t g^h(\tau) d\tau + h^2 |B|^2 + \\ 3\delta \int_{\tau_i}^t F^h(\tau) d\tau + 2\delta^2 (1 + |B|^2) \int_{\tau_i}^t \{|w_\tau^h(\tau)|_H^2 + |x_{r\tau}(\tau)|_H^2\} d\tau, \end{aligned} \quad (25)$$

where $F^h(\tau) = |u_*(\tau)|_U^2 + |v^h(\tau)|_U^2$. Summing right and left hand of the inequality (25) over i and taking into account Lemmas 1 and 2, the

inclusion $u_*(\cdot) \in L_2(T; U)$, we deduce

$$\begin{aligned} \varepsilon_h(t) &\leq \varepsilon_h(t_0) + b_1 h(1 + h/\delta) + b_2 \delta + b_3 \delta \int_{t_0}^t F^h(\tau) d\tau \leq \\ &\varepsilon_h(t_0) + b_1 h(1 + h/\delta) + b_4 \delta + b_3 \delta^2 \sum_{j=0}^{i(t)} |v_j^h|^2, \end{aligned} \quad (26)$$

where the symbol $i(t)$ denotes an integer part of number t . Besides, by the rule of definition of v_i^h (see (23)), we have

$$|v_i^h|_U^2 \leq 2|B|^2(\mu_i^h + h^2)\alpha^{-2}, \quad (27)$$

where $\mu_i^h = \mu^h(\tau_i) = x_r(\tau_i) - w^h(\tau_i)$. From (26), (27) and inequalities $\varepsilon_h(t_0) \leq 4h^2$, $h\delta^{-1}(h) \leq C$, follows the estimation

$$\begin{aligned} \mu_i^h &\leq \varepsilon_h(t_0) + b_1 h(1 + h/\delta) + b_3 \delta + \alpha |u_*(\cdot)|_{L_2(T; U)}^2 + \\ &b_2 \delta^2 \sum_{j=0}^{i-1} 2|B|^2(\mu_j^h + h^2)\alpha^{-2} \leq b_5(h + \delta + \alpha) + b_6 \delta^2 \alpha^{-2} \sum_{j=0}^{i-1} \mu_j^h. \end{aligned}$$

Taking into account Gronwall inequality, and inequality $\delta(h)\alpha^{-2}(h) \leq C$, we deduce

$$\mu_i^h \leq b_5(h + \delta + \alpha) \exp\{b_6(\vartheta - t_0)\delta/\alpha^2\} \leq b_7(h + \delta + \alpha). \quad (28)$$

Summing left hand of the inequality (27) over i , we obtain from (28)

$$\delta^2 \sum_{j=0}^{m_h-1} |v_j^h|^2 \leq 2|B|^2 \sum_{j=0}^{m_h-1} (\mu_j^h + h^2)\alpha^{-2} \leq b_8 \delta \alpha^{-2} (\alpha + h + \delta). \quad (29)$$

Due to (24) without commonness loss, suppose $\delta(h)\alpha^{-1}(h) \leq 1$. Therefore, from (26), (29) we derive the estimation

$$\varepsilon_h(t) \leq b_7(h + \delta + \delta^2 \alpha^{-2} + h\delta \alpha^{-2}) \leq b_8(h + \delta). \quad (30)$$

Therefore $|v^h(\cdot)|_{L_2(T; U)}^2 \leq |u_*(\cdot)|_{L_2(T; U)}^2 + b_8(h + \delta)\alpha^{-1}$. This means that the inequality (9) takes place if we have $k_1(h) = 1$, $k_2(h) = (b_8(h + \delta(h))\alpha^{-1}(h))^{1/2}$. From (30) we deduce (10), where functional $\Lambda^0(\cdot, \cdot)$ is defined by (15) and $k_3(h) = b_9(h + \delta(h) + \alpha(h))$. The theorem is proved.

The next theorem is proved analogously to Theorem 3:

Theorem 5 *Let conditions of Theorem 3 hold. Then the estimate*

$$|u_*(\cdot) - v^h(\cdot)|_{L_2(T; H)} \leq K\{(h + \delta(h))^{1/2} + (h + \delta(h))\alpha^{-1}(h)\}$$

is true.

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