NONCONSERVATIVE SCHRÖDINGER EQUATIONS WITH UNOBSERVED NEUMANN B.C.: GLOBAL UNIQUENESS AND OBSERVABILITY IN ONE SHOT*

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1. Introduction. Problem statement

1.1. Problem statement. Assumptions

Let Ω be an open bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega = \Gamma$ of class C^2 , consisting of the closure of two disjoint parts: Γ_0 (uncontrolled or unobserved part) and Γ_1 (controlled or observed part), both relatively open in Γ : $\partial\Omega = \Gamma \equiv \overline{\Gamma_0 \cup \Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. In this paper, we consider the following Schrödinger equation in the (complex-valued) unknown $w(t, x)$ defined on Ω :

$$\mathcal{P}w \equiv iw_t + \Delta w = F(w) + f \quad \text{in } Q \equiv (0, T] \times \Omega. \quad (1.1.1)$$

In (1.1.1), we have set

$$F(w) \equiv q_1(t, x) \cdot \nabla w(t, x) + q_0(t, x)w(t, x), \quad (1.1.2)$$

subject to the following preliminary *standing assumption* on the coefficients: We let $|q_1|, q_0 \in L_\infty(Q)$, so that the following pointwise estimate

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holds true:

$$|F(w)|^2 \leq C_T[|\nabla w|^2 + |w|^2], \quad \forall (t, x) \in Q. \quad (1.1.3)$$

We assume throughout that the non-homogeneous term f satisfies

$$f \in L_2(Q). \quad (1.1.4)$$

This is an announcement of results in the full-length paper [8].

Main assumptions. The main focus of the present paper refers to the case where Eqn. (1.1.1) is supplemented by *purely Neumann* B.C.: $\frac{\partial w}{\partial \nu}|_{\Sigma} \equiv 0$, while observation or control takes place only on a *subportion* Γ_1 of the boundary Γ . This problem is of interest on both physical grounds (it arises in the structural acoustic problem) and on mathematical grounds (the Lopatinski's condition is not satisfied). For this key case, we shall need the following assumptions just as in [7] in the corresponding wave equation case. In addition to the standing hypotheses (1.1.3) on $F(w)$ and (1.1.4) on f , the following assumptions are postulated throughout.

(A.1) Given the triple $\{\Omega, \Gamma_0, \Gamma_1\}$, $\partial\Omega = \overline{\Gamma_0 \cup \Gamma_1}$, there exists a strictly convex (real-valued) non-negative function $d : \overline{\Omega} \rightarrow \mathbb{R}^+$, of class $C^3(\overline{\Omega})$, such that, if we introduce the (conservative) vector field $h(x) \equiv \nabla d(x)$, $x \in \Omega$, then the following two properties hold true:

(i)

$$\left. \frac{\partial d}{\partial \nu} \right|_{\Gamma_0} = \nabla d \cdot \nu = h \cdot \nu = 0 \text{ on } \Gamma_0; \quad h \equiv \nabla d. \quad (1.1.5_N)$$

(ii) the (symmetric) Hessian matrix $\mathcal{H}_d$ of $d(x)$ [i.e., the Jacobian matrix J_h of $h(x)$] is strictly positive definite on $\overline{\Omega}$: there exists a constant $\rho > 0$ such that for all $x \in \overline{\Omega}$:

$$\mathcal{H}_d(x) = J_h(x) = [d_{x_i x_j}] = \left[\frac{\partial h_i}{\partial x_j} \right] \geq \rho I, \quad i, j = 1, \dots, n. \quad (1.1.6)$$

(A.2) A working assumption, which can be relaxed (see Remark 1.1.1), is that $d(x)$ has no critical point on $\overline{\Omega}$:

$$\inf_{x \in \Omega} |h(x)| = \inf_{x \in \Omega} |\nabla d(x)| = p > 0. \quad (1.1.7)$$

Remark 1.1.1. Assumption (A.2) can, in fact, be removed as in [7, Sect. 10], [19, Sec. 10], [8, Sect. 10], by splitting Ω as $\Omega = \Omega_1 \cup \Omega_2$, for two suitable overlapping sets Ω_1 and Ω_2 , and working with two strictly convex functions d_1 and d_2 . Since the full statement of results without

(A.2) requires a rather lengthy preparatory background, we shall here retain assumption (A.2) and refer to the above references for its removal.

Remark 1.1.2. (Neumann) Assumption (1.1.5) is due to the purely Neumann B.C. $\frac{\partial w}{\partial \nu}|_{\Sigma} \equiv 0$, as in the case of the wave equation [7, Remark 1.1.2]. It was introduced in [16, Section 5]. Reference [7, Appendices A-C] provides, by different mathematical techniques, *several classes of triples* $\{\Omega, \Gamma_0, \Gamma_1\}$ in $\mathbb{R}^n$, $n \geq 2$, where assumptions (A.1) and (A.2) are satisfied. (In light of Remark 1.1.1, only assumption (A.1) is the critical one.) For instance, just to quote one general result: one can construct *explicitly* [7, Theorem A.4.1, p. 301] the required strictly convex function $d(x)$ satisfying (A.1) if the (Euclidean) bounded domain $\Omega \in \mathbb{R}^n$ is (i) convex (respectively, concave) on the side of the portion Γ_0 of its boundary, and (ii) there exists a radial vector field $(x - x_0)$ for some $x_0 \in \mathbb{R}^n$ which is entering (respectively, exiting) Ω through Γ_0 . Moreover, [7, Corollary A.4.2, p. 306] the previously constructed strictly convex function $d(x)$ has the following additional property: its gradient $\nabla d|_{\Gamma_0}$, once restricted on the portion Γ_0 of the boundary, vanishes at the unique point $x \in \Gamma_0$, if such exists on Γ_0 , where the vector field $x - x_0$ is orthogonal to Γ_0 . The above condition on the existence of such $d(x)$ is only sufficient. It holds true also in the case where Ω is a bounded set of a finite-dimensional Riemann manifold [19, Appendix B]. Other classes are given in [7, Appendices A-C] satisfying assumption (A.1): for instance, where the portion Γ_0 of the boundary is logarithmic concave [7, Lemma A.2.2, p. 294].

(Dirichlet) Even though the purely Neumann B.C. case: $\frac{\partial w}{\partial \nu}|_{\Sigma} \equiv 0$ will be the central focus of this paper, our treatment will allow us to also include the purely Dirichlet B.C. case: $w|_{\Sigma} \equiv 0$. Here, however, hypothesis (1.1.5_N) can be dispensed with. It will be replaced by the much weaker condition

$$h \cdot \nu \leq 0 \quad \text{on } \Gamma_0. \quad (1.1.5_D)$$

A specific example is $d(x) = \frac{1}{2}\|x - x_0\|^2$, with x_0 outside Ω , where then $h(x) = \nabla d(x) = (x - x_0)$ is radial. [A combination of Dirichlet/Neumann B.C. is also included in our treatment.]

Remark 1.1.3. We expect the techniques and results of the present note (paper [8]) to be extended to the fully general Euclidean case, where the Euclidean Laplacian operator Δ in (1.1.1) is replaced by a second-order elliptic operator with *variable coefficients* (in space), of class, say, $C^2(\overline{\Omega})$; or, more generally, by the Laplace-Beltrami operator Δ_g , in case Ω is a bounded set of a finite-dimensional Riemann manifold $\{M, g\}$, with metric g . This extension would be accomplished, in the present

Schrödinger case, in the same way as the Euclidean wave equation paper [7] (respectively, [4]) was extended to the Riemann wave equation in [19] (no lower-order terms in the estimates) (respectively, [5], with lower-order terms in the estimates); or the Euclidean Schrödinger equation paper [17] was extended to the Riemann Schrödinger equation in [18] (with lower-order terms in the estimates). That is, by replacing the Euclidean metric of the present paper with the appropriate Riemann metric.

At any rate, throughout Section 2.1, we shall merely deal with smooth solutions of Eqn. (1.1.1) with no B.C. imposed, subject *only* to hypotheses (1.1.6) and (1.1.7). Then, assumption (1.1.5_N): $h \cdot \nu \equiv 0$ on Γ_0 [resp. (1.1.5_D): $h \cdot \nu \leq 0$ on Γ_0] will be introduced only when analyzing purely Neumann B.C. [resp. purely Dirichlet B.C.].

Pseudo-convex function $\varphi(x, t)$. Having chosen, by assumption (A.1), a strictly convex potential function $d(x) \geq 0$, we next introduce the pseudo-convex function $\varphi : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ of class C^3 :

$$\varphi(x, t) = d(x) - c \left(t - \frac{T}{2} \right)^2; \quad 0 \leq t \leq T, \quad x \in \Omega, \quad (1.1.8a)$$

where $T > 0$ is arbitrary, and where then $c = c_T$ is chosen large enough as to have

$$cT^2 > 4 \max_{x \in \overline{\Omega}} d(x), \text{ so that } cT^2 > 4 \max_{x \in \overline{\Omega}} d(x) + 4\delta \quad (1.1.8b)$$

for a suitably small $\delta > 0$, henceforth kept fixed. Unless otherwise explicitly noted, $\varphi(x, t)$ is selected as described above and kept fixed henceforth. Such function $\varphi(x, t)$ has the following properties:

(a) for the constant $\delta > 0$, fixed in (1.1.8b), we have

$$\varphi(x, 0) \equiv \varphi(x, T) = d(x) - c \frac{T^2}{4} \leq -\delta, \text{ uniformly in } x \in \Omega; \quad (1.1.9)$$

(b) there are t_0 and t_1 , with $0 < t_0 < \frac{T}{2} < t_1 < T$, such that

$$\min_{x \in \overline{\Omega}, t \in [t_0, t_1]} \varphi(x, t) \geq -\frac{\delta}{2}, \quad (1.1.10)$$

since $\varphi(x, \frac{T}{2}) = d(x) \geq 0$ for all $x \in \Omega$ (in fact, only the weaker property: $\min \varphi(x, t) \geq \sigma > -\delta$ is actually needed).

Throughout this paper, we set

$$E(t) \equiv \int_{\Omega} |\nabla w(t)|^2 d\Omega;$$

$$\mathbb{E}(t) = \int_{\Omega} [|\nabla w(t)|^2 + |w(t)|^2] d\Omega = \|w(t)\|_{H^1(\Omega)}^2. \quad (1.1.11)$$

Goals. As already mentioned, we consider, at first, sufficiently smooth solutions $w(t, x)$ of the Schrödinger Eqn. (1.1.1).

First goal. Then, our *first* goal is to establish Carleman-type inequalities with w in $H^1(\Omega)$ —basic energy level—for these solutions *without lower-order terms*. Carleman inequalities, however, with (interior) lower-order terms were obtained in [17, Theorems 2.1.1 and 2.1.2, pp. 464–466] for the Schrödinger Eqn. (1.1.1), and in [18, Theorems 3.3 and 3.4, pp. 640–641] for Eqn. (1.1.1) with the (Euclidean) Laplacian Δ replaced by a variable coefficient (in space) uniformly elliptic operator; or, with essentially the same effort, for the Schrödinger Eqn. (1.1.1) with the Euclidean Laplacian Δ replaced by the Laplace-Beltrami operator Δ_g , defined on a bounded set $\Omega \subset M$ with boundary, of a Riemann manifold $\{M, g\}$. In the present work, as well as in the prior references [17] and [18], the boundary terms (traces of w) of the solutions w of Eqn. (1.1.1) which appear in the Carleman estimates are given *explicitly*.

Second goal. As a consequence of Carleman estimates without lower-order term and explicit boundary terms (our first goal), we then achieve our second goal: that is, we obtain global uniqueness results as well as continuous observability/uniform stabilization inequalities, with w in the basic energy level $H^1(\Omega)$, in one shot, as part of the same flow of arguments. This is in contrast with prior Carleman estimates, hence continuous observability/uniform stabilization inequalities polluted by lower-order terms as in [12], [18] and, for $F(w) \equiv 0$, in [3]. In this latter case, a first disadvantage is the necessity to require an independent global uniqueness result in order to absorb, and hence eliminate, the lower-order term from the sought-after estimates. Moreover, a second disadvantage is the lack of control on the constants arising in the final estimates, as the aforementioned absorption process proceeds by contradiction.

In short, the key aim of the present paper is to eliminate lower-order terms from the sought-after continuous observability/uniform stabilization estimates, therefore avoiding the two disadvantages cited above.

Third goal. Our third goal is to obtain lower-level energy estimate (more precisely, w in $L_2(\Omega)$ rather than in $H^1(\Omega)$), such as they are needed in the problem of uniform stabilization with $L_2(0, \infty; L_2(\Gamma))$ -Dirichlet feedback control. See Section 2.5.

Literature. As to the literature, we note that prior works on Schrödinger equations [3],[11],[10],[12]–[13],[17],[1],[18] under various B.C. and degree of generality obtain the continuous observability/uniform stabili-

zation inequalities *polluted by lower-order terms*. In fact this is the case for almost all papers on exact controllability/uniform stabilization of the literature, regardless of the specific evolution equation. To our knowledge, the only exceptions where lower-order terms do not appear in the observability/stabilization estimates are [2],[7] for second-order hyperbolic equations with Δ as principal part; [19] for Δ_g as principal part; and [15] for general evolution equations mostly in the case of Dirichlet B.C. The present paper ([8]) – and its successor with Δ replaced by Δ_g – emphasizes the purely Neumann B.C. case of (1.1.1), as is the case for [7].

2. Main results under (A.1) and (A.2)

2.1. Carleman estimates without lower-order terms for $H^{2,2}(Q)$ -solutions

Theorem 2.1.1. (First version) *Let $T > 0$ arbitrary and let $c = c_T$ be defined by (1.1.8b). Let $d(x) \in C^3(\overline{\Omega})$ be the non-negative, real, strictly convex function satisfying assumptions (A.1(ii)) = (1.1.6) and (A.2) = (1.1.7). Define $\varphi(x, t)$ by (1.1.8). Let w be a solution of Eqn. (1.1.1) [with no boundary conditions imposed] in the class:*

$$w \in H^{2,2}(Q) \equiv L_2(0, T; H^2(\Omega)) \cap H^2(0, T; L_2(\Omega)), \quad (2.1.1)$$

where (1.1.1) is subject to the standing assumption (1.1.3) for $F(w)$ and (1.1.4) for f . Then, for all τ sufficiently large, the following one-parameter family of estimates holds true:

$$\begin{aligned} B_\Sigma(w) + 4 \int_0^T \int_\Omega e^{2\tau\varphi} |f|^2 d\Omega dt \\ \geq \left[4\tau\rho - \frac{1}{2} - 4C_T \right] \int_0^T \int_\Omega e^{2\tau\varphi} |\nabla w|^2 d\Omega dt \\ + [4\tau^3 \rho p + \mathcal{O}(\tau^2) - 4C_T] \int_0^T \int_\Omega e^{2\tau\varphi} |w|^2 d\Omega dt \\ - c_{d,T} \tau e^{-2\tau\delta} [\mathbb{E}(T) + \mathbb{E}(0)] \\ \geq [k_\tau^2] e^{-\delta\tau} \int_{t_0}^{t_1} \mathbb{E}(t) dt - c_\varphi \tau e^{-2\tau\delta} [\mathbb{E}(T) + \mathbb{E}(0)], \end{aligned} \quad (2.1.2)$$

where $\rho > 0$, $p > 0$, $\delta > 0$ are defined by (1.1.6), (1.1.7), (1.1.8b) and k_τ^2 is a positive constant $\geq k_{\tau_0}^2$, $\forall \tau \geq \tau_0 > 0$. Moreover, $\mathbb{E}(t)$ is defined by (1.1.11), while t_0, t_1 are as in (1.1.10). Finally, setting $h \equiv \nabla d$ as in (1.1.5_N), then the boundary terms $B_\Sigma(w)$ are given explicitly as follows,

where $\xi = \operatorname{Re} w$, $\eta = \operatorname{Im} w$:

$$\begin{aligned}
 B_{\Sigma}(w) &= 2\tau \int_0^T \int_{\Gamma} e^{2\tau\varphi} [2\tau^2|h|^2]|w|^2 h \cdot \nu \, d\Gamma \, dt \\
 &\quad - 4c\tau \int_0^T \int_{\Gamma} e^{2\tau\varphi} \left(t - \frac{T}{2}\right) \left[\eta \frac{\partial \xi}{\partial \nu} - \xi \frac{\partial \eta}{\partial \nu}\right] d\Gamma \, dt \\
 &\quad - 2\tau \int_0^T \int_{\Gamma} e^{2\tau\varphi} [\xi_t \eta - \xi \eta_t] h \cdot \nu \, d\Gamma \, dt \\
 &\quad + \int_0^T \int_{\Gamma} e^{2\tau\varphi} [2\tau^2|h|^2 - \tau \Delta d] \left[\bar{w} \frac{\partial w}{\partial \nu} + w \frac{\partial \bar{w}}{\partial \nu}\right] d\Gamma \, dt \\
 &\quad + 2\tau \int_0^T \int_{\Gamma} e^{2\tau\varphi} h \cdot \left[\nabla \bar{w} \frac{\partial w}{\partial \nu} + \nabla w \frac{\partial \bar{w}}{\partial \nu}\right] d\Gamma \, dt \\
 &\quad - 2\tau \int_0^T \int_{\Gamma} e^{2\tau\varphi} |\nabla w|^2 h \cdot \nu \, d\Gamma \, dt.
 \end{aligned} \tag{2.1.3a}$$

$$\begin{aligned}
 B_{\Sigma}(w) &\leq C_{\tau} \left\{ \int_0^T \int_{\Gamma} e^{2\tau\varphi} \left[|w|^2 + \left|\frac{\partial w}{\partial \nu}\right|^2\right] d\Sigma \right. \\
 &\quad \left. + \int_0^T \left[\|w_t\|_{H^{-1}(\Gamma)}^2 + \|w\|_{H^1(\Gamma)}^2\right] dt. \right.
 \end{aligned} \tag{2.1.3b}$$

Theorem 2.1.1 is proved in [8, Sections 3 through 5].

Remark 2.1.1. The Carleman estimate (2.1.2) of Theorem 2.1.1 is essentially the one in [17, Theorem 2.1.1, p. 464] later generalized in [18, Theorem 3.3, p. 640] to the case where the Euclidean Laplacian Δ is replaced by the Laplace-Beltrami operator Δ_g on a Riemann manifold $\{M, g\}$, *except for the critical improvement* that our present version (2.1.2) does *not* include an interior lower-order term, unlike the estimates of the aforementioned references [17], [18].

As in [17], [18], in order to refine Theorem 2.1.1, we need to specialize the first-order differential operator $F(w)$ by imposing a *structural* property, as stated by the following assumption:

(A.3) Let $F(w)$ in (1.1.2) be specialized as follows:

$$\begin{cases} q_1(t, x) = \nabla \pi(t, x) - ir_1(t, x), & \text{for a real-valued scalar function} \\ & \pi(t, x) \text{ and a real-valued vector function } r_1(t, x), \\ r_1 \in L_{\infty}(0, T; [H^1(\Omega)]^n), \quad q_0 \in L_{\infty}(0, \infty; H^1(\Omega)). \end{cases} \tag{2.1.4}$$

[At the price of a change of variable, one may take $\pi(t, x) \equiv 0$. Then, the structural property $iq_1 = r_1$ (real) is *critical*, but we are not attempting to extract the minimal possible regularity of r_1 and q_0 .]

The use of assumption (A.3) = (2.1.4) is seen in [8, Lemma 6.1(ii),(iv), Eqn. (6.7), (6.9), estimate (6.12) and Remark 6.1]. Such assumption, which is related to well-posedness, will permit us to obtain a second version, more refined, of the Carleman estimate (2.1.2) of Theorem 2.1.1.

Theorem 2.1.3. (Second version) *Assume the setting (in particular, (1.1.6), (1.1.7)), and the notation of Theorem 2.1.1. In addition, we assume hypothesis (A.3) = (2.1.4) and that $f \in L_2(0, T; H^1(\Omega))$. Let w be a solution of Eqn. (1.1.1) in the class $H^{2,2}(Q)$ in (2.1.1). Then, for all $\tau > 0$ sufficiently large, the following one-parameter family of estimates holds true, where $k_{\varphi, \tau} > 0$,*

$$\begin{aligned} & \tilde{B}_\Sigma(w) + 4 \int_0^T \int_\Omega e^{2\tau\varphi} |f|^2 d\Omega dt + C_{p,q,\rho,\tau} \|f\|_{L_2(0,T;H^1(\Omega))}^2 \\ & \geq \left\{ \left[4\tau\rho - \frac{1}{2} - 4C_T \right] e^{-\delta\tau} \frac{(t_1 - t_0)}{2} e^{-C_T T - c_{d,T}\tau} e^{-2\tau\delta} \right\} [\mathbb{E}(T) + \mathbb{E}(0)] \quad (2.1.5) \\ & \geq k_{\varphi,\tau} [\mathbb{E}(T) + \mathbb{E}(0)] \geq k_{\varphi,\tau} [E(T) + E(0)], \quad (2.1.6) \end{aligned}$$

with explicit constant which is noted in [8, Eqn. (6.21)]. $\mathbb{E}(t)$ and $E(t)$ are defined in (1.1.11). Moreover, the boundary terms $B_\Sigma(w)$ in (2.1.5) are given by

$$\begin{aligned} \tilde{B}_\Sigma(w) = & B_\Sigma(w) + C_{\varphi,p,q} \left\{ \int_0^T \int_\Gamma \left| \frac{\partial w}{\partial \nu} \right| \left[|w_t| + \left| \frac{\partial w}{\partial \mu} \right| |r_1 \cdot \mu| \right. \right. \\ & + \left. \left| \frac{\partial w}{\partial \nu} \right| |r_1 \cdot \nu| + |q_0 w| + |w| + |f| \right] d\Gamma dt \\ & + \left. \int_0^T \int_\Gamma |\nabla w|^2 |r_1 \cdot \nu| d\Gamma dt \right\} + \int_0^T \int_{\Gamma_1} |w|^2 d\Gamma_1 dt, \quad (2.1.7) \end{aligned}$$

with constant $C_{\varphi,p,q}$ given explicitly in the left side of [8, (6.35)], where $\frac{\partial}{\partial \mu}$ denotes the tangential derivative, μ is a unit tangent vector. [Moreover, the last boundary term $\int_0^T \int_{\Gamma_1}$ in (2.1.7) may be omitted, if it is known that w vanishes on a portion of the boundary of positive measure, so that Poincaré inequality holds true.]

Theorem 2.1.2 is proved in [8, Section 6] (for $H^{2,2}(Q)$ -solutions).

2.2. Extension of Carleman estimates to finite energy solutions

Theorem 2.2.1. *Assume the hypotheses of Theorem 2.1.2: (1.1.3), (A.3) = (2.1.4) for $F(w)$, $f \in L_2(0, T; H^1(\Omega))$, in addition to (A.1(ii)) = (1.1.6) and (A.2) = (1.1.7). Then, estimate (2.1.2) of Theorem 2.1.1*

and estimate (2.1.6) of Theorem 2.1.2 can be extended to finite energy solutions in the class

$$w \in C([0, T]; H^1(\Omega)); \quad \frac{\partial w}{\partial \nu} \in L_2(0, T; L_2(\Gamma)), \quad w_t \in L_2(0, T; H^{-1}(\Gamma)), \quad (2.2.1)$$

provided that the (unbounded) term $F(w) = -ir_1 \cdot \nabla w + q_0 w$ satisfies the following additional hypothesis:

(A.4) (this is only for convenience)

$$\text{the coefficients } r_1 \text{ and } q_0 \text{ are time-independent;} \quad (2.2.2)$$

The proof of Theorem 2.2.1 is given in [8, Section 7.2].

2.3. Continuous observability. Global uniqueness. Dirichlet B.C.

Purely Dirichlet problem. Here we consider the following problem:

$$\begin{cases} iw_t + \Delta w = F(w) + f & \text{in } (0, T] \times \Omega \equiv Q; \\ w(0, \cdot) = w_0 & \text{in } \Omega; \\ w|_{\Sigma} = 0 & \text{in } (0, T] \times \Gamma = \Sigma, \end{cases} \quad \begin{array}{l} (2.3.1a) \\ (2.3.1b) \\ (2.3.1c) \end{array}$$

In the case of Dirichlet B.C. the extension Theorem 2.2.1 is *not* needed. Indeed the extension of the final sought-after continuous observability inequality (2.3.2) below can be readily accomplished from $H^{2,2}(Q)$ -solutions to $H^{1,1}(Q)$ -solutions just by virtue of the regularity theorem ('reverse inequality') obtained in [3] and reported also in [4]: the Neumann trace $\frac{\partial w}{\partial \nu}$ is dominated by the $H^1(\Omega)$ -norm of the I.C. w_0 . Thus the additional assumption (2.2.2) of Theorem 2.2.1 may be dispensed with. We obtain

Theorem 2.3.1. *Let w be a solution of problem (2.3.1) with I.C. $w_0 \in H_0^1(\Omega)$, and with $f \in L_2(0, T; H^1(\Omega))$, under the standing assumption (1.1.3) on $F(w)$, as well as (A.3) = (2.1.4). Assume, further, hypotheses (A.1)(ii) = (1.1.6), (A.2) = (1.1.7), for $d(x)$, see Remark 1.1.2. Define now Γ_0 as in (1.1.5_D), i.e., by: $h \cdot \nu \leq 0$, $h = \nabla d$, on Γ_0 . Let $\Gamma_1 = \Gamma \setminus \Gamma_0$. Let $T > 0$ be arbitrary. Then:*

(a) *there exists a constant $c_T > 0$, such that the following continuous observability inequality holds true:*

$$c_T E(0) \leq \int_0^T \int_{\Gamma_1} \left| \frac{\partial w}{\partial \nu} \right|^2 d\Gamma_1 dt + \|f\|_{L_2(0, T; H^1(\Omega))}^2. \quad (2.3.2)$$

The constant c_T is explicit, see [8].

(b) Let now $f = 0$ in (2.3.1). Then, the following global uniqueness result holds true: with $T > 0$, let w be a $H^{1,1}(Q)$ -solution of Eqn. (2.3.1a) and over-determined B.C.'s:

$$w|_{\Sigma} \equiv 0 \quad \text{and} \quad \frac{\partial w}{\partial \nu} \Big|_{\Sigma_1} = 0, \quad \text{where } h \cdot \nu \leq 0, \quad \text{on } \Gamma_0 = \Gamma \setminus \Gamma_1, \quad (2.3.3)$$

$\Sigma_1 = (0, T] \times \Gamma_1$. Then, in fact, $w \equiv 0$ in Q , indeed in $\mathbb{R}_t \times \Omega$.

The proof of Theorem 2.3.1 is given in [8, Section 8].

2.4. Continuous observability. Global uniqueness. Neumann B.C.

Purely Neumann problem. Here we consider the following problem:

$$\begin{cases} iw_t + \Delta w = F(w) + f & \text{in } (0, T] \times \Omega \equiv Q; \\ w(0, \cdot) = w_0 & \text{in } \Omega; \\ \frac{\partial w}{\partial \nu} \Big|_{\Sigma} = 0 & \text{in } (0, T] \times \Gamma \equiv \Sigma, \end{cases} \quad \begin{matrix} (2.4.1a) \\ (2.4.1b) \\ (2.4.1c) \end{matrix}$$

As explained in more detail in [8, Section 7], it is for the Neumann B.C. case that the extension Theorem 2.2.1 is called for. Accordingly, we inherit its additional assumption (2.2.2).

Theorem 2.4.1. *Let w be the solution of problem (2.4.1) with I.C. $w_0 \in H^1(\Omega)$, and with $f \in L_2(0, T; H^1(\Omega))$, under the standing assumption (1.1.3) on $F(w)$. Assume, further, hypotheses (A.1), (A.2), (A.3) [that is, (1.1.5_N), (1.1.6), (1.1.7), (2.1.4)], as well as the additional hypotheses as (A.4) = (2.2.2). Let $\Gamma_1 = \Gamma \setminus \Gamma_0$, where $h \cdot \nu = 0$ on Γ_0 as defined by (1.1.5_N), and let $T > 0$ be arbitrary. Then:*

(a) *there exists a constant $c_T > 0$, such that the following continuous observability inequality holds true:*

$$c_T E(0) \leq \int_0^T \int_{\Gamma_1} [|w|^2 + |w_t|^2] d\Gamma_1 dt + \|f\|_{L_2(0, T; H^1(\Omega))}^2. \quad (2.4.2)$$

(b) Let now $f = 0$ in (2.4.1). Then, the following global uniqueness result holds true: with $T > 0$, let w be a $H^{1,1}(Q)$ -solution of Eqn. (2.4.1a) and over-determined B.C.'s:

$$\frac{\partial w}{\partial \nu} \Big|_{\Sigma} \equiv 0 \quad \text{and} \quad w|_{\Sigma_1} \equiv 0, \quad \text{where } h \cdot \nu = 0 \quad \text{on } \Gamma_0 = \Gamma \setminus \Gamma_1, \quad (2.4.3)$$

$\Sigma_1 = (0, T] \times \Gamma_1$. Then, in fact, $w \equiv 0$ in Q , indeed in $\mathbb{R}_t \times \Omega$.

Indeed, [8] proves first the global uniqueness statement of part (b) of Theorem 2.4.1 in Section 8, as a direct consequence of the Carleman

estimate without lower-order terms of Theorem 2.1.2, first for smooth solutions and next extended to $H^{1,1}(Q)$ -solutions as in Theorem 2.2.1. Next, part (b) is used to establish part (a), in [8, Theorem 9.2 of Section 9] by virtue also of the trace Lemma 9.1 in [8].

Remark 2.4.1. Theorem 2.1.1 also yields a global uniqueness result in the purely Neumann case for solutions in the class (2.1.1).

2.5. Lower-level energy estimates

In this section, we formulate an estimate at a lower level of energy, that is, at the $L_2(\Omega)$ -level for w . To this end, we introduce anisotropic Sobolev spaces:

$$H_a^s(\Sigma) \equiv H^{s,s/2}(\Sigma) \equiv L_2(0, T; H^s(\Gamma)) \cap H^{s/2}(0, T; L_2(\Gamma)),$$

the latter Sobolev spaces being classical time-space spaces defined in [10, vol. II]. Then, $H_a^{-1}(\Sigma)$ is the dual space to $H_a^1(\Sigma)$, with respect to $L_2(\Sigma)$ as a pivot/space. We then have in the prior notation:

Theorem 2.5.1. *Let w be a sufficiently smooth solution of the Schrödinger equation (1.1.1) with $f \in L_2(Q)$. Then, for any $T > 0$, the following inequality holds true: there exists a constant C_T such that*

$$\begin{aligned} & \int_0^T \left[\|w\|_{L_2(\Omega)}^2 + \|w_t\|_{H^{-2}(\Omega)}^2 \right] dt + \|w(0)\|_{L_2(\Omega)}^2 + \|w_t(0)\|_{H^{-2}(\Omega)}^2 \\ & \leq C_T \left\{ \|w\|_{L_2(\Sigma_1)}^2 + \left\| \frac{\partial w}{\partial \nu} \right\|_{H_a^{-1}(\Sigma)}^2 + \|w\|_{H^{-1}(Q)}^2 + \|f\|_{L_2(Q)}^2 \right\}. \end{aligned}$$

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