ON THE SOBOLEV BOUNDARY VALUE PROBLEM WITH SINGULAR AND REGULARIZED BOUNDARY CONDITIONS FOR ELLIPTIC EQUATIONS

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Abstract In the domain G bounded by the exterior $(n-1)$ -dimensional boundary Γ_0 and interior smooth n_k -dimensional $(0 \leq n_k \leq n-1)$ manifold Γ_k ($k = 1, \dots, \chi$) without bord the boundary value problem (BVP) with boundary conditions (BC) on Γ_k is considered. For solutions $u(x)$ with power singularity on Γ_k of the Dirichlet problem for Laplace operator the asymptotical representation of $u(x)$ near Γ_k is obtained. Basing on this representation singular boundary condition on the Γ_k is formulated, the integral representation of the solution $u(x)$ of the Sobolev BVP with singular BVC on the Γ_k $k = 1, \dots, \chi$) is obtained.

Keywords: Sobolev boundary problem, elliptic equation, singular and regularized boundary conditions.

Introduction

Let $G_0 \subset R^n$ be a bounded domain with $(n-1)$ -dimensional boundary $\Gamma_0 \in C^\infty$ and the n_k -dimensional manifolds Γ_k without boundary lying inside of Γ_0 , $0 \leq n_k \leq n-1$. Assume that $\Gamma_k \in C^\infty$, ($k = 1, \dots, \chi$) and $\Gamma_k \cap \Gamma_j = \emptyset$ for $k \neq j$. Denote by G the domain $G_0 \setminus \bigcup_{j=1}^{\chi} \Gamma_j$,

$\Gamma = \bigcup_{j=0}^{\chi} \Gamma_j$ the boundary of domain G .

The Sobolev boundary value problem in the classical formulation [1,2] is a boundary value problem in the domain G with differential boundary

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conditions on Γ_k ($k = 0, \dots, \chi$):

$$L(x, D)u(x) = f(x) \quad (x \in G, \text{ord} L = 2m), \quad (1)$$

$$B_{j0}(x, D)u(x)|_{\Gamma_0} = \varphi_{j0}(x), \quad j = 1, \dots, m, \text{ord} B_{j0} = m_j, \quad (2)$$

$$B_{jk}(x, D)u(x)|_{\Gamma_k} = \varphi_{jk}(x), \quad j = 1, \dots, q_k, \quad k = 1, \dots, \chi \quad (3)$$

The boundary conditions on the Γ_k is understood as equalities of traces $B_{jk}u|_{\Gamma_k}$ to given functions φ_{jk} , $D = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$.

In such formulation the number of boundary conditions (3) on each variety Γ_k depends on the degree of smoothness of the solution $u(x)$ near Γ_k . This problem in the spaces $H_2^s(G)$ of sufficiently smooth solutions has been investigated in [1,2]. The Sobolev problem in the Banach spaces $H_2^s(G)$ of distributions was studied in [3,4]. Using the Green formulas, the authors introduced the notion of generalized solution of the Sobolev problem, by defining the functional spaces $\tilde{H}_p^s(G)$ of Sobolev type for solutions and proving the completeness collection of isomorphisms generated by the operator of boundary value problem. It has been proved that for $\forall s \in R^1$ the operator of Sobolev problem realizes the isomorphism between the spaces $\tilde{H}_p^s(G)$ (of solutions) and $\tilde{K}_p^s(G, \Gamma)$ (of compatible right hand sides).

In the theory mentioned above the boundary value problem (1)–(3) was studied in a closed domain $\overline{G}$, since the operator A of the boundary value problem in a open domain has nontrivial kernel. The dimension of the kernel $\ker A$ depends on the degrees of smoothness of the solution $u(x)$ near and on the varieties Γ_k .

In the constructed theory the boundary conditions (3) have sense as traces on Γ_k and for smooth solutions overdetermine the boundary problem (1),(2) in $\overline{G}$. Those conditions (3) which lose sense as traces on Γ_k , by no means are not connected with the comportment of the solution $u(x)$ near Γ_k . However, in many problems from applied sciences the necessity for studying of solutions, having power singularities on manifolds of various dimensions [6,7], arises.

In this case is necessary to give on Γ_k other natural conditions. These conditions are concerned either with the asymptotic behavior of the solution $u(x)$ near the manifold Γ_k or with giving on Γ_k of a regularized solution. For simplicity here we consider the Dirichlet problem in the half-space R_+^n for Laplace operator with singular boundary conditions on the variety $R^q \subset R_+^n$ and the boundary value problem with singular boundary conditions on Γ_k , $k = 1, \dots, \chi$.

The Dirichlet problem in R_+^n/R^q .

Let $x = (x_1, \dots, x_n) \in R^n$, $n \geq 3$, and let $G_0 = R_+^n = \{x \in R^n : x_n > 0\}$ be the half-space $x_n > 0$ in R^n , $x = (x', x'')$, $x' \in R^q$, $x'' \in R^{n-q}$, where

R^{n-q} is the $(n-q)$ -dimensional subspace, orthogonal to R^q in R^n . Denote by $H^S(G), H^S(\Gamma)$ the Sobolev spaces in G and Γ respectively, $s \in R^1$, with norms $\|\cdot, G\|_s$, and $\|\cdot, \Gamma_1\|_s$.

Consider the following problem. Find the solution $u(x) \in H^s(G)$ of the equation

$$\Delta u(x) = f(x) \in H^{s-2}(G), \quad (4)$$

under the boundary condition on Γ_0 :

$$u(x)|_{\Gamma_0} = \varphi(x) \in H^{s-1/2}(\Gamma_0). \quad (5)$$

Evidently the solution of the problem (4),(5) in G must be defined as the restriction on G of the corresponding solution of the Dirichlet problem (4),(5) in $\overline{G}$ for arbitrary $\overline{f}(x)$. If $\overline{f}(x)$ is an expansion of $f(x)$ on G_0 , then the solution $\overline{u}(x)$ of the Dirichlet problem (4),(5) in $\overline{G}$ with $\overline{f}(x)$ exists and

$$\|\overline{u}(x), \overline{G}\|_s < C \left(\|\overline{f}(x), \overline{G}\|_{s-2} + \|\varphi(x), \Gamma_0\|_{s-1/2} \right)$$

with the constant C not depending of $\overline{f}(x)$. Therefore, the general solution of the problem (4),(5) in G has the form $\overline{u}(x) = u_0(x) + u'(x)$, $u_0(x)$ and $u'(x)$ are the solutions of the Dirichlet problem (4),(5) for given $\overline{f}_0(x)$ in $\overline{G}$ and $f'(x)$ in $\overline{G}$ with $\text{supp } f'(x) \subset R^q$. Usually $\overline{f}_0(x)$ is a sufficiently smooth expansion of $f(x)$ from G to $\overline{G}$, $f'(x)$ is a singular distribution. In particular, $u'(x)$ is a solution of problem (4),(5) with $f(x) \equiv 0$ in G , $\varphi \equiv 0$ on Γ_0 . It is known [5] that the distribution $f'(x)$ with $\text{supp } f'(x) \subset R^q$ has the form

$$f'(x) = \sum_{\sigma} f_{\sigma}(x') \times D_{x''}^{\sigma} \delta(x'' - x''_0), \quad (6)$$

where $\delta(x'' - x''_0)$ is the Dirac measure concentrated on the R^q and the symbol $\times$ means the direct product of distributions.

For sufficiently smooth $f_0(x)$ and $\varphi(x)$ the solution $u_0(x)$ of the Dirichlet problem (1),(2) in $\overline{G}$ can be represented with the help of the Green function $\mathcal{G}(x, y)$ by formulae [9]

$$u(x) = \int_{R_+^n} \mathcal{G}(x, y) f_0(y) dy - \int_{\Gamma_0} \frac{\partial}{\partial \nu_y} \mathcal{G}(x, y') \varphi(y') dy', \quad y' \in R^{n-1}. \quad (7)$$

From this equality in a formal way we obtain

$$u'(x) = \sum_{\sigma} \int_{R_+^n} D_{x''}^{\sigma} \mathcal{G}(x', y', x'', x''_0) f_{\sigma}(y') dy', \quad (8)$$

where $dy' = dy_1 \cdots dy_q$, $D_{x''}^\sigma$ is the derivative with respect to the variables x'' from the ortogonal subspace R^{n-q} .

For simplicity of the analitical verification, at first we consider only term of the function (6) with $\sigma = 0$: $(f'(x) = \rho(x') \times \delta(x'' - x''_0))$. Then

$$\begin{aligned} u'(x) &= \int_{R^q} E_n(x' - y', x'' - x''_0) \rho(y') dy' + \\ &+ \int_{R^q} g_n(x, y', \overline{x''_0}) \rho(y') dy' = I_1 + I_2. \end{aligned} \quad (9)$$

Here and in what follows

$$E_n(x', x'') = (2 - n) \Omega_n^{-1} r^{2-n} \equiv a_n r^{2-n}, \quad r^2 = |x' - y'|^2 + |x'' - x''_0|^2,$$

$g_n(x, y', x''_0) = a_n \tilde{r}^{2-n}$, where $\tilde{r}^2 = |x' - y'|^2 + |x'' - \overline{x''_0}|^2$; $x''_0 = (x_{q+1}^0, \dots, x_{n-1}^0, -x_n^0)$, $\Omega_n = 2\pi^{n/2} \Gamma(\frac{n}{2})$ is the area of the surface of the sphere $|x| = 1$ in R^n , Γ is the Euler's function. Now we pass to study the behaviour of the solution $u'(x)$ near the manifold $\Gamma_1 = R^q$. For this it is necessary to introduce some notations.

Let $n \geq 3$, $n - q \leq 2$, $\rho(x') \in C_0^{n-q-1}(R^q)$. Denote by $\alpha = n - q - 2$,

$$P_\alpha(x', z') \rho = \sum_{\lambda=0}^{\alpha} \sum_{|k|=\lambda} \frac{\rho^{(k)}(x')}{k!} (-z')^k \equiv \sum_{\lambda=0}^{\alpha} \mathcal{P}_\lambda(x', z') \rho$$

is the segment of the Taylor expansion of the function $\rho(x' - z')$ at the point $z' = 0$,

$$\begin{aligned} Q_\lambda(D') \rho(x') &= \sum_{|k|=\lambda} A_k \frac{f^{(k)}(x')}{k!}, \quad A_k = \int_{|\omega'|=1} (-\omega')^k d\omega', \\ I_{11}(x) &= \int_{R^q} E_n(z', x'' - x''_0) [\rho(x' - z') - P_{\alpha-1}(x', z') \rho - \\ &\quad - \theta(z') \mathcal{P}_\alpha(x', z') \rho] dz', \end{aligned} \quad (10)$$

where $k = (k_1, \dots, k_q)$, $D' = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_q})$, $\theta(z')$ is the characteristic function of the ball $B_1 = \{z' : |z'| < 1\}$ in R^q : $\theta(z') = 1$ for $|z'| < 1$ and $\theta(z') = 0$ for $|z'| > 1$.

The following assertion holds true.

Theorem 1. Let $\rho(x') \in C_0^{\alpha+1}(\Gamma_1)$. Then near the manifold Γ_1 the solution $u'(x)$ is represented by the following equality

$$\begin{aligned} u'(x) = & I_{11}(x) + \sum_{\lambda=0}^{\alpha-1} Q_\lambda(D')\rho(x') \cdot r_2^{-(n-q-1-\lambda)} \\ & + a_n \left(\ln \frac{1}{r_2} + \gamma \right) Q_\alpha(D')\rho(x') + I_2(x', x''_0) + o(r_2), \end{aligned} \quad (11)$$

$$o(r_2) \rightarrow 0, r_2 \rightarrow 0$$

where γ is a constant, defined in the proof of theorem.

Proof. For establishing the representation (11) we investigate each integral I_1, I_2 in neighbourhood of the manifold Γ_1 .

1. We begin with the integral I_1 . It is easy to see that

$$\begin{aligned} I_1(x) = & I_{11}(x) + a_n \int_{R^q} r^{-n+2} \cdot P_{\alpha-1}(x', z') \rho dz' + \\ & + \int_{|z'| < 1} \mathcal{P}_\alpha(x', z') \rho dz' = I_{11}(x) + I_{12}(x) + I_{13}(x). \end{aligned} \quad (12)$$

a) Since $\rho(x') \in C_0^{\alpha+1}(R^q)$ the integral $I_{11}(x)$ is absolutely and uniformly convergent on each compact $K = \{x \in R_+^n, |x| < R\}$. Therefore, $I_1(x)$ defines a continuous function near the variety Γ_1 and it is clear that

$$I_{11}(x) = I_{11}(x', x''_0) + o(r_2), \quad o(r_2) = o(|x'' - x''_0|) \rightarrow 0, r_2 \rightarrow 0. \quad (13)$$

b) Now consider $I_{12}(x)$. Firstly, making the substitution $z' = r_2 \cdot \zeta'$, after passing to spherical coordinates (ω', r_1) , we obtain

$$\begin{aligned} I_{12}(x) &= a_n \cdot \sum_{\lambda=0}^{\alpha-1} Q_\lambda(D')\rho(x') \cdot \int_0^\infty r^{2-n} \cdot r_1^{q+\lambda-1} dr_1 \\ &= \sum_{\lambda=0}^{\alpha-1} Q_\lambda(D')\rho(x') \cdot a_n I_\lambda \cdot r_2^{-(n-q-2-\lambda)}, \quad r^2 = r_1^2 + r_2^2. \end{aligned} \quad (14)$$

In I_λ we make the substitution $r_1^2 \cdot (1 + r_1^2)^{-1} = v$, then use the relation between Euler's B and Γ functions and the relation $z\Gamma(z) = \Gamma(z+1)$ [10]. Thus $a_n I_\lambda = 2^{-1} \pi^{-n/2} \Gamma((q+\lambda)/2) \cdot \Gamma((\alpha-\lambda)/2) \stackrel{def}{=} \beta_\lambda$. Therefore,

$$I_{12}(x) = \sum_{\lambda=0}^{\alpha-1} \beta_\lambda Q_\lambda(D')\rho(x') \cdot r_2^{-(\alpha-\lambda)}. \quad (15)$$

c) Now we calculate the integral $I_{13}(x)$.

$$I_{13}(x) = \sum_{|k|=\alpha} \frac{\rho^{(k)}(x')}{k!} a_n \int_{|z'|<1} r^{2-n} \cdot (-z')^k dz' \stackrel{def}{=} \sum_{|k|=\alpha} a_n B_k \cdot \frac{\rho^{(k)}(x')}{k!}. \quad (16)$$

Firstly, we make the substitution of variables $z' = r_2 \cdot \zeta'$, then pass to spherical coordinate (ω, r_1) and separate the finite part of the integral [8]. We obtain

$$\begin{aligned} B_k &= a_n A_k \int_0^{r_2^{-1}} r_1^{\alpha+q-1} (1+r_1)^{(2-n)/2} dr_1 \\ &= a_n A_k \left[\int_0^1 \xi^{n-2} r_1^{-1} dr_1 + \int_1^{r_2^{-1}} (\xi^{n-2} - 1) \frac{dr_1}{r_1} \right] + a_n A_k \int_1^{r_2^{-1}} \frac{dr_1}{r_1} \\ &= \gamma a_n A_k + a_n A_k \ln \frac{1}{r_2} + o(r_2), \end{aligned} \quad (17)$$

where $\xi^2 = r_1^2(1+r_1^2)^{-1}$, γ stands for the expression from the square brackets in (17).

2. Now consider $I_2(x)$. Evidently, $I_2(x)$ is a continuous function near the variety Γ_1 and, therefore,

$$I_2(x) = I_2(x', x''_0 - \bar{x}''_0) + o(x), \quad o(x) \xrightarrow{r_2 \rightarrow 0} 0, \quad r_2 \rightarrow 0. \quad (18)$$

Placing (13), (14), (15), (16) in (12) and (12), (18) in (9) we obtain the desired representation (11). ■

Now we make the following remarks.

1) Evidently, the representation (11) contains only operators $Q_\lambda(D')\rho(x')$ of even orders;

2) The representation (11) contains the function $\ln 1/r_2$ only when the number $n - q$ is even;

3) The expression $I_{11}(x) = I_{11}(x', x'' - x''_0) = \int_{R^q} E_n(z', 0) [\rho(x' - z') - P_{\alpha-1}(x', z')\rho + \theta(z')\mathcal{P}_\alpha(x', z')\rho] dz'$ is a regularization of the trace of the fundamental solution $E_n(x)$ on the variety $\Gamma_1 = R^q$;

4) The main (singular) member of the representation (11) is the fundamental solution $E_{n-q}(x'')$ of the operator $\Delta_{x''}$ in the orthogonal subspace R^{n-q} ;

5) Establishing representations for the derivatives $\frac{\partial^s u'(x)}{\partial r_2^s}$ near R^q is quite similar.

On the Sobolev problem with singular and regularized boundary conditions on γ_k .

The representation (11) is the sum of the singular part $u'_s(x)$ and the regular part $u'_r(x)$ of the solution $u'(x)$. In the cases when the boundary

conditions (3) on Γ_k lose the sense as traces on Γ_k , it is necessary to change these conditions with other natural boundary conditions. As it was noted before, these conditions are about the asymptotic behaviour of the solution near Γ_k or about giving on Γ_k of a regularized part. Other boundary conditions on Γ_k can be imposed. Here we consider two simple cases of singular boundary conditions on Γ_k .

a) Let G be a bounded domain defined before, Γ_k is smooth q_k -dimensional manifold, locally diffeomorphic to R^{q_k} , $x=(x', x'')$, where $x' \in R^{n_k}$ ($n_k = n - q_k$).

Consider the follows boundary value problem. Find in G the solution of the equation

$$\Delta u(x) = f(x) \quad (19)$$

with usually condition

$$u(x) |_{\Gamma_0} = \varphi_0(x) \quad (20)$$

and singular boundary condition on Γ_k :

$$\lim_{x \rightarrow (x', x_0'') \in \Gamma_k} u(x) E_{n_k}^{-1}(x'' - x_0'') = \rho_k(x'), \quad k = 1, \dots, \chi. \quad (21)$$

The following statement is true

Theorem 2. Let Γ_k be locally isomorphic to R^{q_k} , $f(x) \in C^0(G)$, $\varphi_0(x) \in C^2(\Gamma_0)$, $\rho_k(x') \in C^0(\Gamma_k)$ and $u(x)$ solution of boundary value problem (19)–(21). Then

$$\begin{aligned} u(x) = & \int_G \mathcal{G}(x, y) f(y) dy + \int_{\Gamma_0} \frac{\partial \mathcal{G}}{\partial \nu}(x, y) \cdot \varphi_0(y') dy' + \\ & + \sum_{k=1}^{\chi} \int_{\Gamma_k} \mathcal{G}(x, y', x_0'') \rho_k(y') dy', \end{aligned} \quad (22)$$

where $\mathcal{G}(x, y)$ is the Green function of Dirichlet problem (19), (20) in $\overline{G}$.

Proof is similar to proof of Riemann–Green formulae [9, chap. 2]. Denote by S_k^ε the surface $|x - x_0| = \varepsilon$, where $x_0 \in \Gamma_k$, G_ε is the domain with boundary $\Gamma_\varepsilon = \Gamma_0 \cup \left\{ \bigcup_{k=1}^{\chi} S_k^\varepsilon \right\}$, ν_k is the exterior normal to Γ_ε in the point $x \in S_k^\varepsilon$. Writing the Green formulae [9] for function $\mathcal{G}(x, y)$ and $u(x)$ in the domain G_ε , passing to limit when $\varepsilon \rightarrow 0$ and taking in consideration (21), we obtain (22). ■

b) Let $G = R_+^n / R^q$, Γ_0 be the hiperplan $x_n = 0$ in R^n , $\Gamma_1 = R^q$ from the problem (21), (4), (5). Consider the follows boundary problem. Find in G the solution $u(x)$ of the equation (19) with the boundary condition (20) on Γ_0 and the additional boundary condition on the Γ_1 :

$$u_{reg}(x) = h(x'), \quad (23)$$

where $u_{reg}(x)$ is the regular (finite) part of solution $u(x)$. The conditions (23) mean that

$$I_{11}(x') + \gamma a_n Q_\alpha(D')\rho(x') + I_2(x', x''_0) = h(x'), \quad (24)$$

where $h(x')$ is given function.

The equation (24) is a hipersingular integro-differential equation [11], whose type depends on the number $n - q$. Some classes of such equation is investigated in [11].

The solvability of the equation (24) means that BPV with regularized conditions (23) is solvable. Each solution of the equation (24) generated a solution of BVP with regularized BC (23).

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