# SMOOTH MAPPINGS AND NON $\mathcal{F}_T$-ADAPTED SOLUTIONS ASSOCIATED WITH HAMILTON-IACOBI STOCHASTIC EQUATIONS

Daniela Ijacu
*Academy of Economic Studies, Bucharest, Romania*

Constantin Varsan
*Institute of Mathematics, Romanian Academy*
*P.O. Box 1-764, RO-70700 Bucharest, Romania*
cvarsan@imar.ro

**Abstract** Stochastic partial differential equations of Hamilton-Iacobi type including non $\mathcal{F}_t$-adapted solutions are studied. Using a Stratonovich type stochastic integral and an orbit solutions in a finite dimensional Lie algebra, we are dealing with non $\mathcal{F}_t$-adapted solutions associated with an extended characteristic system of stochastic differential equations.

**Keywords:** Stochastic partial differential equations, Stratonovich type stochastic integral and non $\mathcal{F}_t$-adapted solutions.

## 1. Introduction

The analysis is concentrated on stochastic partial differential equations (SPDEs) driven by the folowing dynamics:

$$(1)\quad \begin{cases} d_t u = g_0^\omega(x, u, \partial_x u)dt + \sum_{j=1}^{m} \chi_\tau(t) g_j(x, u, \partial_x u) \otimes dw_j(t) \\ u(0) = u_0^\omega(x), \quad (t, x, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}, \\ (u, \partial_x u) \in B(0, \rho) \subseteq \mathbb{R}^{n+1} \end{cases}$$

where $u_0^\omega(\cdot) \in C_b^2(\mathbb{R}^n)$ and the continuous scalar function $g_0^\omega(\cdot) \in C_b^2(\mathbb{R}^n \times B(0, \rho))$ are only $\mathcal{F}$-measurable on the parameter $\omega \in \Omega$ in a complete probability space $\{\Omega, \mathcal{F}, P\}$.

Here $w(t) = (w_1, \ldots, w_m(t)) \in \mathbb{R}^m$, $t \in [0, T]$, is a standard $m$-dimensional Wiener process and $\tau(\omega) : \Omega \to [0, T]$ is a stopping time allowing

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one to define non $\mathcal{F}_t$-adapted and bounded solutions $(u, \partial_x u) \in B(0, \rho) \subseteq \mathbb{R}^{n+1}$ on a complete filtered probability space $\{\Omega, \mathcal{F}, P; \{\mathcal{F}_t\} \nearrow \mathcal{F}\}$. The stochastic integral "$\otimes$" appearing in the equation (1) coincides with the usual Fisk-Stratonovich integral "$\bigcirc$" provided the dependence on $\omega \in \Omega$ of $u_0(\cdot)$ and $g_0(\cdot)$ is omitted.

As far as the given $u_0^\omega(\cdot)$ and $g_0^\omega(\cdot)$ are not $\mathcal{F}_t$-adapted we need to define a special type of stochastic integral "$\otimes$" (Stratonovich type) encompassing non $\mathcal{F}_t$-adapted solutions $(u(t,x,\omega), \partial_x u(t,x,\omega))$, $t \in [0,T]$ and fulfilling (1) along to the corresponding trajectories $x = \hat{x}(t, \lambda, \omega)$, $t \in [0,T]$ contained in the characteristic system. It can be accomplished using the Langevin's approximation $w^\varepsilon(t)$, $t \in [0,T]$, $\varepsilon \in (0,1]$ of the original Wiener process and an adequate rule of stochastic differentiation. From now on we shall not mention the explicite dependence on $\omega \in \Omega$ and write $u_0(\cdot)$, $g_0(\cdot)$.

On the other hand, the dependence on the gradient $\partial_x u \in \mathbb{R}^n$ of the given $g_i(\cdot)$, $i \in \{0, 1, \ldots, m\}$, is obstructing the martingale approach and to achieve the goal we use some smooth mappings generated by the finite dimensional Lie algebra $L(Z_1, \ldots, Z_m)$ associated with the given smooth diffusion coefficients $\{g_1, \ldots, g_m\}$.

A motivation for considering such problems may appear from describing a non $\mathcal{F}_t$-adapted solutions fulfilling the corresponding Hamilton-Iacobi stochastic differential equations associated with an optimal solution $(\tilde{x}(t,\omega), \tilde{u}(t,\omega))$, $t \in [0,T]$; they are based on an augmented Lagrangean $H(t,x,u,\psi; dt, dw(t))$ defined as a stochastic differential form

$$H(t,x,u,\psi; dt, dw(t)) = [\psi f(t,x,u) + f_0(t,x,u)]dt + \sum_{j=1}^{m} \psi g_j(t,x) \otimes dw_j(t)$$

Here the adjoint row vector function $\psi = \tilde{\psi}(t,\omega)$ has to be determined as a solution of a stochastic differential equation

$$(2) \quad \begin{cases} d_t\tilde{\psi} = -\dfrac{\partial H}{\partial x}(t, \tilde{x}(t,\omega), \tilde{u}(t,\omega), \tilde{\psi}; dt, dw(t)), \quad t \in [0,T] \\ \tilde{\psi}(t_f, \omega) = \partial_x F(\tilde{x}(T,\omega)) \end{cases}$$

and the optimal pair $(\tilde{x}(t,\omega), \tilde{u}(t,\omega))$ obeys to

$$(3) \quad \begin{cases} d_t\tilde{x} = \dfrac{\partial H}{\partial \psi}(t, \tilde{x}, \tilde{u}(t,\omega), \tilde{\psi}(t,\omega); dt, dw(t)), \quad t \in [0,T] \\ \tilde{x}(0) = x_0 \in X \subseteq \mathbb{R}^n \end{cases}$$

$$(4) \quad \begin{cases} \min\limits_{u \in U \subseteq R^k} \tilde{\psi}(t,\omega) f(t, \tilde{x}(t,\omega), u) + f_0(t, \tilde{x}(t,\omega), u) = \\ \tilde{\psi}(t,\omega) f(t, \tilde{x}(t,\omega), \tilde{u}(t,\omega)) + f_0(t, \tilde{x}(t,\omega), \tilde{u}(t,\omega)) \\ \text{a.e. } (t,\omega) \in [0,T] \times \Omega \end{cases}$$

Using a smooth mapping $x=G(p,\hat{x})$, $\hat{x}\in B(x_0,\rho_0)\subseteq\mathbb{R}^n$, $p\in B(0,\rho)\subseteq\mathbb{R}^M$, we decompose the non $\mathcal{F}_t$-adapted solutions $(\tilde{x}(t,\omega),\tilde{\psi}(t,\omega))$, $t\in[0,T]$ into a continuous and $\mathcal{F}_t$-adapted process valued in the place of smooth diffeomorfisms and a corresponding continously differentiable proccess $(\tilde{x}(t,\omega),\tilde{\psi}(t,\omega))$, $t\in[0,T]$ such that (2) and (3) are satisfied provided an adequate stochastic integral "$\otimes$" is defined.

## 2. Some auxiliary lemmas and main results

Everywhere in this paper we assume that smooth deterministic functions $g_j(x,u,p)$ are given such that $g_j\in C_b^\infty(\mathbb{R}^n\times B(0,\rho))$, $j\in\{1,\dots,m\}$ where the ball $B(0,\rho)\subseteq\mathbb{R}^{n+1}$ is fixed. Denote $z=(u,p,x)\in\mathbb{R}^{2n+1}$, $D\overset{\text{def}}{=}B(0,\rho)\times\mathbb{R}^n$ and define smooth vector fields $Z_j(z)=\begin{pmatrix}Y_j(z)\\X_j(z)\end{pmatrix}$, with $X_j(z)=-\partial_p g_j(x,u,p)$ in $\mathbb{R}^n$, $Y_j(z)=\begin{pmatrix}g_j(x,u,p)-\langle p,\partial_p g_j(x,u,p)\rangle\\ \partial_x g_j(x,u,p)+p\partial_u g_j(x,u,p)\end{pmatrix}$ in $\mathbb{R}^{n+1}$. A solution for SPDEs (1) is derived using the corresponding stochastic system of characteristics.

In addition we have to start with a local solution associated with the reduced stochastic differential system:

$$(5)\qquad \begin{aligned} &d_t z=\sum_{j=1}^m \chi_T(t)Z_j(z)\bigcirc dw_j(t),\ t\in[0,T],\ z\in D=B(0,\rho)\times\mathbb{R}^n\\ &z(0)=z_0\in D_0=B(0,\rho_0)\times\mathbb{R}^n,\ 0<\rho_0<\rho\end{aligned}$$

where the Fisk-Stratonovich integral "$\bigcirc$" is used and $w(t)=(w_1(t),w_2(t),\dots,(w_m(t))\in\mathbb{R}^m$ is a standard $m$-dimensional Wiener process on a given filtered probability space $\{\Omega,\mathcal{F},P;\{\mathcal{F}_t\}\nearrow\mathcal{F}\}$.

A local solution fulfilling (5) is found as a continuous and $\mathcal{F}_t$-adapted process valued in the space of smooth mappings $z\in C_b^\infty(D_0;\mathbb{R}^{2n+1})$ and it is done assuming

($\mathcal{H}$) The Lie algebra $L(Z_1,\dots,Z_m)\subseteq C_b^\infty(D_0;\mathbb{R}^{2n+1})$ determined by the vector fields $\{Z_1,\dots,Z_m\}$ is finite dimensional.

The assumption ($\mathcal{H}$) allow us to fixe a system of generators $\{Z_1,\dots,Z_m, Z_{m+1},\dots,Z_M\}\subseteq L(Z_1,\dots,Z_m)$ and to define the corresponding orbit of smooth mappings.

$$(6)\qquad \begin{cases} S(p,z_0)=S_1(t_1)\bigcirc\dots\bigcirc S_M(t_M)(z_0)\\ p=(t_1,\dots,t_M)\in D_M=\prod_{j=1}^M[-a_j,a_j],\ \text{for } z_0\in D_0=B(0,\rho_0)\times\mathbb{R}^n\end{cases}$$

where $S_j(t,z_0)$, $t\in[-a_j,a_j]$, $z_0\in D_0$, is a local flow generated by the vector field $Z_j$, $j\in\{1,\dots,M\}$.

Using the nonsingular algebraic representation of the associated gradient system given in (7) we are able to recover the original vector fields $\{Z_1, \dots, Z_M\}$ along to the orbit solution (6) and some analitic vector fields $q_j \in A(D_M; \mathbb{R}^M)$, $j \in \{1, \dots, M\}$, are defined such that

$$(7) \quad \begin{cases} \dfrac{\partial S}{\partial p}(p, z_0) q_j(p) = Z_j(S(p, z_0)) \ j \in \{1, \dots, M\} \ p \in D_M, \ Z_0 \in D_0 \\ \text{the } (M \times M) \text{ matrix } Q(p) = (q_1(p), \dots, q_M(p)), \ p \in D_M \\ \text{is a non singular one} \end{cases}$$

A local solution for the stochastic differential system (5) is constructed using the mapping $S(p, z_0)$ in (6) provided an $\mathcal{F}_t$-adapted continuous process $p = p(t) \in D_M$, $t \in [0, T]$ is defined as a solution of the following system:

$$(8) \quad d_t p = \sum_{j=1}^{m} \alpha(p) q_j(p) \circ dw_j(t), \ p(0) = 0, \ p \in \mathbb{R}^M$$

where the smooth scalar function $\alpha \in C^\infty(\mathbb{R}^M; [0, 1])$ is taken adequately and fulfilling $\alpha(p) = 0$ for $p \in \mathbb{R}^M \setminus B(0, 2\hat{\rho})$, $\alpha(p) = 1$ for $p \in B(0, \hat{\rho})$, where $\hat{\rho} > 0$ is fixed such that $B(0, 2\hat{\rho}) \subseteq D_M$. Let $\tau(\omega) : \Omega \longrightarrow [0, T]$ be a stopping time defined by $\tau(\omega) = \inf\{t \in [0, T];\ |p(t)| > \hat{\rho}\}$ and associated with the solution $p = p(t)$, $t \in [0, T]$, globally defined in (8). It is easily seen that $\hat{p}(t) = p(t \wedge \tau) \in B(0, \hat{\rho})$, $t \in [0, T]$, is obeying to the following stochastic system:

$$\hat{p}(t) = \sum_{j=1}^{m} \int_0^t \chi_\tau(s) q_j(\hat{p}(s)) \circ dw_j(s) \ \hat{p}(0) = 0, \ t \in [0, T]$$

where $\chi_\tau(t) = 1$ for $\tau > t$ and $\chi_\tau(t) = 0$ for $\tau \leqslant t$, $t \in [0, T]$.

The $\mathcal{F}_t$-adapted and continuous process

$$z(t, z_0) = S(p(t), z_0), \ t \in [0, T], \ z_0 \in D_0 = B(0, \rho_0) \times \mathbb{R}^n$$

will be a local solution for the stochastic system in (5) fulfilling the following integral equations:

$$Z(t, z_0) = z_0 + \sum_{j=1}^{m} \int_0^t \chi_\tau(s) Z_j(z(s, z_0)) \circ dw_j(s) \ t \in [0, T], \ z_0 \in D_0$$

A local solution for SPDEs (1) can be constructed provided a continuously differentiable process $z_0 = z_0(t, \lambda) \in D_0$, $t \in (0, a]$, $0 < a \leq T$, is defined such that

$$(9) \quad z(t, \lambda) = S(p(t); z_0(t, \lambda)) \in D, \ t \in [0, a], \ \lambda \in \mathbb{R}^n$$

is a local solution of an extended system of characteristics

$$(10)\quad \begin{cases} d_t z = Z_0(z)dt + \sum_{j=1}^{m} \chi_T(t) Z_j(z) \otimes dw_j, \ t \in [0,T], \ z \in B(0,\rho) \times \mathbb{R}^n \\ z(0) = z_0(\lambda) = (u_0(\lambda), \partial_\lambda u_0(\lambda)) \in B(0,\rho_1) \times \mathbb{R}^n = D_1, \ 0 < \rho_1 < \rho \end{cases}$$

where $z = (u, \partial_x u, x) = (u, p, x)$ and the smooth vector field $Z_0(z) = \begin{pmatrix} Y_0(z) \\ X_0(z) \end{pmatrix}$ is associated with the drift part $g_0$ in (1) as follows:

$$(11)\quad \begin{cases} X_0(z) = -\partial_p g_0(x,u,p) \in \mathbb{R}^n, \\ Y_0(z) = \begin{pmatrix} g_0(x,u,p) + < p, X_0(z) > \\ \partial_x g_0(x,u,p) + p \partial_u g_0(x,u,p) \end{pmatrix} \end{cases}$$

In addition the Stratonovich type integral "$\otimes$" is computed passing to the limit $\varepsilon \searrow 0$ in an ordinary rule of derivation applied to the smooth mapping

$$(12)\quad z^\varepsilon(t,\lambda) = S(p^\varepsilon(t); z_0(t,\lambda)), \ t \in [t', t''] \subset (0,a], \ 0 < a \leq T$$

where $p = p^\varepsilon(t)$, $t \in [0,T]$, is fulfilling the following system of ordinary differential equations:

$$\frac{dp}{dt} = \sum_{j=1}^{m} \chi_T(t) \alpha(p) q_j(p) \frac{dw_j^\varepsilon(t)}{dt}, \ t \in [0,T], \ p(0) = 0$$

provided the Langevin's smooth approximation

$$w^\varepsilon(t) = \int_0^t y^\varepsilon(s) ds = w(t) - \eta(t,\varepsilon), \ d_t y^\varepsilon(t) = -\frac{1}{\varepsilon} y^\varepsilon(t) dt + \frac{1}{\varepsilon} dw(t)$$

for $t \in [0,T]$, $\varepsilon \in (0,1]$, is used.

As a consequence we may and do write the following

**Definition 1** A stochastic integral "$\otimes$" appearing in (10) is computed as follows:

$$\int_{t'}^{t''} \chi_T(t) Z_j(z(t,\lambda)) \otimes dw_j(t) = \left[ \int_{t'}^{t''} \chi_T(t) Z_j(S(\hat{p}(t); z_0)) \bigcirc dw_j(t) \right]_{z_0 = z_0(t',\lambda)} +$$
$$\int_{t'}^{t''} \left[ \int_t^{t''} \frac{\partial}{\partial z_0} (\chi_T(\sigma) Z_j(S(\hat{p}(\sigma); z_0)) \bigcirc dw_j(\sigma) \right]_{z_0 = z_0(t,\lambda)} \frac{dz_0}{dt}(t,\lambda) dt$$

$j \in \{1, \ldots, m\}$, where the Fisk-Stratonovich stochastic integral "$\bigcirc$" associated with continuous $\mathcal{F}_t$-adapted process valued in the space of smooth mappings $C^\infty(D_0; \mathbb{R}^{2n+1})$ is used.

**Definition 2** Let $\varphi \in C_b^{0,3}([0,a]\times D;\mathbb{R})$ and $\hat{z}(t,\lambda) = S(\hat{p}(t))$, $z_0(t,\lambda)\in D$, $t\in[0,a]$, $\lambda\in\mathbb{R}^n$ be defined as in (9). Then

$$\int_{t'}^{t''} \chi_T(t)\varphi(t,\hat{z}(t,\lambda)) \otimes dw_j(t) = \left[\int_{t'}^{t''} \chi_T(t)\varphi(t,S(\hat{p}(t);z_0)) \bigcirc dw_j(t)\right]_{z_0=z_0(t',\lambda)} + \int_{t'}^{t''}\left[\int_{t}^{t''} \frac{\partial}{\partial z_0}\left(\chi_T(\sigma)\varphi(\sigma,S(\hat{p}(\sigma);z_0))\right) \bigcirc dw_j(\sigma)\right]_{z_0=z_0(t,\lambda)} \frac{dz_0}{dt}(t,\lambda)dt$$

The folowing stochastic rule of derivation holds true.

**Lemma 1** *Let $f\in C_b^{1,3}([0,T]\times D;\mathbb{R})$ be given and consider the solution $\hat{z}(t,\lambda) = S(\hat{p}(t);z_0(t,\lambda))$, $t\in[0,a]$, fulfilling the following integral equations*

$$\hat{z}(t'',\lambda) - \hat{z}(t',\lambda) = \int_{t'}^{t''} \frac{\partial S}{\partial z_0}(\hat{p}(t);z_0(t,\lambda))\frac{dz_0}{dt}(t,\lambda)dt + \sum_{j=1}^{m}\int_{t'}^{t''} \chi_T(t)Z_j(\hat{z}(t,\lambda)) \otimes dw_j(t),\ [t',t'']\subseteq[0,a]$$

*Then $E \overset{def}{=} f(t'',\hat{z}(t'',\lambda)) - f(t',\hat{z}(',\lambda)) \overset{def}{=} \lim_{\varepsilon\to 0}\int_{t'}^{t''}\frac{d}{dt}[f(t,z^\varepsilon(t,\lambda)]dt$ can be expressed as follows*

$$E = \int_{t'}^{t''} [\partial_t f(t,\hat{z}(t,\lambda)) + \langle \partial_z f(t,\hat{z}(t,\lambda)), \partial_{z_0}S(\hat{p}(t);z_0(t,\lambda))\frac{dz_0}{dt}(t,\lambda)\rangle]dt + \sum_{j=1}^{m}\int_{t'}^{t''} \chi_T(t)\langle \partial_z f(t,\hat{z}(t,\lambda),Z_j(\hat{z}(t,\lambda))\rangle \otimes dw_j(t)$$

*where the stochastic integral "$\otimes$" is acting as in the Definition 2, and $z^\varepsilon(t,\lambda) = S(p^\varepsilon(t);z_0(t,\lambda))$ is defined in (12).*

Relying on the integral equations given in Lemma 1 we may and do choose $z_0 = z_0(t,\lambda)$ such that the characteristic system in (10) is fulfilled. In this respect, let $z_0 = z_0(t,\lambda)$, $t\in[0,a]$, $0<a\le T$, be the unique solution of the following system of ordinary differential equations

$$(13)\quad \begin{cases} \dfrac{dz_0}{dt}(t,\lambda) = \left[\dfrac{\partial S}{\partial z_0}(\hat{p}(t);z_0(t,\lambda)\right]^{-1} Z_0(S(\hat{p}(t);z_0(t,\lambda)) \\ z_0(0,\lambda) = z_0(\lambda) = (y_0(\lambda);\lambda)\in B(0,\rho_1)\times\mathbb{R}^n,\ 0<\rho_1<\rho \end{cases}$$

where the vector field $Z_0\in C_b^1(D,\mathbb{R}^{2n+1})$ is defined in (11). By a direct computation we get the following

**Lemma 2** *Assume the hypothesis* $(\mathcal{H})$ *is fulfilled and define* $\hat{z}(t,\lambda) = S(\hat{p}(t); z_0(t,\lambda))$, *where* $z_0(t,\lambda), t \in [0,a], \lambda \in \mathbb{R}^n$ *is the unique solution associated with* (13). *Then* $z = \hat{z}(t,\lambda), t \in [0,a], \lambda \in \mathbb{R}^n$, *is a local solution of* (10) *obeying to*

$$\hat{z}(t,\lambda) = z_0(\lambda) + \int_0^t Z_0(\hat{z}(s,\lambda))ds + \sum_{j=1} \int_0^t \chi_T(s) Z_j(\hat{z}(s,\lambda)) \otimes dw_j(s), \quad \forall t \in [0,a],\ \lambda \in \mathbb{R}^n$$

By definition $\hat{z}(t,\lambda) = (\hat{y}(t,\lambda); \hat{x}(t,\lambda))$ obey to the integral equation in Lemma 2. We may and do solve the following algebraic equations: $\hat{x}(t,\lambda) = x$ and find $\lambda = \psi(t,x)$, $t \in [0,a]$, $x \in \mathbb{R}^n$ such that

(14) $\hat{x}(t,\psi(t,x)) = x,\ \psi(t,\hat{x}(t,\lambda)) = \lambda,\ t \in [0,a],\ x \in \mathbb{R}^n$

Denote $y(t,x) = \hat{y}(t,\psi(t,x)) = (u(t,x), p(t,x))$, $t \in [0,a]$, $x \in \mathbb{R}^n$ and one sees easily that

(15) $u(t,\hat{x}(t,\lambda) = \hat{u}(t,\lambda), p(t,\hat{x}(t,\lambda) = \hat{p}(t,\lambda)$

A local solution for SPDEs (1) is asimilated with the above given continuous process $u{=}u(t,x)$, $t{\in}[0,a]$, $\lambda, x{\in}\mathbb{R}^n$ provided we are able to show

(16) $\partial_x u(t,x) = p(t,x),\ t \in [0,a],\ x \in \mathbb{R}^n$

and the stochastic differential $[d_t u(t,x)]_{x=\hat{x}(t,\lambda)}$ along to $x = \hat{x}(t,\lambda)$ is acting as the following integral shows:

(17) $$\int_{t'}^{t''} [d_t u(t,x)]_{x=\hat{x}}(t,\lambda) = \int_{t'}^{t''} [d_t \hat{u}(t,\lambda) - \langle \hat{p}(t,\lambda), d_t \hat{x}(t,\lambda) \rangle],$$

for any $[t',t''] \subseteq [0,a]$, where the left hand side in (17) is defined as

(18) $$\int_{t'}^{t''} [d_t u(t,x)]_{x=\hat{x}}(t,\lambda) = \lim_{\varepsilon \to 0} \int_{t'}^{t''} [d_t u^\varepsilon(t,x)]_{x=x^\varepsilon(t,\lambda)}\, dt$$

Here $u^\varepsilon(t,x) = u^\varepsilon(t,\psi^\varepsilon(t,x))$, and the smooth approximation $z^\varepsilon(t,\lambda) = (y^\varepsilon(t,\lambda), x^\varepsilon(t,\lambda)) = S(p^\varepsilon(t); z_0(t,\lambda)$ is a continuously differentiable mapping with respect to both variables $t \in [0,a]$, $\lambda \in \mathbb{R}^n$, and $\lambda = \psi^\varepsilon(t,x)$ is the unique solution of the algebraic equations

(19) $x^\varepsilon(t,\lambda) = x \in \mathbb{R}^n,\ \psi^\varepsilon(t, x^\varepsilon(t,\lambda)) = \lambda,\ t \in [0,a]$

Using the continuously differentiable process $z^\varepsilon(t,\lambda)$ we get the equations (16) and (17) fulfilled and expressed as follows

**Lemma 3** *Under the same conditions as in Lemma 2 define* $y(t,x) = y(t,\psi(t,x)) = (u(t,x),p(t,x))$, $t \in [0,a]$, $x \in \mathbb{R}^n$ *as in* (15). *Then* (16) *and* (17) *hold true, i.e.*

$$\partial_x u(t,x) = p(t,x),\ t \in [0,a],\ x \in \mathbb{R}^n$$

$$\int_{t'}^{t''} [d_t u(t,x)]_{x=\hat{x}(t,\lambda)} = \int_{t'}^{t''} d_t\hat{u}(t,\lambda) + \int_{t'}^{t''} \langle \hat{p}(t,\lambda), \partial_p g_0(\hat{z}(t,\lambda))\rangle dt + \sum_{j=1}^{m} \int_{t'}^{t''} \chi_T(t)\langle \hat{p}(t,\lambda), \partial_p g_j(\hat{z}(t,\lambda))\rangle \otimes dw_j(t)$$

*for any* $[t',t''] \subseteq [0,a]$, *where the left hand side is given in* (18).

The following theorem is a direct consequence of the results stated in Lemmas 2 and 3.

**Theorem 1** *Let* $g_i(x,u,p)$, $i \in \{0,1,\dots,m\}$, *be given such that the hypothesis* $(\mathcal{H})$ *is fulfilled. Let* $\hat{z}(t,\lambda) = (\hat{y}(t,\lambda), \hat{x}(t,\lambda))$, $(t,\lambda) \in [0,a] \times \mathbb{R}^n$, *be the local solution associated with the integral equations given in Lemma* 2. *Let* $u(t,x) = \hat{u}(t,\psi(t,x))$ *and* $p(t,x) = \hat{p}(t,\psi(t,x))$ *where* $\hat{y}(t,\lambda) = (\hat{u}(t,\lambda),\ \hat{p}(t,\lambda))$ *and* $\lambda = \psi(t,x)$ *is the unique solution fulfilling* (14).

*Then* $\partial_x u(t,x) = p(t,x)$ *and* $u = u(t,x)$ *is a local solution of the SPDE* (1) *along to* $x = \hat{x}(t,\lambda)$, *i.e.* $u(0,x) = u_0(x)$, $x \in \mathbb{R}^n$, *and*

$$[d_t u(t,x)]_{x=\hat{x}(t,\lambda)} = g_0(\hat{z}(t,\lambda))dt + \sum_{j=1}^{m} \chi_T(t) g_j(\hat{z}(t,\lambda)) \otimes dw_j(t)$$

*for any* $t \in [0,a]$, $\lambda \in \mathbb{R}^n$, *where*

$$\int_{t'}^{t''} [d_t u(t,x)]_{x=\hat{x}(t,\lambda)} = \hat{u}(t'',\lambda) - \hat{u}(t',\lambda) + \int_{t'}^{t''} \langle \hat{p}(t,\lambda), \partial_p g_0(\hat{z}(t,\lambda))\rangle dt + \sum_{j=1}^{m} \int_{t'}^{t''} \chi_T(t)\langle \hat{p}(t,\lambda), \partial_p g_j(\hat{z}(t,\lambda))\rangle \otimes dw_j(t),$$ *for any* $[t',t''] \subseteq [0,a]$.

**Comment.** A SPDE of parabolic type is obtained from the equation (1) replacing the drift $g_0$ by $[\Delta_x u + f(x,u,\partial_x u)]$ where the Laplacian $\Delta_x u$ has to be computed along to the continuous process $x = \hat{x}(t,\lambda)$ wich may involve new difficulties unless we assume, in addition, $\partial_p g_i(x,u,p) = b_i \in \mathbb{R}^n$, $i \in \{1,2,\dots,m\}$.

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