DETERMINING A SEMILINEAR PARABOLIC PDE FROM FINAL DATA *

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Abstract Given the following boundary value problem

$$\begin{cases} y_t - \Delta y + a(x)b(y) = u\chi_{\omega_{in}} & \text{in } \Omega \times (0, T), \\ y = 0 & \text{on } \partial\Omega \times (0, T), \\ y(0) = y_0 & \text{in } \Omega, \end{cases}$$

we are concerned with the following *inverse problem*: assuming that for each source term $u(x, t) \in L^2(\omega_{in} \times (0, T))$ the solution at the final time $y(x, T)$ is known over ω_{out} , we want to determine the nonlinear term $a(x)b(y)$ over $\Omega \times \mathbb{R}$. We suppose that ω_{in} and ω_{out} are (probably small) fixed open subsets of Ω with $\omega_{in} \cap \omega_{out} \neq \emptyset$ and $\chi_{\omega_{in}}$ denotes the characteristic function of ω_{in} .

Keywords: inverse problem, semilinear parabolic PDE, parameter estimation.

1. Introduction

Let us consider the following boundary value problem, associated with a semilinear parabolic equation

$$\begin{cases} y_t(x, t) - \Delta y(x, t) + a(x)b(y(x, t)) = u(x, t)\chi_{\omega_{in}}(x) & \text{in } Q, \\ y(x, t) = 0 & \text{on } \Sigma, \\ y(x, 0) = y_0(x) & \text{in } \Omega, \end{cases} \quad (1)$$

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where Ω is a bounded open subset of $\mathbb{R}^n$ with a boundary $\partial\Omega$ of class C^2 , $Q = \Omega \times (0, T)$, $\Sigma = \partial\Omega \times (0, T)$ and $T > 0$.

Given the functions a , b , u and y_0 , the classical direct problem consists of determining $y(x, t)$ over Q . Here we are concerned with the following *inverse problem*: assuming that the initial datum y_0 is known and that for each source term $u \in L^2(\omega_{in} \times (0, T))$ the solution at the final time $y(x, T)$ is known over $\omega_{out} \subset \Omega$, we want to determine the nonlinear term $a(x)b(y)$.

Our work is somehow related to the recovery of a nonlinear term $f(x, t, y)$ through the knowledge of the Dirichlet-to-Neumann map associated to the semilinear PDE, see [4, Chapter 9] and the bibliography therein.

On the other hand, in [2], we have studied the case where the source terms u are acting on the whole domain Q (i.e. $\omega_{in} = \Omega$). In fact, in that case, we have proved the simultaneous identification of the nonlinearities A , $\vec{b}$ and c appearing in the quasilinear parabolic equation

$$y_t - \operatorname{div}(A(y)\nabla y + \vec{b}(y)) + c(y) = u(x, t) \quad \text{in } Q,$$

that are unique in an appropriate class.

The main objective of this work is to deal with the case $\omega_{in} \neq \Omega$. To this end, we restrict ourselves to the semilinear PDE given in (1), although the linear part of the operator can be taken in a more general way. In the first part of the work, we prove the uniqueness of $a(x)b(y)$ assuming that a is analytic on $\overline{\Omega}$ without changing the sign and $b \in C^1(\mathbb{R})$ with $b' \in L^\infty(\mathbb{R})$. This property can be deduced by using the approximate controllability for this type of equations, when the function u is viewed as a control.

In the second part, we present a finite-dimensional optimization problem that allows the practical reconstruction of $a(x)b(y)$. In order to illustrate the applicability of the method, some numerical results are shown in the one-dimensional space case (i.e. $n = 1$). The influence of noise in the data will be also taken into account.

2. Uniqueness of the nonlinear term

Let us recall the space

$$W(0, T) = \{y \in L^2(0, T; H_0^1(\Omega)) : y_t \in L^2(0, T; H^{-1}(\Omega))\},$$

where $H_0^1(\Omega)$ denotes the classical Sobolev space and $H^{-1}(\Omega)$ its dual.

Let us introduce the hypotheses that will be assumed on the nonlinear term of the semilinear parabolic operator:

H1) a is a real analytic function on $\overline{\Omega}$ and $a(x) \geq 0 \quad \forall x \in \overline{\Omega}$.

H2) $b \in C^1(\mathbb{R})$ with $b' \in L^\infty(\mathbb{R})$.

The sign condition in H1) is just made to fix ideas: the important point here is that a does not change the sign in Ω (see the end of the proof of Theorem 2).

For the sake of brevity, we designate $\mathcal{C} = \{(a, b) \text{ verifying } H1) - H2)\}$.

Given $(a, b) \in \mathcal{C}$, $u \in L^2(\omega_{in} \times (0, T))$ and $y_0 \in L^2(\Omega)$, it is well known that the problem (1) has a unique solution y in $W(0, T)$, see [5, Theorem 6.7, pp. 466]. Taking into account that the initial datum y_0 will remain fixed through the rest of the paper, we will designate by $y_{u,a,b}$ the unique solution of problem (1) in $W(0, T)$.

Let us now recall an approximate controllability result for parabolic equations:

Theorem 1 *Let us fix $y_0 \in L^2(\Omega)$, $(a, b) \in \mathcal{C}$ and ω_{in} a nonempty open subset of Ω . Then, the set $R(T) = \{y_{u,a,b}(x, T) : u \in L^2(\omega_{in} \times (0, T))\}$ is dense in $L^2(\Omega)$.*

This is a particular case of a more general result (see for instance [1] and [3]). At this level, the assumption H1) on a can be weakened to $a \in L^\infty(\Omega)$.

With the help of this controllability result, we can establish the following uniqueness theorem:

Theorem 2 (Uniqueness) *Let us fix $y_0 \in L^2(\Omega)$ and $\omega_{in}, \omega_{out}$ two open subsets of Ω such that $\omega_{in} \cap \omega_{out} \neq \emptyset$. Suppose that there exist two pairs $(a, b), (\tilde{a}, \tilde{b}) \in \mathcal{C}$ such that $y_{u,a,b}(T) = y_{u,\tilde{a},\tilde{b}}(T)$ in ω_{out} for each $u \in L^2(\omega_{in} \times (0, T))$. Then, $a(x)b(s) = \tilde{a}(x)\tilde{b}(s)$ for all $(x, s) \in \Omega \times \mathbb{R}$.*

Proof. Given $u, v \in L^2(\omega_{in} \times (0, T))$ and $\lambda \in (0, 1)$, let us consider $y_{u+\lambda v,a,b}$ and $y_{u,a,b}$. Moreover we define $z_{\lambda,a,b} = (y_{u+\lambda v,a,b} - y_{u,a,b})/\lambda$. Analogously, from $y_{u+\lambda v,\tilde{a},\tilde{b}}$ and $y_{u,\tilde{a},\tilde{b}}$ we introduce $z_{\lambda,\tilde{a},\tilde{b}} = (y_{u+\lambda v,\tilde{a},\tilde{b}} - y_{u,\tilde{a},\tilde{b}})/\lambda$. By hypothesis we know that $y_{u,a,b}(T) = y_{u,\tilde{a},\tilde{b}}(T)$ in ω_{out} and $y_{u+\lambda v,a,b}(T) = y_{u+\lambda v,\tilde{a},\tilde{b}}(T)$ in ω_{out} for each λ . Consequently, $z_{\lambda,a,b}(T) = z_{\lambda,\tilde{a},\tilde{b}}(T)$ in ω_{out} . By using the Mean Value Theorem, it can be proved that $z_{\lambda,a,b} \rightarrow z_{v,a,b}$ in $W(0, T)$ as $\lambda \rightarrow 0$, where $z_{v,a,b}$ is the unique solution in $W(0, T)$ of the problem

$$\begin{cases} z_t(x, t) - \Delta z(x, t) + a(x)b'(y_{u,a,b})z(x, t) = v(x, t)\chi_{\omega_{in}}(x) & \text{in } Q, \\ z(x, t) = 0 & \text{on } \Sigma, \\ z(x, 0) = 0 & \text{in } \Omega. \end{cases}$$

Thanks to H2), let us point out that $a(x)b'(y_{u,a,b}(x, t)) \in L^\infty(Q)$. Hence, it is clear that previous problem is well posed in $W(0, T)$.

The same argumentation leads us to show that $z_{\lambda, \tilde{a}, \tilde{b}} \rightarrow z_{v, \tilde{a}, \tilde{b}}$ in $W(0, T)$ as $\lambda \rightarrow 0$, where $z_{v, \tilde{a}, \tilde{b}}$ is the unique solution in $W(0, T)$ of the problem

$$\begin{cases} z_t(x, t) - \Delta z(x, t) + \tilde{a}(x)\tilde{b}'(y_{u, \tilde{a}, \tilde{b}})z(x, t) = v(x, t)\chi_{\omega_{in}}(x) & \text{in } Q, \\ z(x, t) = 0 & \text{on } \Sigma, \\ z(x, 0) = 0 & \text{in } \Omega. \end{cases}$$

Furthermore, $z_{v, a, b}(T) = z_{v, \tilde{a}, \tilde{b}}(T)$ in ω_{out} . Now, we introduce $w = z_{v, a, b} - z_{v, \tilde{a}, \tilde{b}}$ that satisfies $w(T) = 0$ in ω_{out} and it can be viewed as the unique solution in $W(0, T)$ of the linear problem

$$\begin{cases} w_t - \Delta w + ab'(y_{u, a, b})w = (\tilde{a}\tilde{b}'(y_{u, \tilde{a}, \tilde{b}}) - ab'(y_{u, a, b}))z_{v, \tilde{a}, \tilde{b}} & \text{in } Q, \\ w(x, t) = 0 & \text{on } \Sigma, \\ w(x, 0) = 0 & \text{in } \Omega. \end{cases} \quad (2)$$

For each $\phi \in L^2(\omega_{out})$, we consider the following adjoint problem:

$$\begin{cases} -p_t(x, t) - \Delta p(x, t) + a(x)b'(y_{u, a, b}(x, t))p(x, t) = 0 & \text{in } Q, \\ p(x, t) = 0 & \text{on } \Sigma, \\ p(x, T) = \phi(x)\chi_{\omega_{out}}(x) & \text{in } \Omega, \end{cases} \quad (3)$$

whose unique solution p_ϕ also belongs to $W(0, T)$.

Multiplying the equation (2) by p_ϕ , integrating by parts and taking into account that $w(T)p(T) = 0$ in Ω , we obtain

$$\int_Q (\tilde{a}(x)\tilde{b}'(y_{u, \tilde{a}, \tilde{b}}(x, t)) - a(x)b'(y_{u, a, b}(x, t)))z_{v, \tilde{a}, \tilde{b}}(x, t)p_\phi(x, t)dxdt = 0, \quad (4)$$

for all $v \in L^2(\omega_{in} \times (0, T))$ and all $\phi \in L^2(\omega_{out})$.

Finally, let us introduce a new boundary problem associated with the previous ones, given by

$$\begin{cases} -q_t - \Delta q + \tilde{a}\tilde{b}'(y_{u, \tilde{a}, \tilde{b}})q = (\tilde{a}\tilde{b}'(y_{u, \tilde{a}, \tilde{b}}) - ab'(y_{u, a, b}))p_\phi & \text{in } Q, \\ q(x, t) = 0 & \text{on } \Sigma, \\ q(x, T) = 0 & \text{in } \Omega. \end{cases} \quad (5)$$

Once more, multiplying the PDE in (5) by $z_{v, \tilde{a}, \tilde{b}}$, integrating by parts, taking into account (4) and the PDE satisfied by $z_{v, \tilde{a}, \tilde{b}}$, we derive

$$\int_Q v(x, t)\chi_{\omega_{in}}(x)q(x, t)dxdt = 0, \text{ for all } v \in L^2(\omega_{in} \times (0, T)). \quad (6)$$

Therefore, $q(x, t) = 0$ in $\omega_{in} \times (0, T)$ and using the PDE in (5) we get

$$(\tilde{a}(x)\tilde{b}'(y_{u,\tilde{a},\tilde{b}}(x, t)) - a(x)b'(y_{u,a,b}(x, t)))p_\phi(x, t) = 0, \quad (7)$$

in $\omega_{in} \times (0, T)$, for all $\phi \in L^2(\omega_{out})$.

In particular, taking $t = T$ in (7), we have

$$(\tilde{a}(x)\tilde{b}'(y_{u,\tilde{a},\tilde{b}}(x, T)) - a(x)b'(y_{u,a,b}(x, T)))\phi(x)\chi_{\omega_{out}}(x) = 0 \quad \text{in } \omega_{in},$$

for all $\phi \in L^2(\omega_{out})$. Therefore,

$$\tilde{a}(x)\tilde{b}'(y_{u,\tilde{a},\tilde{b}}(x, T)) = a(x)b'(y_{u,a,b}(x, T)) \quad \text{in } \omega_{in} \cap \omega_{out}. \quad (8)$$

Taking into account that $y_{u,a,b}(T) = y_{u,\tilde{a},\tilde{b}}(T)$ in ω_{out} for each u in $L^2(\omega_{in} \times (0, T))$ and that Theorem 1 implies the density of

$$\{y_{u,a,b}(T)|_{\omega_{in} \cap \omega_{out}} : u \in L^2(\omega_{in} \times (0, T))\}$$

in $L^2(\omega_{in} \cap \omega_{out})$, for each $s \in \mathbb{R}$ we can select a sequence $\{u_m\} \subset L^2(\omega_{in} \times (0, T))$ such that as m tends to $+\infty$

$$y_{u_m,a,b}(T) \longrightarrow s \quad \text{in } L^2(\omega_{in} \cap \omega_{out}).$$

Using the identity (8) for each u_m and passing to the limit in m , we arrive to $\tilde{a}(x)\tilde{b}'(s) = a(x)b'(s)$ for all $(x, s) \in (\omega_{in} \cap \omega_{out}) \times \mathbb{R}$. When $\tilde{a}(x) \neq 0$ and $b'(s) \neq 0$, we derive the existence of a real constant C_1 such that

$$\frac{a(x)}{\tilde{a}(x)} = \frac{\tilde{b}'(s)}{b'(s)} = C_1 \quad \forall x \in \omega_{in} \cap \omega_{out}, \quad \forall s \in \mathbb{R}.$$

Due to the unique analytic continuation property, $a(x) = C_1\tilde{a}(x)$ for all $x \in \Omega$. Moreover, there exists a real constant $C_2 \in \mathbb{R}$ such that $\tilde{b}(s) = C_1b(s) + C_2$ for all $s \in \mathbb{R}$. Let us point out that $\tilde{a}(x)\tilde{b}(s) = a(x)b(s) + C_2\tilde{a}(x)$ for all $(x, s) \in \Omega \times (0, T)$.

To show that $C_2 = 0$, we fix $u \in L^2(\omega_{in} \times (0, T))$ and introduce $\zeta = y_{u,a,b} - y_{u,\tilde{a},\tilde{b}}$. Arguing as at the beginning of the proof, we derive that ζ is the unique solution of the following linear problem:

$$\begin{cases} \zeta_t(x, t) - \Delta \zeta(x, t) + a(x)c(x, t)\zeta(x, t) = C_2\tilde{a}(x) & \text{in } Q, \\ \zeta(x, t) = 0 & \text{on } \Sigma, \\ \zeta(x, 0) = 0 & \text{in } \Omega, \end{cases} \quad (9)$$

where $c(x, t) = \int_0^1 b'(sy_{u,a,b}(x, t) + (1-s)y_{u,\tilde{a},\tilde{b}}(x, t))ds$. Moreover, by hypothesis we know that $\zeta(x, T) = 0$ in ω_{out} . Once more, given $\phi \in L^2(\omega_{out})$, we consider $\hat{p}$ the unique solution of the adjoint problem

$$\begin{cases} -\hat{p}_t(x, t) - \Delta \hat{p}(x, t) + a(x)c(x, t)\hat{p}(x, t) = 0 & \text{in } Q, \\ \hat{p}(x, t) = 0 & \text{on } \Sigma, \\ \hat{p}(x, T) = \phi(x)\chi_{\omega_{out}}(x) & \text{in } \Omega. \end{cases} \quad (10)$$

Multiplying by $\hat{p}$ the PDE satisfied by ζ , integrating by parts and using that $\hat{p}(T)\zeta(T) = 0$ in Ω , we deduce

$$C_2 \int_Q \tilde{a}(x)\hat{p}(x, t)dxdt = 0. \quad (11)$$

Choosing $\phi(x) \geq 0$ for all $x \in \omega_{out}$, $\phi \not\equiv 0$, the Maximum Principle implies $\hat{p}(x, t) \geq 0$ in Q , $\hat{p} \not\equiv 0$. Noticing that $\tilde{a}(x) \geq 0$ in Ω , the equality (11) implies $C_2 = 0$. ■

Remarks.

- i) It follows from the proof that the functions $a(x)$ and $b(y)$ can be determined separately in a unique way, up to a multiplicative constant, because $a(x)b(y) = \frac{a(x)}{C_1}C_1b(y)$ for all $C_1 > 0$.
- ii) Let us point out that the conclusion of Theorem 2 remains valid if it is supposed that the relation $y_{u,a,b}(T) = y_{u,\tilde{a},\tilde{b}}(T)$ in ω_{out} only holds when $u \in L^2(\omega_{in} \times [T - \delta, T])$, where $\delta \in (0, T)$ is fixed. This is a consequence of the fact that the controllability result also holds in this case, i.e. under the conditions of Theorem 1, the set $R(T) = \{y_{u,a,b}(x, T) : u \in L^2(\omega_{in} \times [T - \delta, T])\}$ is dense in $L^2(\Omega)$, (see [3, Remark 3.1 (c)]). Here, $y_{u,a,b}$ denotes the unique solution of problem (1), when the right hand term of the PDE is given by $u(x, t)\chi_{\omega_{in} \times [T - \delta, T]}(x, t)$.
- iii) Exactly the same technique can be used to prove the uniqueness of the term $a(x)y + b(y)$ for the Dirichlet boundary problem associated to the PDE $y_t - \Delta y + a(x)y + b(y) = u\chi_{\omega_{in}}$, under the assumptions that a is analytic on $\overline{\Omega}$ and H2) (no sign condition has to be imposed on a).

3. The identification process. Numerical experiments.

In this section, problem (1) is considered in the one-dimensional space case (i.e. $n = 1$). We are concerned with getting a numerical approximation $\tilde{a}(x)\tilde{b}(y)$ of the nonlinear term $a(x)b(y)$ from a finite number of observations η_{ij} . Let us consider (1) for a finite set of source terms $\{u_j\}_{j=1}^{n_u}$. For each u_j , we have measurements η_{ij} of $y_{u_j,a,b}(x_i, T)$ at some points $x_i \in \omega_{out}$.

For our experiments we have chosen $y_0(x) = 0.5x(4 - x)$, $T = 1$ and $\Omega = (0, 4)$ with the grid $\{x_i = i/10\}_{i=0}^{40}$. The data η_{ij} correspond with the nodes x_i in $\omega_{out} = (0.4 - 1.e - 12, 2 + 1.e - 12)$ and they were obtained solving (1) with six of the following source terms:

$$u_j(x, t) = \begin{cases} -50(i_j - 2k_j) \sin(i_j \pi t) \sin(2k_j \pi x) & \text{if } x \in (0, 0.5), \\ 0 & \text{if } x \in [0.5, 4), \end{cases} \quad (12)$$

for $i_j, k_j \in \{1, 2, \dots, 15\}$. Note that $\omega_{in} = (0, 0.5)$ and only the node $x_i = 0.4$ belongs to $\omega_{in} \cap \omega_{out}$.

The first step of the identification process is related with the choice of the interval (for the variable y) where we are going to recover the non-linearity $a(x)b(y)$. Our working interval is an enlargement of the interval defined by the data, $I = [\eta_{min} = \min \eta_{ij}, \eta_{max} = \max \eta_{ij}]$, by taking a safety barrier, $M > 0$, in both extremes: $I_w = [\eta_{min} - M, \eta_{max} + M]$. In our calculations we took an equidistant grid $\{y_i\}_{y=0}^{n_y}$ in I_w with a mesh size $h = (\eta_{max} - \eta_{min})/10$.

The reconstruction of $a(x)b(y)$ consists in determining two vectors $\hat{a} = (\hat{a}_0, \dots, \hat{a}_{n_x}) \in \mathbb{R}^{n_x+1}$ and $\hat{b} = (\hat{b}_0, \dots, \hat{b}_{n_y}) \in \mathbb{R}^{n_y+1}$, where $\hat{a}_i$ and $\hat{b}_i$ are the coefficients of the approximations

$$\check{a}(x) = \sum_{i=0}^{n_x} \hat{a}_i B_i(x) \quad \text{and} \quad \check{b}(y) = \sum_{i=0}^{n_y} \hat{b}_i \hat{B}_i(y), \quad (13)$$

being B_i and $\hat{B}_i$ piecewise linear B-splines verifying

$$B_i(x_{2j}) = \delta_{ij} \text{ for } i, j = 0, \dots, n_x \quad \text{and} \quad \hat{B}_i(y_j) = \delta_{ij} \text{ for } i, j = 0, \dots, n_y.$$

Let us point out that we use double mesh size for the function $\check{a}$ than for the discretization of Ω . For inverse problems, some experiments indicate that it is better to take a bigger mesh size for the spatially varying parameters than for the solution y .

Now we are ready to state the finite-dimensional problem for recovering $a(x)b(y)$. Following an output least squares method with regularization, we consider the minimization problem

$$\min_{(\hat{a}, \hat{b}) \in U_{ad}} J(\hat{a}, \hat{b}), \quad (14)$$

where

$$\begin{aligned} J(\hat{a}, \hat{b}) = & \frac{1}{2} \sum_{j=1}^{n_u} \int_{\omega_{out}} (y_{u_j, \check{a}, \check{b}}(x, T) - \eta_j(x))^2 dx + \\ & + \gamma \left(\int_{\Omega \times I_w} \|\nabla(\check{a}(x)\check{b}(y))\|^2 dx dy + \epsilon \left(\int_{\Omega} (\check{a}(x))^2 dx + \int_{I_w} (\check{b}(y))^2 dy \right) \right), \end{aligned} \quad (15)$$

being $\eta_j(x)$ a continuous observation which interpolates the data η_{ij} , $i = 4, \dots, 20$, $\gamma > 0$, $\epsilon \geq 0$ and U_{ad} the set of feasible vectors given by

$$U_{ad} = \{(\hat{a}, \hat{b}) \in \mathbb{R}^{n_x+1} \times \mathbb{R}^{n_y+1} : \hat{a}_i \geq 0, \quad i = 0, \dots, n_x\}. \quad (16)$$

Hence, the objective function is the sum of a least squares term and a regularization term. If $\epsilon > 0$, the last term is useful for proving the coercivity of the functional. On the other hand, if $\epsilon = 0$, we also can obtain the coercivity of J by imposing some constraints, for instance: $\check{a}(x_0)^2 + \check{a}(x_{n_x})^2 \leq c_1$, $\check{b}(y_0)^2 + \check{b}(y_{n_y})^2 \leq c_2$, $\|\hat{a}\|^2 \geq c_3 > 0$ and $\|\hat{b}\|^2 \geq c_3 > 0$; but this kind of formulation needs some a priori knowledge about the functions to recover. In the simpler case $a(x) \equiv 1$, taking $\check{b}(y_0) = 0$, $\check{b}(y_{n_y}) = 0$, J is coercive on $U_{ad} = \mathbb{R}^{n_y-1}$ (see [2]).

The following result states the existence of minimizers.

Theorem 3 *There exists at least one solution to the problem (14).*

Proof. Because of J is continuous (see Appendix of [2]), coercive and the feasible set U_{ad} is non-empty and closed, the result follows. ■

Problem (14) was solved by using the subroutine E04UCF from NAG Library, that implements a sequential quadratic programming (SQP) algorithm. Each evaluation of the objective function was computed by taking (for each j) a linear spline which interpolates the values $\eta_{ij} - y_{u_j, \check{a}, \check{b}}(x_i, T)$, where $y_{u_j, \check{a}, \check{b}}(x_i, T)$ were obtained by solving (1) with a linearized Crank-Nicholson-Galerkin method; to this end, we took a semidiscrete approach with 39 piecewise linear finite elements for the discretization of the spatial domain and the nodes $\{iT/10\}_{i=0}^{10}$ for the discretization of the time variable.

About the stopping test, let us mention that the optimization algorithm terminates successfully if the following conditions are satisfied,

$$\frac{\|(\hat{a}^k - \hat{a}^{k-1}, \hat{b}^k - \hat{b}^{k-1})\|}{1 + \|(\hat{a}^k, \hat{b}^k)\|} < \sqrt{\epsilon}, \quad (17)$$

$$\frac{\|(\nabla J(\hat{a}^k, \hat{b}^k))_{FR}\|}{1 + \max\{1 + |J(\hat{a}^k, \hat{b}^k)|, \|(\nabla J(\hat{a}^k, \hat{b}^k))_{FR}\|\}} \leq \sqrt{\epsilon}, \quad (18)$$

$$(\nabla J(\hat{a}^k, \hat{b}^k))_L \geq 0,$$

where $(\nabla J(\hat{a}^k, \hat{b}^k))_{FR}$ is the vector with the components of $\nabla J(\hat{a}^k, \hat{b}^k)$ corresponding to the free variables (i.e. not fixed at the bounds) and $(\nabla J(\hat{a}^k, \hat{b}^k))_L$ is formed by the rest of the components of $\nabla J(\hat{a}^k, \hat{b}^k)$.

Concerning the bound conditions described in H1) and H2), let us point out that the lower bounds in (16) ensure $\check{a}^k(x) \geq 0$ for all k

and all $x \in \Omega$. On the other hand, from $J(\hat{a}^k, \hat{b}^k) \leq J(\hat{a}^0, \hat{b}^0)$, when $\int_{\Omega} (\check{a}^k(x))^2 dx \geq c_1 > 0$, it follows that

$$((\check{b}^k)'(y))^2 = \left(\sum_{i=0}^{n_y} \hat{b}_i \hat{B}'_i(y) \right)^2 \leq \frac{J(\hat{a}^0, \hat{b}^0)}{\gamma h c_1},$$

for all $y \notin \{y_0, \dots, y_{n_y}\}$ and for all k .

Our numerical experiments were carried out in MATLAB 6 on a DELL PowerEdge 2500 with 1 GB of memory and running under Red Hat Linux, taking $(0.5, \dots, 0.5)$ as the starting point.

To generate the observations η_{ij} , we solve (1) and take

$$\eta_{ij} = (1 + \delta_{ij}) y_{u_j, a, b}(x_i, T),$$

where δ_{ij} are uniformly distributed random numbers in $[-\delta, \delta]$, with δ denoting the noise level.

We have considered two examples.

Example 1 Knowing $a(x) \equiv 1$ in Ω , we try to recover $b(y) = 3 \exp(-y^2)$ from the data obtained with six source terms u_j given by (12) with (i_j, k_j) taking the values $(1, 1), (1, 7), (2, 2), (2, 3), (2, 4)$ and $(2, 6)$. In Table 1 and Figure 1 we present the computed results obtained taking $\epsilon = 0$, $b(y_0) = 0$ and $b(y_{n_y}) = 0$ in the formulation of problem (14), $\varepsilon = 1.e - 9$ for the stopping test. Table 1 summarizes numerical results corresponding to $\delta = 0$ (without explicit data perturbation) and $\delta = 0.1$ (noisy data). For each value of δ , the nodes in I_w are listed in the first column (4 nodes in the safety barrier and 11 nodes on the interval I). For each node y_i , we report the value of the numerical solution, $\check{b}(y_i)$, and the value of the exact solution, $b(y_i)$, in the second and third column respectively, and finally the relative error $|\check{b}(y_i) - b(y_i)| / \max(1, |b(y_i)|)$. At the bottom of the table we show the current values of the penalty parameter γ and the number of iterations required by the optimization routine.

Figure 1 contains two graphics. The nonlinear term $b(y)$ obtained from data with $\delta = 0$ and $\delta = 0.1$ appears on the left and on the right, respectively. Each graphic shows the exact solution b and the numerically identified solution $\check{b}$. We asterisk the values $\check{b}(y_i)$. Moreover, in each graphic we show the data η_{ij} on the axis. Remark that these observations depend on the source terms u_j . Here, the main point is to choose u_j in such a way that the corresponding observations fill as much as possible the interval where we want to recover the coefficient. Hence, it is our experience that, with a good choice, a small number of functions can be enough to recover the nonlinearity.

Table 1. Example 1

Undisturbed data ($\delta = 0$)				Noisy data ($\delta = 0.1$)			
y_i	$\tilde{b}(y_i)$	$b(y_i)$	Error	y_i	$\tilde{b}(y_i)$	$b(y_i)$	Error
-1.8744	0.0309	0.0894	5.854e-2	-1.9181	-0.0374	0.0757	1.132e-1
-1.5088	0.4179	0.3080	1.099e-1	-1.5362	0.2514	0.2832	3.181e-2
-1.1431	0.7212	0.8121	9.095e-2	-1.1544	0.8193	0.7914	2.788e-2
-0.7775	1.6366	1.6390	1.465e-3	-0.7725	1.7270	1.6518	4.556e-2
-0.4119	2.5548	2.5319	9.043e-3	-0.3906	2.6902	2.5755	4.455e-2
-0.0463	3.0884	2.9936	3.166e-2	-0.0088	2.9227	2.9998	2.568e-2
0.3194	2.7510	2.7091	1.546e-2	0.3731	2.6009	2.6101	3.526e-3
0.6850	1.8653	1.8765	5.932e-3	0.7550	1.8487	1.6965	8.968e-2
1.0506	0.9815	0.9948	1.336e-2	1.1369	0.8444	0.8238	2.064e-2
1.4163	0.3796	0.4037	2.407e-2	1.5188	0.1584	0.2988	1.404e-1
1.7819	0.0994	0.1254	2.599e-2	1.9006	0.0732	0.0810	7.731e-3
2.1475	0.0363	0.0298	6.480e-3	2.2825	0.0862	0.0164	6.976e-2
2.5131	-0.0011	0.0054	6.491e-3	2.6644	0.1814	0.0025	1.789e-1
2.8788	0.0102	0.0008	9.404e-3	3.0463	0.6878	0.0003	6.875e-1
3.2444	-0.0106	0.0001	1.072e-2	3.4281	0.5840	0.0000	5.840e-1
$\gamma = 1.e - 12$		ITER=121		$\gamma = 1.e - 4$		ITER=66

Figure 1. Example 1. $b(y) = 3exp(-y^2)$.

From our point of view, in this case, the numerical results are qualitatively good in the interval I defined by the data η_{ij} , even under the presence of a relatively high level of noise (up to 10%). As one can expect, in the safety zone $I_w \setminus I$, the approximation becomes worse, but it seems still acceptable.

Example 2 We consider the recuperation of $a(x)b(y)$ with $a(x) = \sin(\pi x/4)$ and $b(y) = 3exp(-y^2)$ from the data corresponding to the

source terms u_j given by (12) with (i_j, k_j) taking the values $(1, 1)$, $(1, 8)$, $(2, 2)$, $(3, 5)$, $(5, 6)$ and $(15, 4)$. No explicit noise is included here (i.e. $\delta = 0$). In Figure 2, we show the computed results for $\epsilon = 1.e - 12$ and $\varepsilon = 2.e - 9$, assuming $\hat{a}_1 \geq 1.e - 12$. This figure contains four graphics. On the top, the numerical approximation (obtained after 151 iterations) and the exact nonlinearity $a(x)b(y)$ appear on the left and on the right, respectively. On the bottom, the relative errors

$$\frac{|\check{a}(x_i)\check{b}(y_j) - a(x_i)b(y_j)|}{\max(1, |a(x_i)b(y_j)|)}$$

are shown.

Figure 2. Example 2. $a(x) = \sin(\pi x/4)$ and $b(y) = 3\exp(-y^2)$.

In this example, we think that the computed results are relatively satisfactory when $x \in (0, 3.4]$ and they can be improved in $(3.4, 4)$. In our opinion, the deterioration in this zone can be due to the lack of information there: let us remind that ω_{in} and ω_{out} are both included in $[0, 2.1)$, while $\Omega = (0, 4)$. This difficulty does not appear when the nonlinear term is independent of the space variable x .

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