ITERATIVE PROCEDURE FOR STABILIZING SOLUTIONS OF DIFFERENTIAL RICCATI TYPE EQUATIONS ARISING IN STOCHASTIC CONTROL

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Abstract The paper presents a numerical algorithm to determine the stabilizing solutions of the Riccati type differential equations arising in the optimal control of time-varying stochastic systems subjected both to multiplicative white noise and to Markov jumps.

Keywords: Stochastic linear systems, optimal control, stabilizing solutions of Riccati type systems, iterative procedures

1. Riccati equations arising in stochastic control

In the present paper the following Riccati type system of differential equations is considered:

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$$\begin{aligned}
& \frac{d}{dt}X(t, i) + A_0^*(t, i)X(t, i) + X(t, i)A_0(t, i) + \sum_{k=1}^r A_k^*(t, i)X(t, i)A_k(t, i) \\
& + \sum_{j=1}^d q_{ij}X(t, j) - (X(t, i)B_0(t, i) + \sum_{k=1}^r A_k^*(t, i)X(t, i)B_k(t, i) \\
& + L(t, i))(R(t, i) + \sum_{k=1}^r B_k^*(t, i)X(t, i)B_k(t, i))^{-1}(B_0^*(t, i)X(t, i) + \\
& \sum_{k=1}^r B_k^*(t, i)X(t, i)A_k(t, i) + L^*(t, i)) + M(t, i) = 0, i \in \mathcal{D}.
\end{aligned} \tag{1}$$

This system arises in optimal control problems associated with the time-varying stochastic systems subjected both to multiplicative white noise and to Markovian jumping:

$$\begin{aligned}
dx(t) &= [A_0(t, \eta(t))x(t) + B_0(t, \eta(t))u(t)] dt \\
&+ \sum_{k=1}^r [A_k(t, \eta(t))x(t) + B_k(t, \eta(t))u(t)] dw_k(t)
\end{aligned} \tag{2}$$

where $t \in \mathbf{R}_+$, with the state vector $x \in \mathbf{R}^n$ and with the control inputs $u \in \mathbf{R}^m$, $\eta(t), t \geq 0$ is a right Markov chain with the state space $\mathcal{D} = \{1, \dots, d\}$ and the probability transition matrix $P(t) = [p_{ij}(t)] = e^{Qt}$, $t \geq 0$ in which the elements q_{ij} of Q have the property that $\sum_{i=1}^r q_{ij} = 0, i \in \mathcal{D}$ and $q_{ij} \geq 0$ if $i \neq j$. $w(t) = (w_1(t), \dots, w_r(t))^*$ is an r -dimensional standard Wiener process. Throughout the paper it is assumed that the coefficients of (1) are bounded and continuous matrix valued functions. By $\mathcal{S}_n^d$ it is denoted the $\mathcal{S}_n \oplus \dots \oplus \mathcal{S}_n$, $\mathcal{S}_n$ being the subspace of symmetric $n \times n$ matrices. For the next developments the following two definitions are reminded:

Definition 1. The system $(A_0, A_1, \dots, A_r; Q)$ is exponentially stable in mean square (ESMS) if there exist $\alpha > 0, \beta \geq 1$ such that

$$E[|\Phi(t, t_0)x_0|^2 | \eta(t_0) = i] \leq \beta e^{-\alpha(t-t_0)} |x_0|^2, \forall t \geq t_0 \geq 0, x_0 \in \mathbf{R}^n, i \in \mathcal{D},$$

where $\Phi(t, t_0)$ is the fundamental matrix of the linear differential equation obtained from (2) with $u \equiv 0$.

Definition 2. A solution $\tilde{X} : \mathbf{R}_+ \rightarrow \mathcal{S}_n^d$ of the equation (1) is called *stabilizing solution* if it has the following properties:

$$a) \inf_{t \geq 0} |\det[R(t, i) + \sum_{k=1}^r B_k^*(t, i)\tilde{X}(t, i)B_k(t, i)]| > 0, i \in \mathcal{D}.$$

b) The system $(A_0 + B_0\tilde{F}, A_1 + B_1\tilde{F}, \dots, A_r + B_r\tilde{F}; Q)$ is stable, where $\tilde{F}(t) = (\tilde{F}(t, 1), \tilde{F}(t, 2), \dots, \tilde{F}(t, d))$,

$$\begin{aligned} \tilde{F}(t, i) &= -[R(t, i) + \sum_{k=1}^r B_k^*(t, i)\tilde{X}(t, i)B_k(t, i)]^{-1}[B_0(t, i)\tilde{X}(t, i) \\ &+ \sum_{k=1}^r B_k^*(t, i)\tilde{X}(t, i)A_k(t, i) + L^*(t, i)], \quad (t, i) \in \mathbf{R}_+ \times \mathcal{D}. \end{aligned}$$

Consider the cost function:

$$\begin{aligned} J(t_0, x_0, u) &= E \int_{t_0}^{\infty} [x_u^*(t) M(t, \eta(t)) x_u(t) + x_u^*(t) L(t, \eta(t)) u(t) \\ &+ u^*(t) L^*(t, \eta(t)) x_u(t) + u^*(t) R(t, \eta(t)) u(t)] dt \end{aligned} \quad (3)$$

where $M(t, i) = M^*(t, i)$; $R(t, i) = R^*(t, i)$, $(t, i) \in \mathbf{R}_+ \times \mathcal{D}$ and $x_u(t)$ denotes the solution of the system (2) corresponding to the input $u(\cdot)$ with the initial condition $(t_0, x_0) \in \mathbf{R}_+ \times \mathbf{R}^n$. In [4] it is proved that the solution of the optimal control problem requiring to minimize the above cost function under the constraint $\lim_{t \rightarrow \infty} E|x_u(t, t_0, x_0)|^2 = 0$ depends on the stabilizing solution of the Riccati system (1). In the particular case when $A_k = 0, B_k = 0, k = 1, \dots, r$ and $\mathcal{D} = \{1\}$ equation (1) reduces to the well-known deterministic control Riccati equation considered in the pioneering work of Kalman [8]. If $A_k = 0, B_k = 0, k = 1, \dots, r$ and $d > 2$, one obtains the case when the system is subjected only to Markovian jumps for which the quadratic optimization problem has been studied for instance in [1], [7], [9], [10] and [11]. If $\mathcal{D} = \{1\}$ one gets the optimal control problem for stochastic systems with multiplicative white noise, previously considered in [2], [3], [5], [6], [12], [13].

2. An iterative procedure to compute the stabilizing solution of Riccati type systems

In this section an iterative procedure to compute the stabilizing solution of (1) is presented. The Riccati type system (1) can be rewritten in a compact form as:

$$\frac{d}{dt}X(t) + \mathcal{L}^*(t)X(t) - \mathcal{P}^*(t, X(t))\mathcal{R}^{-1}(t, X(t))\mathcal{P}(t, X(t)) + M(t) = 0 \quad (4)$$

$\mathcal{L}^*(t)$ being the adjoint of the operator $\mathcal{L}$ defined as $\mathcal{L}(t) : \mathcal{S}_n^d \rightarrow \mathcal{S}_n^d$ by

$$\begin{aligned} (\mathcal{L}(t)X)(i) &= A_0(t, i)X(i) + X(i)A_0^*(t, i) + \\ &+ \sum_{k=1}^r A_k(t, i)X(i)A_k^*(t, i) + \sum_{j=1}^d q_{ji}X(j) \end{aligned} \quad (5)$$

$t \geq 0, i \in \mathcal{D}, X \in \mathcal{S}_n^d$, and $\mathcal{P}, \mathcal{R}$ and M are defined as:

$$X \rightarrow \mathcal{P}(t, X) : \mathcal{S}_n^d \rightarrow \mathcal{M}_{m,n}^d$$

$$\begin{aligned}
\mathcal{P}(t, X) &= (\mathcal{P}_1(t, X), \mathcal{P}_2(t, X), \dots, \mathcal{P}_d(t, X)), \\
\mathcal{P}_i(t, X) &= B_0^*(t, i)X(i) + \sum_{k=1}^r B_k^*(t, i)X(i)A_k(t, i) + L^*(t, i) \\
X &\rightarrow \mathcal{R}(t, X) : \mathcal{S}_n^d \rightarrow \mathcal{S}_m^d \text{ by} \\
\mathcal{R}(t, X) &= (\mathcal{R}_1(t, X), \mathcal{R}_2(t, X), \dots, \mathcal{R}_d(t, X)), \\
\mathcal{R}_i(t, X) &= R(t, i) + \sum_{k=1}^r B_k^*(t, i)X(i)B_k(t, i), \\
M(t) &= (M(t, 1), M(t, 2), \dots, M(t, d)) \in \mathcal{S}_n^d.
\end{aligned}$$

The concept of stochastic stabilizability is introduced in standard manner (see, e.g. [4]).

Lemma 3. Assume that the system (2) is stochastically stabilizable. Let $\tilde{F}_0(t) = (\tilde{F}_0(t, 1), \tilde{F}_0(t, 2), \dots, \tilde{F}_0(t, d))$ be a stabilizing feedback gain and let $X_0(t) = (X_0(t, 1), \dots, X_0(t, d))$ be a bounded with bounded derivative solution of the linear differential inequality on $\mathcal{S}_n^d$:

$$\frac{d}{dt}X_0(t) + \mathcal{L}_{\tilde{F}_0}^*(t)X_0(t) + M_0(t) \leq 0 \quad (6)$$

where $M_0(t) = (M_0(t, 1), M_0(t, 2), \dots, M_0(t, d))$, $M_0(t, i) = M(t, i) + \varepsilon I_n + L(t, i)\tilde{F}_0(t, i) + \tilde{F}_0^*(t, i)L^*(t, i) + \tilde{F}_0(t, i)R(t, i)\tilde{F}_0(t, i)$, $\varepsilon > 0$ be fixed.

Under the considered assumptions,

$$X_0(t) - \hat{X}(t) \gg 0 \quad (7)$$

for any bounded solution $\hat{X}(t) : \mathbf{R}_+ \rightarrow \mathcal{S}_n^d$ of the differential inequality on $\mathcal{S}_n^d$:

$$\frac{d}{dt}X(t) + \mathcal{L}^*(t)X(t) - \mathcal{P}^*(t, X(t))\mathcal{R}^{-1}(t, X(t))\mathcal{P}(t, X(t)) + M(t) \gg 0 \quad (8)$$

satisfying the condition

$$\mathcal{R}_i(t, \hat{X}(t)) \geq \hat{\rho}I_n \quad (9)$$

for all $(t, i) \in \mathbf{R}_+ \times \mathcal{D}$, $\hat{\rho} > 0$ being a constant.

Proof. If $\hat{X}(t)$ is a bounded solution of (8) which verifies (9) one can define $\hat{M}(t) = (\hat{M}(t, 1), \hat{M}(t, 2), \dots, \hat{M}(t, d))$ by

$$\hat{M}(t) = \frac{d}{dt}\hat{X}(t) + \mathcal{L}^*(t)\hat{X}(t) - \mathcal{P}^*(t, \hat{X}(t))\mathcal{R}^{-1}(t, \hat{X}(t))\mathcal{P}(t, \hat{X}(t)) + M(t), \quad (10)$$

$t \in \mathbf{R}_+$. Clearly $\hat{M}(t) \geq 0$, by direct computations it follows that that

$$\begin{aligned} & \frac{d}{dt} \hat{X}(t) + \mathcal{L}_{\tilde{F}_0}^*(t) \hat{X}(t) + M(t) + L(t) \tilde{F}_0(t) + \tilde{F}_0^*(t) L(t) + \tilde{F}_0^*(t) R(t) \tilde{F}_0(t) \\ & - \hat{M}(t) - (\hat{F}(t) - \tilde{F}_0(t))^* \mathcal{R}(t, \hat{X}(t)) (\hat{F}(t) - \tilde{F}_0(t)) = 0 \end{aligned} \quad (11)$$

where $\hat{F}(t) = (\hat{F}(t, 1), \hat{F}(t, 2), \dots, \hat{F}(t, d))$ with

$$\hat{F}(t, i) = -\mathcal{R}_i^{-1}(t, \hat{X}(t)) \mathcal{P}_i(t, \hat{X}(t)), \quad t \in \mathcal{I}, \quad i \in \mathcal{D}. \quad (12)$$

From (11) and (6) one gets:

$$\begin{aligned} & \frac{d}{dt} (X_0(t) - \hat{X}(t)) + \mathcal{L}_{\tilde{F}_0}^*(t) (X_0(t) - \hat{X}(t)) + (\tilde{F}_0(t) - \hat{F}(t))^* \mathcal{R}(t, \hat{X}(t)) \\ & (\tilde{F}_0(t) - \hat{F}(t)) + \varepsilon J^d + \hat{M}(t) \leq 0, \quad t \geq 0. \end{aligned}$$

from which it follows that $X_0(t) - \hat{X}(t) \geq Y(t)$ where $t \rightarrow Y(t) = (Y(t, 1), Y(t, 2), \dots, Y(t, d))$ is the unique bounded solution of the Lyapunov type equation

$$\frac{d}{dt} Y(t) + \mathcal{L}_{\tilde{F}_0}^*(t) Y(t) + \varepsilon J^d = 0. \quad (13)$$

Let $T_0(t, s)$ be the linear evolution operator on $\mathcal{S}_n^d$, defined by the linear differential equation:

$$\frac{d}{dt} S(t) = \mathcal{L}_{\tilde{F}_0}(t) S(t).$$

Since $\tilde{F}_0(t)$ is a stabilizing feedback gain, then there exist positive constants β_0, α_0 such that $\|T_0(t, s)\| \leq \beta_0 e^{-\alpha_0(t-s)}$, $(\forall) t \geq s \geq 0$. Therefore the unique bounded solution of the equation (13) is uniform positive and the proof is complete. Based on (7) one deduces that there exists

$\mu_0 > 0$, such that $\mathcal{R}_i(t, X_0(t)) \geq \mu_0 I_n, t \in \mathcal{I}, i \in \mathcal{D}$. Hence the feedback gain $F_0(t) = (F_0(t, 1), \dots, F_0(t, d))$ is well defined by

$$F_0(t, i) = -\mathcal{R}_i^{-1}(t, X_0(t)) \mathcal{P}_i(t, X_0(t)), \quad i \in \mathcal{D}, t \in \mathcal{I}. \quad (14)$$

Further one will prove that $F_0(t)$ is a stabilizing feedback gain for the system (2). To this end one considers $\hat{X}(t)$ a bounded solution of (8) which verifies condition (9). By direct computation and using (10) and (14) one obtains:

$$\begin{aligned} & \frac{d}{dt} \hat{X}(t) + \mathcal{L}_{F_0}^*(t) \hat{X}(t) + M(t) + L(t) F_0(t) + F_0^*(t) L^*(t) + \\ & F_0^*(t) R(t) F_0(t) - (\hat{F}(t) - F_0(t))^* \mathcal{R}(t, \hat{X}(t)) (\hat{F}(t) - F_0(t)) - \hat{M}(t) = 0. \end{aligned} \quad (15)$$

Further (6) may be rewritten as:

$$\begin{aligned} \frac{d}{dt}X_0(t) + \mathcal{L}_{F_0}^*(t)X_0(t) + M(t) + L(t)F_0(t) + F_0^*(t)L^*(t) + F_0^*(t)R(t)F_0(t) \\ + (F_0(t) - \tilde{F}_0(t))^* \mathcal{R}(t, X_0(t))(F_0(t) - \tilde{F}_0(t)) + \varepsilon J^d \leq 0. \end{aligned} \quad (16)$$

From (15), (16) and (7) it follows that $t \rightarrow X_0(t) - \hat{X}(t)$ is a bounded and uniform positive solution of the linear differential inequality on $\mathcal{S}_n^d$:

$$\frac{d}{dt}X(t) + \mathcal{L}_{F_0}^*(t)X(t) + \frac{\varepsilon}{2}J^d \ll 0.$$

and then, since $X_0(t) - \hat{X}(t) \gg 0$ it results that the system $(A_0 + B_0F_0, A_1 + B_1F_0, \dots, A_r + B_rF_0; Q)$ is stable which shows that $F_0(t)$ is a stabilizing feedback gain. As a consequence we deduce that for each $i \in \mathcal{D}$, the zero state equilibrium of the linear differential equation on $\mathbf{R}^n$,

$$\frac{d}{dt}X(t) = (A_0(t, i) + \frac{1}{2}q_{ii}I_n + B_0(t, i)F_0(t, i))X(t)$$

is exponentially stable. Particularly in the time invariant case it follows that the eigenvectors of the matrices $A_0(i) + \frac{1}{2}q_{ii}I_n + B_0(i)F_0(i)$ are located in the half plane $\text{Re}\lambda < 0$.

Taking $X_0(t), F_0(t)$ as a first step, one constructs iteratively the sequences: $\{X_l(t, i)\}_{l \geq 0}$, $\{F_l(t, i)\}_{l \geq 0}$, $i \in \mathcal{D}$ as follows: $t \rightarrow X_{l+1}(t, i)$ is the unique bounded solution of the Lyapunov equation

$$\begin{aligned} \frac{d}{dt}X_{l+1}(t, i) + \left[\tilde{A}_0(t, i) + B_0(t, i)F_l(t, i) \right]^* X_{l+1}(t, i) \\ + X_{l+1}(t, i) \left[\tilde{A}_0(t, i) + B_0(t, i)F_l(t, i) \right] + M_{l+1}(t, i) = 0 \end{aligned} \quad (17)$$

where $M_{l+1}(t) = (M_{l+1}(t, 1) \dots M_{l+1}(t, d))$ with

$$\begin{aligned} M_{l+1}(t, i) &= M(t, i) + \frac{\varepsilon}{l+2}I_n + L(t, i)F_l(t, i) \\ &\quad + F_l^*(t, i)L^*(t, i) + F_l^*(t, i)R(t, i)F_l(t, i) \\ &\quad + \sum_{k=1}^r [A_k(t, i) + B_k(t, i)F_l(t, i)]^* X_l(t, i) \\ &\quad \times [A_k(t, i) + B_k(t, i)F_l(t, i)] + \sum_{j \neq i} q_{ij}X_l(t, j) \end{aligned} \quad (18)$$

$$\tilde{A}_0(t, i) = A_0(t, i) + \frac{1}{2}q_{ii}I_n$$

$$F_{l+1}(t, i) = -(R(t, i) + \sum_{k=1}^r B_k^*(t, i)X_l(t, i)B_k(t, i))^{-1}(B_0^*(t, i)X_{l+1}(t, i)$$

$$+ \sum_{k=1}^r B_k^*(t, i) X_l(t, i) A_k(t, i) + L^*(t, i)), \quad l \geq 0, i \in \mathcal{D}.$$

Further one proves that

- a) $X_l(t, i) - \hat{X}(t, i) \geq \mu_l I_n > 0$ for all integers $l \geq 0$, $i \in \mathcal{D}$, $t \in \mathcal{I}$, $\hat{X}(t) = (\hat{X}(t, 1) \dots \hat{X}(t, d))$ being an arbitrary bounded solution of (8) which verifies (9) and μ_l is a positive constant which not depend upon $\hat{X}(t)$.
 b) The zero state equilibrium of the linear differential equation on $\mathbf{R}^n$

$$\frac{d}{dt}x(t) = [\tilde{A}_0(t, i) + B_0(t, i)F_l(t, i)]x(t)$$

is exponentially stable for each $i \in \mathcal{D}$, $l \geq 0$.

- c) $X_l(t, i) \geq X_{l+1}(t, i)$, $\forall l \geq 0$, $(t, i) \in \mathcal{I} \times \mathcal{D}$.

Notice that the properties a) and b) have been proved for $l = 0$. It will be shown by induction that a), b), c) are fulfilled for every $l \geq 0$. To this end assume that a), b), c) are fulfilled for the first $l - 1$ terms of the sequences defined by (17) and (18). By direct computation one obtains that if $\hat{X}(t)$ is a bounded solution of the inequality (8) which verifies (9) then

$$\begin{aligned} & \frac{d}{dt} \hat{X}(t, i) + [\tilde{A}_0(t, i) + B_0(t, i)F_{l-1}(t, i)]^* \hat{X}(t, i) \\ & + \hat{X}(t, i)(\tilde{A}_0(t, i) + B_0(t, i)F_{l-1}(t, i)) \\ & + \sum_{k=1}^r [A_k(t, i) + B_k(t, i)F_{l-1}(t, i)]^* \hat{X}(t, i) [A_k(t, i) + B_k(t, i)F_{l-1}(t, i)] \\ & + \sum_{j=1, j \neq i}^d q_{ij} \hat{X}(t, j) + M(t, i) + L(t, i)F_{l-1}(t, i) + F_{l-1}^*(t, i)L^*(t, i) \\ & + F_{l-1}^*(t, i)R(t, i)F_{l-1}(t, i) - \hat{M}(t, i) - (\hat{F}(t, i) - F_{l-1}(t, i))^* \mathcal{R}_i(t, \hat{X}(t)) \\ & \times (\hat{F}(t, i) - F_{l-1}(t, i)) = 0, \end{aligned} \tag{19}$$

$\hat{M}(t, i)$, $\hat{F}(t, i)$ being defined in (10) and (12) respectively.

Using (17) with l replaced by $l - 1$ one gets

$$\begin{aligned} & \frac{d}{dt} [X_l(t, i) - \hat{X}(t, i)] + [\tilde{A}_0(t, i) + B_0(t, i)F_{l-1}(t, i)]^* [X_l(t, i) - \hat{X}(t, i)] \\ & + [X_l(t, i) - \hat{X}(t, i)] [\tilde{A}_0(t, i) + B_0(t, i)F_{l-1}(t, i)] + \frac{\varepsilon}{l+1} I_n + \Delta_l(t, i) = 0 \end{aligned} \tag{20}$$

where

$$\begin{aligned} \Delta_l(t, i) = & \sum_{k=1}^r [A_k(t, i) + B_k(t, i)F_{l-1}(t, i)]^* [X_{l-1}(t, i) - \hat{X}(t, i)] \\ & \times [A_k(t, i) + B_k(t, i)F_{l-1}(t, i)] + \sum_{j=1, j \neq i}^d q_{ij} (X_{l-1}(t, j) - \hat{X}(t, j)) \\ & + \hat{M}(t, i) + (\hat{F}(t, i) - F_{l-1}(t, i))^* \mathcal{R}_i(t, \hat{X}(t)) (\hat{F}(t, i) - F_{l-1}(t, i)). \end{aligned}$$

Since $X_{l-1}(t, i) - \hat{X}(t, i) \geq \mu_{l-1}I_n$ we get $\Delta_l(t, i) \geq 0$. Taking into account that $\tilde{A}_0(t, i) + B_0(t, i)F_{l-1}(t, i)$ generates an exponentially stable evolution we may conclude that the equation (20) has a unique bounded solution which is uniform positive definite. Hence there exists $\mu_l > 0$, such that $X_l(t, i) - \hat{X}(t, i) \geq \mu_l I_n$ and thus a) is fulfilled. Further it follows that $\mathcal{R}_i(t, X_l(t)) \geq \nu_l I_m > 0$.

Using (18) one may write

$$\begin{aligned} & \frac{d}{dt}X_l(t, i) + [\tilde{A}_0(t, i) + B_0(t, i)F_l(t, i)]^*X_l(t, i) + X_l(t, i)[\tilde{A}_0(t, i) \\ & + B_0(t, i)F_l(t, i)] + \sum_{k=1}^r[A_k(t, i) + B_k(t, i)F_l(t, i)]^*X_{l-1}(t, i)[A_k(t, i) \\ & + B_k(t, i)F_l(t, i)] + \sum_{j=1, j \neq i}^d q_{ij}X_{l-1}(t, j) + M(t, i) + \frac{\varepsilon}{l+1}I_n \\ & + L(t, i)F_l(t, i) + F_l^*(t, i)L^*(t, i) + F_l^*(t, i)R(t, i)F_l(t, i) + [F_l(t, i) \\ & - F_{l-1}(t, i)]^*\mathcal{R}_i(t, X_{l-1}(t))[F_l(t, i) - F_{l-1}(t, i)] = 0. \end{aligned} \quad (21)$$

It is easy to see that $t \rightarrow \hat{X}(t, i)$ verifies:

$$\begin{aligned} & \frac{d}{dt}\hat{X}(t, i) + (\tilde{A}_0(t, i) + B_0(t, i)F_l(t, i))^*\hat{X}(t, i) + \hat{X}(t, i)(\tilde{A}_0(t, i) \\ & + B_0(t, i)F_l(t, i)) + \sum_{k=1}^r(A_k(t, i) + B_k(t, i)F_l(t, i))^*\hat{X}(t, i)(A_k(t, i) \\ & + B_k(t, i)F_l(t, i)) + \sum_{j=1, j \neq i}^d q_{ij}\hat{X}(t, j) + M(t, i) + F_l^*(t, i)L^*(t, i) \\ & + L(t, i)F_l(t, i) + F_l^*(t, i)R(t, i)F_l(t, i) - \hat{M}(t, i) \\ & - (\hat{F}(t, i) - F_l(t, i))\mathcal{R}_i(t, \hat{X}(t))(\hat{F}(t, i) - F_l(t, i)) = 0. \end{aligned} \quad (22)$$

Thus it results that for each $i \in \mathcal{D}$, $t \rightarrow X_l(t, i) - \hat{X}(t, i)$ is a bounded and uniformly positive definite solution of the linear differential inequality:

$$\begin{aligned} & \frac{d}{dt}Y(t, i) + [\tilde{A}_0(t, i) + B_0(t, i)F_l(t, i)]^*Y(t, i) \\ & + Y(t, i)[\tilde{A}_0(t, i) + B_0(t, i)F_l(t, i)] + \frac{\varepsilon}{2(l+1)}I_n < 0 \end{aligned}$$

which allows to conclude that the zero state equilibrium of the linear differential equation

$$\frac{d}{dt}x(t) = (\tilde{A}_0(t, i) + B_0(t, i)F_l(t, i))x(t) \quad (23)$$

is exponentially stable and b) is fulfilled.

Subtracting (17) from (21) one gets that $t \rightarrow X_l(t, i) - X_{l+1}(t, i)$ is a bounded solution of the equation

$$\begin{aligned} & \frac{d}{dt}(X_l(t, i) - X_{l+1}(t, i)) + (\tilde{A}_0(t, i) + B_0(t, i)F_l(t, i))^*(X_l(t, i) - X_{l+1}(t, i)) \\ & + (X_l(t, i) - X_{l+1}(t, i))(\tilde{A}_0(t, i) + B_0(t, i)F_l(t, i)) + \hat{\Delta}_l(t, i) = 0 \end{aligned} \quad (24)$$

where

$$\begin{aligned}
 \hat{\Delta}_l(t, i) = & \frac{\varepsilon}{(l+1)(l+2)} I_n + [F_l(t, i) - F_{l-1}(t, i)]^* \mathcal{R}_i(t, X_{l-1}(t)) \\
 & \times [F_l(t, i) - F_{l-1}(t, i)] + \sum_{k=1}^r [A_k(t, i) + B_k(t, i) F_l(t, i)]^* \\
 & \times (X_{l-1}(t, i) - X_l(t, i)) [A_k(t, i) + B_k(t, i) F_l(t, i)] \\
 & + \sum_{j=1, j \neq i}^d q_{ij} (X_{l-1}(t, j) - X_l(t, j)),
 \end{aligned} \tag{25}$$

for $l \geq 1$ and

$$\hat{\Delta}_l(t, i) \geq \frac{\varepsilon}{2} I_n + (F_0(t, i) - \tilde{F}_0(t, i))^* \mathcal{R}_i(t, X_0(t)) (F_0(t, i) - \tilde{F}_0(t, i))$$

for $l = 0$.

Since $\hat{\Delta}_0(t, i) \geq 0$ and the zero state equilibrium of (23) for $l = 0$ is exponentially stable it follows from (24) for $l = 0$ that $X_0(t, i) - X_1(t, i) \geq 0$ and further by induction one obtains that $\hat{\Delta}_l \geq 0$ for $l \geq 1$ which leads to $X_l(t, i) - X_{l+1}(t, i) \geq 0$ and c) is fulfilled.

From a) and c) one concludes that the sequences $\{X_l(t, i)\}_{l \geq 0}, i \in \mathcal{D}$ are convergent. More precisely we have:

Theorem 1 Assume that system $(\mathbf{A}, \mathbf{B}; Q)$ is stochastically stabilizable and that the differential inequality (8) has a bounded with bounded derivative solution $\tilde{X}(t)$ which satisfies (9). Then for any choice of a stabilizing feedback gain $\tilde{F}_0(t) = (\tilde{F}_0(t, 1), \tilde{F}_0(t, 2), \dots, \tilde{F}_0(t, d))$, the sequences $\{X_l(t, i)\}_{l \geq 0}, i \in \mathcal{D}$, constructed as solutions of (17) (the first terms $X_0(t, i)$ obtained by solving (6)) are convergent. If

$$\tilde{X}(t, i) = \lim_{l \rightarrow \infty} X_l(t, i), \quad (t, i) \in \mathcal{I} \times \mathcal{D} \tag{26}$$

then $\tilde{X}(t) = (\tilde{X}(t, 1), \tilde{X}(t, 2), \dots, \tilde{X}(t, d))$ is the stabilizing bounded solution of the system (1) verifying (9).

Remark 1. a) Excepting the first step, when to obtain $X_0(t, i)$ one needs to solve a system of linear inequalities of higher dimension, namely (6), further to obtain the next terms of the sequences $\{X_l(t, i)\}_{l \geq 1}, i \in \mathcal{D}$, one needs to solve a system of d uncoupled Lyapunov equations. Note that to compute the gains $F_l(t, i)$ in (18) we need both the value of $X_l(t, i)$ and the value of $X_{l-1}(t, i)$.

b) Based on the uniqueness of the bounded solution of a Lyapunov equation, it follows that when the system (2) and the cost function (3) are in the time invariant case, then the matrices X_l and F_l do not depend upon t and they are obtained as solutions of algebraic Lyapunov equations.

c) Also from the uniqueness of the bounded solution of a Lyapunov equation we deduce that if the coefficients of the system (1) are θ -periodic functions defined on $\mathbf{R}$, then the bounded solution of (17) are

θ -periodic function too. Hence it is sufficient to compute the values of $X_l(t, i)$, $F_l(t, i)$ on the interval $[0, \theta]$. At each step l , the initial condition $X_l(0, i)$ is obtained by solving the linear equation

$$X_l(0, i) = \Phi_{l,i}^*(\theta, 0)X_l(0, i)\Phi_{l,i}(\theta, 0) + \int_0^\theta \Phi_{l,i}^*(s, 0)M_l(s, i)\Phi_{l,i}(s, 0)ds$$

$\Phi_{l,i}(t, s)$ being the fundamental matrix solution of (23). For the first step $X_0(t, i)$ is chosen as a periodic solution of the Lyapunov type equation on $\mathcal{S}_n$

$$\frac{d}{dt}X_0(t) + \mathcal{L}_{\tilde{F}_0}^*(t)X_0(t) + M_0(t) = 0$$

where $M_0(t) = (M_0(t, 1), M_0(t, 2), \dots, M_0(t, d))$,

$$\begin{aligned} M_0(t, i) &= M(t, i) + \varepsilon I_n + L(t, i)\tilde{F}_0(t, i) + \tilde{F}_0^*(t, i)L^*(t, i) \\ &\quad + \tilde{F}_0^*(t, i)R(t, i)\tilde{F}_0(t, i). \end{aligned}$$

If $T_0(t, t_0)$ is the linear evolution operator defined by the linear differential equation on $\mathcal{S}_n^d$:

$$\frac{d}{dt}S(t) = \mathcal{L}_{\tilde{F}_0}(t)S(t) \quad (27)$$

then the initial condition $X_0(0) = (X_0(0, 1), X_0(0, 2), \dots, X_0(0, d))$ is given by

$$X_0(0) = [\tilde{J} - T_0^*(\theta, 0)]^{-1} \int_0^\theta T_0^*(s, 0)M_0(s)ds$$

where $\tilde{J}$ is the identity operator on $\mathcal{S}_n^d$; $\tilde{J} - T_0^*(\theta, 0)$ is invertible due to the exponential stability of the evolution defined by the differential equation (27).

3. Numerical examples

The above iterative numerical procedures will be illustrated considering the linear time-invariant stochastic system of order $n = 2$, subjected to both multiplicative noise and Markovian jumps with $r = 1$ and $\mathcal{D} = \{1, 2\}$ having:

$$\begin{aligned} A_0(1) &= \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, & A_0(2) &= \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \\ A_1(1) &= \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, & A_1(2) &= \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} B_0(1) &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}, & B_0(2) &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ L_0(1) &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}, & L_0(2) &= \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \\ M_0(1) &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, & M_0(2) &= \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}, \\ R(1) &= 1, & R(2) &= 2. \end{aligned}$$

The purpose is to solve the system (1) correspondind to the above numerical values using the iterative procedure indicated in the statement of Theorem 1. Three dinstinct cases have been considered: the case when the system is subjected only to Markov jumps, the case when the system is subjected only to multiplicative white noise, and the case when the system is perturbed with both multiplicative white noise and Markovian jumps.

Case a). The Markovian jumping case:

$A_1(i) = 0$, $B_1(i) = 0$, $i \in \mathcal{D}$. For the stabilizing gains

$$\tilde{F}_0(1) = [0.5923 \quad -0.7004], \quad \tilde{F}_0(2) = [-0.0330 \quad 0.0653],$$

solving (6) one obtains:

$$X_0(1) = 10^3 \begin{bmatrix} 1.5519 & -0.0524 \\ -0.0524 & 1.7776 \end{bmatrix}, \quad X_0(2) = 10^3 \begin{bmatrix} 1.1139 & 0.2680 \\ 0.2680 & 1.3970 \end{bmatrix}.$$

The solution of (1) for this case was determined solving iteratively (17). For an imposed level of accuracy $\|X_{l+1}(i) - X_l(i)\| < 10^{-6}$ we obtained after 69 iterations:

$$X(1) = \begin{bmatrix} 30.7868 & 24.3960 \\ 24.3960 & 26.2218 \end{bmatrix}, \quad X(2) = \begin{bmatrix} 21.5504 & -11.7226 \\ -11.7226 & 19.2254 \end{bmatrix}.$$

Case b). The multiplicative white noise perturbations case: $\mathcal{D} = \{1\}$, $A_i = A_i(1)$; $B_i = B_i(1)$, $i = 0, 1$.

In this case one obtains the initial values:

$$\tilde{F}_0 = [-0.4094 \quad 0.8482], \quad X_0 = \begin{bmatrix} 292.8945 & 163.9337 \\ 163.9337 & 140.9240 \end{bmatrix}$$

and, after 202 iterations, the solution of (1):

$$X = \begin{bmatrix} 1.0782 & 1.0307 \\ 1.0307 & 0.5878 \end{bmatrix}.$$

Case c). The case when the system is subjected to both Markovian jumps and multiplicative white noise.

In this situation the following initial values are obtained:

$$\tilde{F}_0(1) = [-0.3852 \quad 0.8594], \quad \tilde{F}_0(2) = [-0.9000 \quad 0.5763],$$

$$X_0(1) = 10^8 \begin{bmatrix} 5.8005 & -4.5733 \\ -4.5733 & -3.7733 \end{bmatrix},$$

$$X_0(2) = 10^8 \begin{bmatrix} -0.7123 & -0.5110 \\ 0.5110 & -4.8453 \end{bmatrix}.$$

The solution of (1) was obtained after 133 iterations solving (17); thus it results:

$$X(1) = \begin{bmatrix} 2.1893 & 2.0159 \\ 2.0159 & 2.0998 \end{bmatrix}, \quad X(2) = \begin{bmatrix} 0.7940 & -0.4088 \\ -0.4088 & 3.3714 \end{bmatrix}.$$

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