A DIFFUSION APPROXIMATION MODEL OF WEB SERVERS

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Abstract We present a model of Web servers in terms of $G/G/m/N$ service system with autocorrelated (self-similar) input traffic. Diffusion approximation enables introduction of time-varying input and the analysis of transient states.

Keywords: self-similar traffic, diffusion approximation, performance evaluation, Web servers

1. Introduction

Due to the exponential growth of the number of Web servers and of the number of their users, all problems related to the efficient access to these servers become extremely important to performances of the whole Internet. A considerable attention was already paid to model Web traffic and to evaluate the performance of Web servers, cf. [1, 13]. The general conclusion of extensive measurement studies, e.g. [11] is that the pattern of incoming traffic is complex, self-similar in most cases, and that its Hurst parameter is increasing with the traffic intensity, e.g. [7]. The Hurst parameter displays the degree of statistical self-similarity of the observed traffic. Self similarity means that the statistic character of the process does not change with the change in the time scale. Among

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others, this property implies a long-term autocorrelation of the traffic seen as a stochastic process and has a great impact on the performance of networks, enlarging all queues in nodes, increasing probability of loss, and thus deteriorating the quality of services, [14]. A typical queuing model which is used to represent a Web server, e.g. Apache server, is a multichannel service station with a limited queue, see Fig.1, or, following Kendall's notation, [6], a $G/G/m/N$ service system. The number m of parallel service channels corresponds to the maximum number of processes serving persistent connections (parameter *MaxClients* in the Apache server). The incoming traffic represents HTTP traffic at session level: new customers represent session startups, service time in one of parallel channels is a series of active and idle periods. Active time is the transfer time of an entire Web page; the first active time includes also the connection establishing time and slow-start phase in TCP, so it is usually longer. The idle time corresponds to the click idle time (a user think time). The service time is finished when either the session is closed or the connection is timed out (due to *KeepAliveTimeout* in the Apache server) while it is in idle period. In the latter case the session goes back to the queue, via feed-back loop when the next click is done, see [7] for statistics of these times.

However, the queueing theory does not furnish any analytical solution to this model, if we assume general input and general service time distributions, not to mention autocorrelation and self-similarity of the input traffic. Therefore, performance studies are rather based on simulation. Below, we propose a model based on a diffusion approximation and giving analytical, although approximate solution of the problem. An important feature of this model is that it may be used in presence of time-varying correlated input traffic and predicts transient queue distributions. The burstiness of sources makes the steady-state analysis unable to accurately determine the quality of such service parameters as distribution of transmission delay and loss rates, especially during short time periods representing the connection time of a particular user.

We concentrate on the mathematical side of the model. Numerical aspects need careful programming as they include manipulation on very small and very large numbers when Laplace transforms are considered. Purely numerical approach (numerical solution of diffusion equations) is also possible.

2. Diffusion approximation primer

Let us recall the principles of *the diffusion approximation* [10]: the process $N(t)$ which represents the number of customers in a service sta-

Figure 1. Queueing model of a www server

tion is replaced by a diffusion process $X(t)$. The pdf $f(x, t; x_0)$ of the latter is defined by the diffusion equation

$$\frac{\partial f(x, t; x_0)}{\partial t} = \frac{\alpha}{2} \frac{\partial^2 f(x, t; x_0)}{\partial x^2} - \beta \frac{\partial f(x, t; x_0)}{\partial x}. \quad (1)$$

which should be solved with appropriate diffusion parameters α , β and boundary conditions to give an approximation of the queue distribution: $p(n, t; n_0) \approx f(n, t; n_0)$.

Let $A(x)$, $B(x)$ denote the interarrival and service time distributions at a service station. The distributions are general, their means and variances are: $E[A] = 1/\lambda$, $E[B] = 1/\mu$, $\text{Var}[A] = \sigma_A^2$, $\text{Var}[B] = \sigma_B^2$. Squared coefficients of variation are denoted $C_A^2 = \sigma_A^2 \lambda^2$, $C_B^2 = \sigma_B^2 \mu^2$. For the models of G/G/1 and G/G/1/N stations, the parameters α , β are chosen as $\beta = \lambda - \mu$, $\alpha = \sigma_A^2 \lambda^3 + \sigma_B^2 \mu^3 = C_A^2 \lambda + C_B^2 \mu$, see [10].

Boundary conditions with sojourn times in the limiting barriers were proposed by Gelenbe in [5]. If a G/G/1 station is considered, the process $X(t)$ is limited by one boundary situated at $x = 0$; for the G/G/1/N model there are two boundaries at $x = 0$ and $x = N$. When the process comes to $x = 0$, it remains there for a time corresponding to the idle time of the system and then it jumps to $x = 1$; when it comes to $x = N$ it stays there for a time during which the queue is full and then jumps to $x = N - 1$. This finite queue model and the steady-state solution of the diffusion equation with the described boundaries were given in [5] and since then are in current use in computer systems modelling.

To obtain the *transient solution* $f(x, t)$, in [3] we proposed to express the density $f(x, t; x_0)$ with the use of a superposition

$$f(x, t; \psi) = \phi(x, t; \psi) + \int_0^t g_1(\tau) \phi(x, t - \tau; 1) d\tau + \int_0^t g_{N-1}(\tau) \phi(x, t - \tau; N - 1) d\tau \quad (2)$$

of densities $\phi(x, t; x_0)$ of diffusion processes bounded by absorbing barriers at $x = 0$ and $x = N$; the process is finished when it comes to one of them. Such densities are easier to obtain and its form is known, [2]:

$$\phi(x, t; x_0) = \begin{cases} \delta(x - x_0) & \text{for } t = 0 \\ \frac{1}{\sqrt{2\pi\alpha t}} \sum_{n=-\infty}^{\infty} \left\{ \exp \left[\frac{\beta x'_n}{\alpha} - \frac{(x - x_0 - x'_n - \beta t)^2}{2\alpha t} \right] - \exp \left[\frac{\beta x''_n}{\alpha} - \frac{(x - x_0 - x''_n - \beta t)^2}{2\alpha t} \right] \right\} & \text{for } t > 0, \end{cases} \quad (3)$$

where $x'_n = 2nN$, $x''_n = -2x_0 - x'_n$. The function ψ denotes initial condition of the process, $\phi(x, t; \psi) = \int_0^N \phi(x, t; \xi) \psi(\xi) d\xi$.

Functions $g_1(t)$ and $g_{N-1}(t)$ in Eq. (2) are the densities of reinitialization of the diffusion processes with absorbing barriers at $x = 1$ and $x = N - 1$; they are obtained from the balance equations (4)

$$\begin{aligned} \gamma_0(t) &= p_0(0)\delta(t) + \gamma_{\psi,0}(t) + \int_0^t g_1(\tau) \gamma_{1,0}(t - \tau) d\tau + \int_0^t g_{N-1}(\tau) \gamma_{N-1,0}(t - \tau) d\tau, \\ \gamma_N(t) &= p_0(N)\delta(t) + \gamma_{\psi,N}(t) + \int_0^t g_{N-1}(\tau) \gamma_{N-1,N}(t - \tau) d\tau + \int_0^t g_1(\tau) \gamma_{1,N}(t - \tau) d\tau \\ g_1(\tau) &= \int_0^\tau \gamma_0(t) l_0(\tau - t) dt, \quad g_{N-1}(\tau) = \int_0^\tau \gamma_N(t) l_N(\tau - t) dt. \end{aligned} \quad (4)$$

In these equations, $\gamma_a(t)$ is the pdf that a diffusion process enters a barrier at $x = a$, $a = 0, N$, and $\gamma_{x_0,a}(t)$ is pdf of the first passage time from x_0 to a . The latter is obtained from the diffusion process with two absorbing barriers, e.g. $\gamma_{1,0}(t) = \lim_{x \rightarrow 0} [\frac{\alpha}{2} \frac{\partial \phi(x, t; 1)}{\partial x} - \beta \phi(x, t; 1)]$.

$\gamma_{\psi,a}(t)$ denotes the pdf of first passage time to a from the points ξ distributed on the axis x with density $\psi(\xi)$: $\gamma_{\psi,a}(t) = \int_0^N \gamma_{\xi,a}(t) d\xi$.

3. Correlation and time-dependent diffusion parameters

The presented above transient solution refers to the case of constant parameters of the diffusion equation. However, this may be relaxed by the division of the time axis into relatively short periods during which the parameters are kept constant; they are allowed to change at the beginning of each interval. At each interval j the diffusion equation is solved for its parameters β_j , α_j characteristic for this interval and depending on the input parameters λ_j , C_{Aj}^2 ; at the end of interval j this

solution becomes the initial condition for the next interval where the parameters are $\beta_{j+1}, \alpha_{j+1}$.

Below, we use the same approach for the diffusion model in case of correlated and time varying input stream. The correlated input is frequently modelled by a Markov modulated Poisson process: a Markov chain of M states defines the parameter of Poisson input process. If the chain is in state i , the parameter is λ_i . If $M = 2$ we have an on-off source. Using the diffusion approximation we may say more generally that the Markov chain determines the phases during which the input, not necessarily Poisson, is characterised by its two first moments of interarrival times: during phase i they are $E[A_i] = \lambda_i^{-1}$ and $E[A^2]$.

Hence, if $\pi_i(t)$ is the probability that a Markov process is at the state i at the moment t , the interarrival times distribution has the first two moments at the time t

$$E[A(t)] = \frac{\sum_{i=1}^M \pi_i(t) \lambda_i E[A_i]}{\sum_{i=1}^M \pi_i(t) \lambda_i} = \frac{\sum_{i=1}^M \pi_i(t)}{\sum_{i=1}^M \pi_i(t) \lambda_i} = \frac{1}{\sum_{i=1}^M \pi_i(t) \lambda_i} = \frac{1}{\lambda(t)} \quad (5)$$

$$E[A(t)^2] = \frac{\sum_{i=1}^M \pi_i(t) \lambda_i E[A_i^2]}{\sum_{i=1}^M \pi_i(t) \lambda_i}. \quad (6)$$

If $\lambda_i = 0$, i.e. the phase i is an off period, the term $E[A_i^2]$ in Eq. (6) is replaced by the second moment of the phase i distribution and $\lambda_i = 0$ is replaced by the reciprocal of the mean length of this phase.

The solution of the equation system

$$\frac{d\pi_i(t)}{dt} = \sum_{\text{all } k} \pi_k(t) q_{ki}$$

where q_{ki} are transition rates from a state k to state i , gives the transient state probabilities of the Markov chain underlying the input evolution we use $\pi_i(t)$ to determine the time-dependent diffusion parameters: $\beta(t) = \lambda(t) - \mu$, $\alpha(t) = \lambda(t)C_A^2(t) + \mu C_B^2$, where

$$\lambda(t) = \frac{1}{E[A(t)]}, \quad C_A^2(t) = \frac{E[A(t)^2]}{E[A(t)]^2} - 1. \quad (7)$$

In the simplest case of two-state Markov chain (an on-off source), if the rate of leaving state j is r_j , $j = 1, 2$, the transient probabilities of the state $j = 1$ are

$$\begin{aligned} \pi_1(t | \pi_1(0) = 1) &= \frac{r_1}{r_1 + r_2} \exp[-(r_1 + r_2)t] + \frac{r_2}{r_1 + r_2}, \\ \pi_1(t | \pi_1(0) = 0) &= \frac{r_2}{r_1 + r_2} [1 - \exp[-(r_1 + r_2)t]], \end{aligned} \quad (8)$$

where $\pi_i(t|\pi_i(0) = 1)$ is the probability of state i at time t if at $t = 0$ the chain was at this state, and $\pi_2(t|\pi_2(0) = 1) = 1 - \pi_1(t|\pi_1(0) = 0)$, $\pi_2(t|\pi_2(0) = 0) = 1 - \pi_1(t|\pi_1(0) = 1)$.

The correlation should be regarded within the time scale as defined for the function $R(t, \tau) = E[A(t)A(t + \tau)] - E[A(t)^2]$:

$$R(t, \tau) = \sum_{i=1}^M \pi_i(t) \left[\pi_i(\tau|\pi_i(0) = 1) \lambda_i^2 + \sum_{j=1, j \neq i}^M \pi_j(\tau|\pi_i(0) = 1) \lambda_i \lambda_j \right]. \quad (9)$$

We may also consider a superposition of K independent, on-off (or modulated by various Markov chains) sources. Denote by $\lambda^{(k)}(t)$, $C_A^{(k)^2}(t)$ the parameters of the k -th on-off source. The resulting input stream $\lambda(t)$ is obviously the sum of all $\lambda^{(k)}(t)$ and, as the number of source k customers that arrived within a considered period (which is short compared to the rate of input stream changes but sufficiently long to receive a certain number of customers), is normally distributed with variance $\lambda^{(k)}(t)C_A^{(k)^2}(t)$; the sum of independent randomly distributed variables has also normal distribution with variance which is the sum of component variances, hence

$$C_A^2(t) = \sum_{k=1}^K \frac{\lambda^{(k)}(t)}{\lambda(t)} C_A^{(k)^2}(t). \quad (10)$$

If the steady-state is considered, the probabilities $\pi_i(t)$ become π_i .

4. Diffusion model of $G/G/m/N$ station with feed-back

If several service channels are busy in parallel, the output stream of the service system depends on the number of busy channels, and if there is a probability, denoted here by r , that the leaving customers return to the queue, also the input stream depends on the number of channels in use. This should be reflected by the diffusion parameters: $\alpha = \alpha(x, t)$, $\beta = \beta(x, t)$. A part $x \in [0, m]$ of the whole diffusion interval $[0, N]$ is divided into subintervals of unitary length and the parameters are kept constant within these subintervals. If $n - 1 < x < n$, $n \leq m$, it is supposed that n channels are busy, hence we choose

$$\alpha(x, t) = \lambda C_A^2(t) + n(1 + r)\mu C_B^2, \quad \beta(x, t) = \lambda(t) - n(1 - r)\mu$$

for $n - 1 < x < n$, $n \leq m$.

For $x \geq m - 1$, $\alpha = \alpha(m, t)$, $\beta = \beta(m, t)$. Jumps from $x = N$ to $x = N - 1$ are performed with density $m\mu$ (m servers are busy).

In transient state we should balance the probability flows between neighbouring intervals with different diffusion parameters. We put imaginary barriers at the borders of these intervals and suppose that the diffusion process entering the barrier at n , $n = 1, 2, \dots, m-1$, from its left side (the process is growing) is absorbed and immediately reappears at $x = n + \varepsilon$. Similarly, the process which is diminishing and enters the barrier from its right side reappears at its other side at $x = n - \varepsilon$. We used previously a similar systems of barriers for the analysis of transient states in a cellular network modelled as a system of $G/G/N/N$ stations, [4]. The density functions for the unitary intervals between $x = 1$ and $x = m - 1$ are as follows:

$$\begin{aligned} f_n(x, t; \psi_n) &= \phi_n(x, t; \psi_n) + \int_0^t g_{n-1+\varepsilon}(\tau) \phi_n(x, t - \tau; n - 1 + \varepsilon) d\tau + \\ &+ \int_0^t g_{n-\varepsilon}(\tau) \phi_n(x, t - \tau; n - \varepsilon) d\tau, \quad n = 2, \dots, m-1, \end{aligned}$$

and the probability mass flows entering the barriers are:

$$\gamma_n^R(t) = g_{n-\varepsilon}(t), \quad \gamma_n^L(t) = g_{n+\varepsilon}(t), \quad n = 1, \dots, m-1$$

and are obtained in the similar way as in $G/G/1/N$ model, e.g.:

$$\begin{aligned} \gamma_n^L(t) &= \gamma_{\psi_n, n}(t) + \int_0^t g_{n-1+\varepsilon}(\tau) \gamma_{n-1+\varepsilon, n}(t - \tau) d\tau + \\ &+ \int_0^t g_{n-\varepsilon}(\tau) \gamma_{n-\varepsilon, n}(t - \tau) d\tau, \\ \gamma_n^R(t) &= \gamma_{\psi_{n+1}, n}(t) + \int_0^t g_{n+\varepsilon}(\tau) \gamma_{n+\varepsilon, n}(t - \tau) d\tau + \\ &+ \int_0^t g_{n+1-\varepsilon}(\tau) \gamma_{n+1-\varepsilon, n}(t - \tau) d\tau, \quad n = 2, \dots, m-1 \end{aligned}$$

where $\gamma_{i,j}(t)$ are the densities of the first passage time between points i, j . The equations for boundary intervals on the left and right sides must include the jumps from barriers, in the same way as in $G/G/1/N$ model. This system of equations is subject to Laplace transformation, the Laplace transforms $\bar{f}_n(x, s; \psi_n)$ are obtained and then inverted with the use of Stehfest algorithm [12]. For each time- and space-subinterval with constant parameters, transient solution is obtained. This approach was programmed and mastered numerically.

4.1. Numerical example

Consider an input traffic modulated by a continuous time birth and death Markov chain of 31 states, Fig. 2, proposed by [9] and then

Figure 2. Markov chain used in numerical examples to represent correlated input

Figure 3. Mean number of connections

frequently used as an example of autocorrelated process. The rate of leaving state j is given by

$$r_j = 1.03077(31 - j) + 2.5923j$$

and transition probabilities are

$$p_{j,j+1} = \frac{1.03077(31 - j)}{1.03077(31 - j) + 2.5923j},$$

$$p_{j,j-1} = 1 - p_{j,j+1}, \quad 2 \leq j \leq 30;$$

$$p_{1,2} = p_{31,30} = 1.$$

The traffic intensity at state j is $\lambda_j = 2.391(j - 1)$. At $t = 0$ the server is empty, the source starts with initial state $j_0 = 17$. During $t \in [800, 1200]$ a second source of the same type is activated. The traffic is cut off at $t = 2000$. Let the maximum number of connections $m = 10$ and $N = 100$. Fig. 3 presents the mean number of connections (served or waiting for execution) as a function of time. Considered mean time T of a single connection 10, 12 and 15 time units with exponential distribution, $r = 0.2$.

5. Conclusions

The model presented here incorporates some important features which are usually difficult to capture by analytical methods: description of transient states, autocorrelated, time-varying input traffic, general service times. The model was implemented and may be useful in the performance evaluation of Web servers. We applied a MMPP process to characterise the input traffic because it is the most frequently used way to present a correlated input. This description may be easily replaced by any other which gives the values of interarrival moments, hence the values of the diffusion parameters, as a function of time.

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