A SIMPLE ALGORITHM FOR COMPUTING SEVERAL SEQUENCES SYNTHESIS

Wang Mingsheng, Qi sihan, Feng denguo

Engineering Research Center for Information Security Technology

Institute of Software, Chinese Academy of Sciences

Beijing 100080, Beijing P.O. Box 8718, P.R. China

wangmsh@ercist.iscas.ac.cn

Abstract In this paper, we use module Grobner bases to Compute several sequences synthesis problem. It is a natural generalization of Fitzpatrick's algorithm.

Keywords: Module Grobner bases, Several sequences synthesis, Buchberger algorithm.

Introduction

Several sequences synthesis problem have been considered by many authors, for example [1], [2], [10]. In [1], [2], the authors considered this problem using Grobner bases method of ideal and their methods involved in solving equations in at least 2 variables. In [9], the author discovered a new approach to solve key equation by using Grobner bases method of submodule of free module over polynomial ring in one variable. In this paper, we extend the method of [9] to the general case.

Let F be a field and $F[x]$ be a polynomial ring in variable x . Let $s_1 = (s_{01}, s_{11}, \dots, s_{n_1 1})$,

$s_2 = (s_{02}, s_{12}, \dots, s_{n_2 2}), \dots, s_t = (s_{0t}, s_{1t}, \dots, s_{n_t t})$ be t linear recurring sequences of finite length in F . A linear feedback shift register of length l with feedback coefficients $(\sigma_1, \sigma_2, \dots, \sigma_l)$ is said to generate $s_1, \dots, s_t$, if

$$s_{ji} = \sum_{r=1}^l \sigma_r s_{j-ri}, \quad i = 1, \dots, t, \quad 1 \leq j \leq l \quad (1.1)$$

An important problem in the theory of coding and cryptography is the synthesis problem:

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Given several sequences in F of finite but possibly different length, find the linear feedback shift register of least length which generate all the sequences.

We know that the synthesis problem have been studied by many authors, but as far as we know, we don't still see methods that use module Grobner bases. In this paper, we give a very natural approach to this problem. Actually, we only need to extend the method of [9] to general case.

1. REVIEW OF MODULE GROBNER BASES

Grobner bases theory for free module of finite rank over the polynomial ring is a basic tool in many applications. Its complete discussions may be found in [3], [4], [5], [6], [7].

Let R be a polynomial ring $F[x_1, \dots, x_n]$ and R^m be a free module of rank m . For $\mathbf{f} = (f_1, \dots, f_m) \in R^m$, $\mathbf{f} = f_1 e_1 + \dots + f_m e_m$, where $\{e_1 = (1, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_m = (0, \dots, 0, 1)\}$ is a standard basis of R^m .

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}$, we set $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$. A monomial $\mathbf{m}$ in R is an element of the form $x^\alpha e_i$ for some e_i . In this case, we say $\mathbf{m}$ contains the standard basis vectore $_i$. every element of $\mathbf{f} \in R$ can be written in a unique way as a F - linear combination of monomials $\mathbf{m}_i$

$$\mathbf{f} = \sum_{i=1}^n c_i \mathbf{m}_i.$$

The product $c \cdot \mathbf{m}$ of a monomial $\mathbf{m}$ with an element $c \in F$ is called a term and c is called its coefficient.

If $\mathbf{m}_1 = x^{\alpha_1} e_i, \mathbf{m}_2 = x^{\alpha_2} e_j$ are monomials, then we say that $\mathbf{m}_1$ divides $\mathbf{m}_2$ if and only if $i = j$ and x^{α_1} divides x^{α_2} . If $\mathbf{m}_1$ divides $\mathbf{m}_2$, we define the *quotient* $\frac{\mathbf{m}_2}{\mathbf{m}_1}$ to $\frac{x^{\alpha_2}}{x^{\alpha_1}}$. In general, we define $LCM(\mathbf{m}_1, \mathbf{m}_2) =$

$$\begin{cases} LCM(x^{\alpha_1}, x^{\alpha_2}) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

We say that a total order $>$ on the monomials of R^m is monomial order (or term order) if

- (1) for every pair of monomials $\mathbf{m}_1, \mathbf{m}_2$, if $\mathbf{m}_1 > \mathbf{m}_2$, then $x^\alpha \mathbf{m}_1 > x^\alpha \mathbf{m}_2$ for every monomial $x^\alpha \in R$, and
- (2) for any monomial $\mathbf{m} \in R^m$ and $x^\alpha \in R$, if $x^\alpha \neq 1$. then $x^\alpha \mathbf{m} > \mathbf{m}$.

Given a monomial order $>$ on monomials, we can write any element $\mathbf{f} \in R^m$ as a sum of terms

$$\mathbf{f} = \sum_{i=1}^t c_i \mathbf{m}_i$$

with $c_i \neq 0$ and $\mathbf{m}_1 > \dots > \mathbf{m}_t$. We define the leading coefficient, leading monomial, and leading term of $\mathbf{f}$ as follows:

$$\begin{aligned} \text{LC}_{>}(\mathbf{f}) &= c_1 \\ \text{LT}_{>}(\mathbf{f}) &= \mathbf{m}_1 \\ \text{LM}_{>}(\mathbf{f}) &= c_1 \mathbf{m}_1. \end{aligned}$$

Given a monomial order in R^m , we can divide by a set G of R^m . For details, we refer to [6], [7].

Division algorithm in R^m . Fix a monomial order $>$ on R^m and let $G = (\mathbf{g}_1, \dots, \mathbf{g}_s)$ be an order s -tuple of elements of R^m . Then every $\mathbf{g} \in R^m$ can be written as

$$\mathbf{g} = b_1 \mathbf{g}_1 + \dots + b_s \mathbf{g}_s + \mathbf{r}$$

where $b_i \in R$, $\mathbf{r} \in R^m$, $\text{LT}(b_i \mathbf{g}_i) \leq \text{LT}(\mathbf{g})$ for all i , and either $\mathbf{r} = 0$, $\mathbf{r}$ is a F -linear combination of monomials none of which is divisible by any of $\text{LT}(\mathbf{g}_1), \dots, \text{LT}(\mathbf{g}_s)$. We call $\mathbf{r}$ the remainder on division by G and denote it by $\bar{g}^G$.

Now, we give the definition of Grobner bases.

Definition 2.1 Let M be a submodule of R^m and $>$ be a monomial order.

A finite set $G = \{g_1, \dots, g_s\}$ is called a Grobner bases for M if $\langle \text{LT}(M) \rangle = \langle \text{LT}(\mathbf{g}_1), \dots, \text{LT}(\mathbf{g}_s) \rangle$, where the symbol $\text{LT}(S)$ denotes the set of leading terms of S , and $\langle S \rangle$ denotes the submodule of R^m generated by S .

Let G be a Grobner bases for M , if $\forall \mathbf{g}_i, \text{LC}(\mathbf{g}_i) = 1$, and $\text{LT}(\mathbf{g}_i) \notin \langle \text{LT}(G \setminus \mathbf{g}_i) \rangle$, then we say that G is a minimal Grobner bases for M .

Once we have a monomial order $>$, for every submodule M of R^m , there is a Grobner bases for M with respect to $>$. If we have a group of generators of M , then we can find a Grobner bases for M by Buchberger algorithm. For details, we refer to [6], [7].

2. COMPUTATIONAL METHODS

Let $\sigma(x) = 1 + \sigma_1 x + \sigma_2 x^2 + \dots + \sigma_l x^l$ and $S_i = s_{0i} + s_{1i}x + s_{2i}x^2 + \dots + s_{n_i i}x^{n_i}$ $1 \leq i \leq t$.

It is easy to see that (1.1) is equivalent to the following formula (3.1)

$$(3.1) \begin{cases} \sigma(x)S_1(x) \equiv r_1(x) \pmod{x^{n_1+1}} \\ \sigma(x)S_2(x) \equiv r_2(x) \pmod{x^{n_2+1}} \\ \dots \\ \sigma(x)S_t(x) \equiv r_t(x) \pmod{x^{n_t+1}} \\ \text{for some polynomials } r_i(x) \text{ such that } \deg r_i \leq l-1, \quad 1 \leq i \leq t. \end{cases}$$

Now, we set $f(x) = x^l \sigma(x^{-1}) = x^l + \sigma_1 x^{l-1} + \sigma_2 x^{l-2} + \dots + \sigma_{l-1} x + \sigma_l$ and $S_i^*(x) = x^{n_i} S_i(x^{-1}) = s_{0i} x^{n_i} + s_{1i} x^{n_i-1} + \dots + s_{n_i i}, 1 \leq i \leq t$.

Hence, it is easily to see that (3.1) is equivalent to the following formula (3.2)

$$(3.2) \begin{cases} f(x)S_1^*(x) \equiv a_1(x) \pmod{x^{n_1+1}} \\ f(x)S_2^*(x) \equiv a_2(x) \pmod{x^{n_2+1}} \\ \dots \\ f(x)S_t^*(x) \equiv a_t(x) \pmod{x^{n_t+1}} \\ \text{for some polynomials } a_i(x) \text{ such that } \deg a_i \leq l-1, \quad 1 \leq i \leq t. \end{cases}$$

Thus, finding linear feedback shift register of least length which generate sequences $s_1, \dots, s_t$. is equivalent to find the polynomial $f(x)$ of least degree satisfying (3.2).

Once we find a polynomial $f(x)$ satisfying (3.2), then $\sigma(x) = x^l f(x^{-1})$ is the polynomial satisfying (3.1), here $l = \deg f(x)$.

So, solving synthesis problem is equivalent to solving the following congruence equation group.

Finding $(a_1(x), a_2(x), \dots, a_t(x), f(x))$ such that

$$(3.3) \begin{cases} f(x)S_1^*(x) \equiv a_1(x) \pmod{x^{n_1+1}} \\ f(x)S_2^*(x) \equiv a_2(x) \pmod{x^{n_2+1}} \\ \dots \\ f(x)S_t^*(x) \equiv a_t(x) \pmod{x^{n_t+1}} \\ \deg a_i(x) < \deg f(x), \quad 1 \leq i \leq t \\ \min(\deg f(x)). \end{cases}$$

Let $K = \{(a_1(x), \dots, a_t(x), f(x)) \in F[x]^{t+1} | f(x)S_i^*(x) \equiv a_i(x) \pmod{x^{n_i+1}}, 1 \leq i \leq t\}$. Then we have the following lemma 3.1.

Lemma 3.1 K is a submodule of free module $F[x]^{t+1}$ over polynomial ring $F[x]$.

Let $e_1 = (1, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_{t+1} = (0, 0, \dots, 0, 1)$ be the standard basis elements of $F[x]^{t+1}$, Namely,

$$F[x]^{t+1} = \bigoplus_{i=1}^{t+1} F[x]e_i.$$

Now, we have the following lemma 3.2

Lemma 3.2 Submodule K is generated by $S_1^*(x)e_1 + \dots + S_t^*(x)e_t + e_{t+1}, x^{n_1+1}e_1, \dots, x^{n_t+1}e_t$.

Proof Let $(a_1(x), \dots, a_t(x), f(x)) \in K$, then there exist $q_1(x), \dots, q_t(x) \in F[x]$ such that $f(x)S_i^*(x) = a_i(x) + q_i(x)x^{n_i+1}, 1 \leq i \leq t$. Hence

$$\begin{aligned} (a_1(x), \dots, a_t(x), f(x)) &= a_1(x)e_1 + \dots + a_t(x)e_t + f(x)e_{t+1} \\ &= (f(x)S_1^*(x) - q_1(x)x^{n_1+1})e_1 \\ &\quad + \dots \\ &\quad + (f(x)S_t^*(x) - q_t(x)x^{n_t+1})e_t + f(x)e_{t+1} \\ &= f(x)(S_1^*(x)e_1 + \dots + S_t^*(x)e_t + e_{t+1} \\ &\quad - q_1(x)(x^{n_1+1}e_1) - \dots - q_t(x)(x^{n_t+1}e_t)). \end{aligned}$$

Hence, the lemma 3.2 is true.

By lemma 3.2, submodule K has an explicit generators group. So, in order to solve problem (3.3), we consider suitable term order on monomial set of $F[x]^{t+1}$.

Now, monomial set of $F[x]^{t+1}$ is $\{x^i e_j | i \geq 0, 1 \leq j \leq t+1\}$. We define a term order $>$ as follows:

$$x^i e_j > x^l e_j \text{ iff } i > j.$$

$$\text{For } k < l, x^i e_k > x^j e_l \text{ iff } i \geq j.$$

It is obvious that $>$ is a term order on monomial set of $F[x]^{t+1}$, and $e_1 > e_2 > \dots > e_{t+1}$.

Now, we have the following lemma 3.3.

Lemma 3.3 Let B be a minimal Grobner bases of K with respect to the above term order $>$. Then B must contain an element $(a_1(x), \dots, a_t(x), f(x))$ whose leading term involves in e_{t+1} , and $f(x)$ is a solution of (3.3).

By lemma 3.3, in order to solve problem (3.3), we only to compute a Grobner bases of K with respect to $>$. As far as we know, there are at least 3 different algorithms to compute a Grobner bases of K .

For the convenience, we give two algorithms to solve this problem.

3.4. Buchberger algorithm.

Input: $F = (S_1^* e_1 + \dots + S_t^* e_t + e_{t+1}, x^{n_1+1} e_1, \dots, x^{n_t+1} e_t)$, the above term order $>$.

Output: a Grobner bases G for $M = \langle F \rangle$ with respect to $>$.

$$G := F$$

REPEAT

$$G' := G$$

FOR each pair $f \neq g$ in G' DO

$$S := \overline{S(f, g)}^{G'}$$

IF $S \neq 0$ THEN $G := G \cup \{S\}$

UNTIL $G = G'$.

Once G is obtained, we can get a minimal Grobner bases of $M = \langle F \rangle$ from G .

Example 3.5. Let $F = GL(2)$, $s_1 = 10101101$, $s_2 = 01010100$. $n_1 = 7, n_2 = 7$.

$$\text{Let } S_1^* = x^7 + 0 \cdot x^6 + x^5 + 0 \cdot x^4 + x^3 + x^2 + 0 \cdot x + 1 = x^7 + x^5 + x^3 + x^2 + 1.$$

$$S_2^* = 0 \cdot x^7 + x^6 + 0 \cdot x^5 + x^4 + 0 \cdot x^3 + x^2 + 0 \cdot x + 0 = x^6 + x^4 + x^2.$$

So, Finding linear feedback shift register of least length synthesizing s_1, s_2 is equivalent to finding a polynomial $f(x)$ of least degree such that for some polynomials $a_1(x), a_2(x)$, the following formula is true.

$$\begin{cases} f(x)S_1^*(x) \equiv a_1(x) \pmod{x^8} \\ f(x)S_2^*(x) \equiv a_2(x) \pmod{x^8} \\ \deg f(x) < \deg a_1(x) \\ \deg f(x) < \deg a_2(x) \end{cases}$$

Let $K = \{(a_1(x), a_2(x), f(x)) | f(x)S_i^*(x) \equiv a_i(x) \pmod{x^8}, 1 \leq i \leq 2\}$. then submodule K of $F[x]^3$ has generators $S_1^*e_1 + S_2^*e_2 + e_3, x^8e_1, x^8e_2$, by lemma 3.2.

Let term order $>$ be defined as follows:

- (1) $x^ie_k > x^je_k$ iff $i > j$.
- (2) For $k < l$, $x^ie_k > x^je_l$ iff $i \geq j$.

To solve synthesis problem for s_1, s_2 , we need to find a minimal Grobner bases of K with respect to the term order $>$, by lemma 3.3. In the following, we make use of Bucherberg algorithm to find a minimal Grobner bases of K with respect to $>$.

Let $F = \{S_1^*e_1 + S_2^*e_2 + e_3, x^8e_1, x^8e_2\}$.

The first step. Let $f_1 = S_1^*e_1 + S_2^*e_2 + e_3, f_2 = x^8e_1, f_3 = x^8e_2$. Then $LT(f_1) = x^7e_1, LT(f_2) = x^8e_1, LT(f_3) = x^8e_2, G := F$.

$$S(f_1, f_2) = f_2 - \frac{LT(f_2)}{LT(f_1)}f_1 = (x^6 + x^4 + x^3 + x, x^7 + x^5 + x^3, x).$$

$$A_1 = \overline{S(f_1, f_2)}^G = (x^6 + x^4 + x^3 + x, x^7 + x^5 + x^3, x).$$

$$G := \{f_1, f_2, f_3, A_1\}.$$

The second step. $LT(f_1) = x^7e_1, LT(f_2) = x^8e_1, LT(f_3) = x^8e_2, LT(A_1) = x^7e_2$.

$$S(f_3, A_1) = f_3 - \frac{LT(f_3)}{LT(A_1)}A_1 = (x^7 + x^5 + x^4 + x^2, x^6 + x^4, x^2).$$

$$A_2 = \overline{S(f_3, A_1)}^G = (x^4 + x^3 + 1, x^2, x^2 + 1).$$

$$G := \{f_1, f_2, f_3, A_1, A_2\}.$$

The third step. $LT(f_1) = x^7e_1, LT(f_2) = x^8e_1, LT(f_3) = x^8e_2$,

$$LT(A_1) = x^7e_2, LT(A_2) = x^4e_2.$$

$$S(f_1, A_2) = f_1 - \frac{LT(f_1)}{LT(A_2)}A_2 = (x^6 + x^5 + x^2 + 1, x^6 + x^5 + x^4 + x^2, x^5 + x^3 + 1).$$

$$A_3 = \overline{S(f_1, A_2)}^G = S(f_1, A_2) - \frac{x^6e_1}{x^4e_1}A_2 = (1, x^6 + x^5 + x^2, x^5 + x^4 + x^3 + x^2 + 1).$$

$$G := \{f_1, f_2, f_3, A_1, A_2, A_3\}.$$

The fourth step. $LT(A_1) = x^7e_2, LT(A_2) = x^4e_1, LT(A_3) = x^6e_2$.

$$S(A_1, A_3) = A_1 - \frac{LT(A_1)}{LT(A_3)}A_3 = (x^6 + x^4 + x^3, x^6 + x^5, x^6 + x^5 + x^4 + x^3).$$

Let $A_4 = \overline{S(A_1, A_3)}^G$. Then

$$A'_4 = S(A_1, A_3) - \frac{x^6e_1}{x^4e_1}A_2 = (x^5 + x^4 + x^3 + x^2, x^6 + x^5 + x^4, x^6 + x^5 + x^3 + x^2).$$

$$A''_4 = A'_4 - \frac{x^5e_1}{x^4e_1}A_2 = (x^3 + x^2 + x, x^6 + x^5 + x^4 + x^3, x^6 + x^5 + x^2 + x).$$

$$A_4 = A''_4 - \frac{x^6e_2}{x^6e_2}A_3 = (x^3 + x^2 + x + 1, x^4 + x^3 + x^2, x^6 + x^4 + x^3 + x + 1).$$

We note that $LT(A_4) = x^6e_3$.

$G = \{f_1, f_2, f_3, A_1, A_2, A_3, A_4\}$ is a Grobner bases of $M = \langle F \rangle$ with respect to $>$, and $\{A_2, A_3, A_4\}$ is a minimal Grobner bases of $M = \langle F \rangle$

with respect to $>$. The least length of linear feedback shift register synthesizing s_1, s_2 is 6.

The characteristic polynomial of the LFSR is $f(x) = x^6 + x^4 + x^3 + x + 1$. and the corresponding LFSR is (in polynomial form) $\sigma(x) = x^6 f(x) = x^6(x^6 + x^4 + x^3 + x^{-1} + 1) = x^6 + x^5 + x^3 + x^2 + 1$.

All the characteristic polynomials of least degree can be given in the set

$$\{f(x) = g_1(x)(x^6 + x^4 + x^3 + x + 1) + g_2(x)(x^5 + x^4 + x^3 + x^2 + 1) + g_3(x)(x^2 + 1) | g_i(x) \in F[x], 1 \leq i \leq 3, \deg f(x) = 6\}.$$

3.6. FGLM algorithm.

Let $r = \max(\deg S_i^*(x))$. We define a term order $>_1$ on monomial set of $F[x]^{t+1}$ as follows:

- (1) $x^i e_j >_1 x^l e_j$ iff $i > l$.
- (2) For $k, l = 1, \dots, t$ and $k < l$, $x^i e_k >_1 x^j e_l$ iff $i \geq j$.
- (3) For $j = 1, \dots, t$, $x^a e_{t+1} >_1 x^b e_j$ iff $a + r \geq b$.

It is easily to see that $>_1$ is a term order and $\{f_1 = x^{n_1+1} e_1, \dots, f_t = x^{n_t+1} e_t,$

$f_{t+1} = S_1^* e_1 + \dots + S_t^* e_t + e_{t+1}\}$ is a Grobner bases of K with respect to $>_1$.

Our problem is to find a Grobner bases of K with respect to $>$. So we make use of FGLM algorithm to convert the above Grobner bases of K with respect to $>_1$ into a Grobner bases of K with respect to $>$.

For a given zero-dimensional ideal I , by FGLM algorithm, a Grobner bases of I with respect to a monomial order can be convert to a Grobner bases of I with respect to another monomial order. For FGLM algorithm, we refer to [8], [4], [6].

In [8], FGLM algorithm was extend to zero-dimensional submodule case. Actually, FGLM algorithm for sumodules is essentially same as FGLM algorithm for ideals. In the following, we outline FGLM algorithm as follows: (Here, we only consider Grobner bases in $F[x]^{t+1}$.)

FGLM algorithm.

Input: $G = \{f_1, f_2, \dots, f_t, f_{t+1}\}$, the monomial order $>$.

Output: a Grobner bases $G_>$ with respect to $>$, a list $B_>$.

(1) Main step. Given the input $x^\alpha e_i$, compute $\overline{x^\alpha e_i}^G$.

(a) If $\overline{x^\alpha e_i}^G$ is linearly dependent on the remainders (on divisin by G) of the monomials in $B_>$, then we have a linear combination:

$$\overline{x^\alpha e_i}^G - \sum_{j_k} c_{j_k} \overline{x^{\alpha(j_k)} e_k}^G = 0.$$

Where $x^{\alpha(j_k)} e_k \in B_>$ and $c_{j_k} \in F$.

In this case, we add $g = x^\alpha e_i - \sum c_{j_k} x^{\alpha(j_k)} e_k$ to $G_>$.

(b) If $\overline{x^\alpha e_i}^G$ is linearly independent from the remainders (on division by G) of the monomials in $B_{>}$, then we add $x^\alpha e_i$ to $B_{>}$.

(2) Next monomial. Replace $x^\alpha e_i$ with the next monomial in $>$ which is not divisible by any of the $LT(g_i), g_i \in G_{>}$.

(3) Termination Test. If the Main loop added a polynomial g to $G_{>}$, then compute $LT(g)$. If for every e_i , we have an element $g \in G_{>}$ such that $LT(g_i)$ involves in e_i , then the algorithm Terminates.

Example 3.7 Let $F = GL(2)$, $s_1 = 10101101, s_2 = 01010100, n_1 = 7, n_2 = 7$.

Let $S_1^*(x), S_2^*(x)$ and K as in example 2.4. With the definition of $>_1$, $G = \{f_1 = x^8 e_1, f_2 = x^8 e_2, f_3 = S_1^* e_1 + S_2^* e_2 + e_3\}$ is a Grobner bases of K with respect to $>_1$. We use FGLM algorithm to compute a minimal Grobner bases of K with respect to the monomial order $>$.

We order monomials by $>$ as follows:

$$e_3 < e_2 < e_1 < x e_3 < x e_2 < x e_1 < x^2 e_3 < x^2 e_2 < x^2 e_1 < x^3 e_3 < x^3 e_2 < x^3 e_1 < x^4 e_3 < \dots$$

We have the following computation steps:

$$1. \overline{e_3}^G = (x^7 + x^5 + x^3 + x^2 + 1) + (x^6 + x^4 + x^2)e_2.$$

$$2. \overline{e_2}^G = e_2.$$

$$3. \overline{e_1}^G = e_1.$$

$$4. \overline{x e_3}^G = (x^6 + x^4 + x^3 + x)e_1 + (x^7 + x^5 + x^3)e_2.$$

$$5. \overline{x e_2}^G = x e_2.$$

$$6. \overline{x e_1}^G = x e_1.$$

$$7. \overline{x^2 e_3}^G = (x^7 + x^5 + x^4 + x^2)e_1 + (x^6 + x^4)e_2.$$

$$8. \overline{x^2 e_2}^G = x^2 e_2.$$

$$9. \overline{x^2 e_1}^G = x^2 e_1.$$

$$10. \overline{x^3 e_3}^G = (x^6 + x^5 + x^3)e_1 + (x^7 + x^5)e_2.$$

$$11. \overline{x^3 e_2}^G = x^3 e_2.$$

$$12. \overline{x^3 e_1}^G = x^3 e_1.$$

$$13. \overline{x^4 e_3}^G = (x^7 + x^6 + x^4)e_1 + x^6 e_2.$$

$$14. \overline{x^4 e_2}^G = x^4 e_2.$$

$$15. \overline{x^4 e_1}^G = x^4 e_1.$$

$$\text{In this case, } \overline{x^4 e_1}^G = \overline{e_3}^G + \overline{x^2 e_3}^G + \overline{x^2 e_2}^G + \overline{x^3 e_1}^G + \overline{e_1}^G.$$

$$\text{So, } x^4 e_1 + e_1 + x^3 e_1 + x^2 e_2 + e_3 + x^2 e_3 \in G_{>}.$$

$$16. \overline{x^5 e_3}^G = (x^7 + x^5)e_1 + x^7 e_2.$$

$$17. \overline{x^5 e_2}^G = x^5 e_2.$$

$$18. \overline{x^6 e_3}^G = x^6 e_1.$$

$$\text{In this case, } \overline{x^2 e_1}^G + \overline{e_1}^G + \overline{x^4 e_2}^G + \overline{x^3 e_2}^G + \overline{x^2 e_2}^G + \overline{x^6 e_3}^G + \overline{x^4 e_3}^G + \overline{x^3 e_3}^G + \overline{x e_3}^G + \overline{e_3}^G = 0.$$

So, $(x^3 + x^2 + x + 1, x^4 + x^3 + x^2, x^6 + x^4 + x^3 + x + 1) \in G_{>}$.

19. $\overline{x^6 e_2}^G = x^6 e_2$.

In this case, $\overline{x^4 e_1}^G + \overline{x^3 e_1}^G + \overline{x^6 e_2}^G + \overline{x^5 e_2}^G + \overline{x^5 e_3}^G + \overline{x^4 e_3}^G + \overline{x^3 e_3}^G = 0$.

So, $(x^4 + x^3, x^6 + x^5, x^5 + x^4 + x^3) \in G_{>}$.

Thus, we get a minimal Grobner bases of K as follows:

$G_{>} = \{g_1 = (x^4 + x^3 + 1, x^2, x^2 + 1), g_2 = (x^4 + x^3, x^6 + x^5, x^5 + x^4 + x^3), g_3 = (x^3 + x^2 + x + 1, x^4 + x^3 + x^2, x^6 + x^4 + x^3 + x + 1)\}$.

We have $LT(g_1) = x^4 e_1$, $LT(g_2) = x^6 e_2$, $LT(g_3) = x^6 e_3$. and $LT(g_2) > LT(g_3) > LT(g_1)$.

Let us give the following definition ([9]).

Definition 3.8 Let K be a non-zero submodule and $>$ be a monomial order. we say that $f \in K$ is a minimal element of K with respect to $>$ if $LT(f)$ is a minimal element of $LT(K)$ with respect to $>$, where we denote by $LT(K)$ the set of leading monomials of elements of K .

When $t = 1$, from [9], the solutions of (3.3) are minimal elements of K . But, when $t > 1$, from example (3.7), we see that the solutions of K are not minimal elements.

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