THE BERNSTEIN BÉZIER SURFACE ON THE SIMPLEX

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Abstract In this paper, the operational symbol representation of the B-B (Bernstein Bézier) surface on the simplex is discussed and the de Boor-Fix formula is derived. In addition, two forms of the interpolation polynomial on the simplex are given. From this, the relation between the Bézier control points and the data points of the B-B surface on the simplex is revealed.

Keywords: B-B surface, deBoor-Fix formula, interpolation polynomial

Introduction

Let $V^i \in \mathbf{R}^m, i = 0, \dots, m$ be $(m + 1)$ affinely independent points in $\mathbf{R}^m$. The barycentric coordinates of $X \in \mathbf{R}^m$ with respect to $V^i (i = 0, \dots, m)$ are denoted by $t := (t_0, t_1, \dots, t_m)$, i.e.,

$$X = \sum_{i=0}^m t_i V_i, \quad \sum_{i=0}^m t_i = 1$$

Thus, $\Omega_m := \{t := (t_0, t_1, \dots, t_m) \in \mathbf{R}^m | 0 \leq t_i \leq 1, i = 0, \dots, m, \sum_{i=0}^m t_i = 1\}$ is the standard m -simplex. We shall use the standard multi-index notation, for any $\alpha := (\alpha_0, \alpha_1, \dots, \alpha_m) \in \mathbf{Z}_+^{m+1}$ and $t := (t_0, t_1, \dots, t_m) \in \Omega_m$, define

$$\begin{aligned} |\alpha| &:= \alpha_0 + \alpha_1 + \dots + \alpha_m, \\ \alpha! &:= \alpha_0! \alpha_1! \dots \alpha_m!, \\ t^\alpha &:= t_0^{\alpha_0} t_1^{\alpha_1} + \dots t_m^{\alpha_m}. \end{aligned}$$

Assume that $f(t)$ is a vector function defined on the simplex Ω_m (i.e., the mapping from Ω_m to $\mathbf{R}^d$), $P_\alpha := f(\frac{\alpha}{n}), |\alpha| = n$, are called as the data points of $f(t)$, and $b^\alpha, |\alpha| = n$, are denoted as the Beizer control points (i.e., the B-net coordinates).

The operational symbol representation of the B-B surface on simplex has been discussed by several authors (see [deBoor, 87]). Due to the shift operator and the difference operator are linear and commutative, they could bring a great convenience for the representation and operation of the B-B surface on the simplex. In this paper the operational symbol representation of the B-B surface on the simplex Ω_m is discussed. Moreover, two forms of the interpolation polynomial on simplex Ω_m are given. Based on this, the relation between the Bézier control points and the data points of the B-B surface on the simplex Ω_m is revealed.

1. OPERATIONAL SYMBOL REPRESENTATION AND DERIVATIVE FORMULA OF THE B-B SURFACE

We first introduce the shift operator $E := (E_0, E_1, \dots, E_m)$, i.e., $b_\alpha := E^\alpha b_0 := E_0^{\alpha_0} E_1^{\alpha_1} \dots E_m^{\alpha_m} b_0$ where b_0 is a nominal symbol, subscript $0 := (0, 0, \dots, 0) \in \mathbf{Z}_+^{m+1}$, b_α is the Bézier control points, and denote that $tE := t_0 E_0 + t_1 E_1 + \dots + t_m E_m$. Then the n -th degree B-B surface on the simplex Ω_m can be expressed by

$$\begin{aligned} B^n(t) &:= \sum_{|\alpha|=n} B_\alpha^n(t) b_\alpha := \sum_{|\alpha|=n} \frac{n!}{\alpha!} t^\alpha b_\alpha \\ &= \sum_{|\alpha|=n} \frac{n!}{\alpha!} t^\alpha E^\alpha b_0 = (tE)^n b_0 \end{aligned} \quad (1)$$

where $B_\alpha^n(t) := \frac{n!}{\alpha!} t^\alpha$ is the Bernstein polynomial basis function of degree n on the simplex Ω_m .

Let $t, \tilde{t} \in \Omega_m$ be two different points. The directional derivatives of $B^n(t)$ in the direction $r := \tilde{t} - t$ is given by the chain rule:

$$D_r B^n(t) := D_\tau B^n(t + \tau r)|_{\tau=0}$$

$$= D_{\tau}((t + \tau r)E)^n|_{\tau=0} = n(rE)(tE)^{(n-1)}b_0 \quad (2)$$

High derivatives are obtained by iteration of this process:

$$D_r^k B^n(t) := \frac{n!}{(n-k)!} (rE)^k (tE)^{(n-k)} b_0 \quad (3)$$

To make use of the conventional notation [Boehm et. al., 84], we denote that $b_0^{(n-k)}(t) := (tE)^{(n-k)}b_0$, $b_{\alpha}^0(t) := b_{\alpha}$ and introduce auxiliary points $b_{\alpha}^{(n-k)}(t) := (E)^{\alpha} b_0^{(n-k)}(t)$, for $|\alpha| = k$, then k -th order directional derivative of $B^n(t)$ can be written in form

$$\begin{aligned} D_r^k B^n(t) &:= \frac{n!}{(n-k)!} (rE)^k (tE)^{(n-k)} b_0 \\ &= \frac{n!}{(n-k)!} \sum_{|\alpha|=k} B_{\alpha}^k(r) E^{\alpha} b_0^{(n-k)}(t) \\ &= \frac{n!}{(n-k)!} \sum_{|\alpha|=k} B_{\alpha}^k(r) b_{\alpha}^{(n-k)}(t) \end{aligned} \quad (4)$$

or alternatively in form

$$\begin{aligned} D_r^k B^n(t) &:= \frac{n!}{(n-k)!} (tE)^{(n-k)} (rE)^k b_0 \\ &= \frac{n!}{(n-k)!} \sum_{|\alpha|=n-k} B_{\alpha}^{(n-k)}(t) b_{\alpha}^k(r) \end{aligned} \quad (5)$$

Expressions (4) and (5) provide two different computation order for $D_r^k B^n(t)$.

2. THE DEBOOR-FIX FORMULA FOR THE B-B SURFACE ON SIMPLEX

The deBoor-Fix formulae are linear functions dual to the B-spline basis [deBoor,78]. In the special case of the Bézier curve, they can be written as follows [Lee,89]:

$$b_i = \frac{1}{n!} \sum_{k=0}^i (n-k)! \binom{i}{k} B^{(k)}(0), i = 0, 1, \dots, n \quad (6)$$

where $B(t) := \sum_{i=0}^n \binom{n}{i} (1-t)^{(n-i)} t^i b_i$ is the Bézier curve of degree n . In the case of the B-B surface on the simplex, [Zhao and Sun, 88] have found the dual bases of multivariate B-B polynomials. Now, we

generalize directly the formulae (6) by using the result obtained in the previous section.

In fact, by making use of the following notations: for any $\alpha, \beta \in \mathbf{Z}_+^{m+1}$, $\beta \leq \alpha$ means that:

$$\begin{aligned} \beta_i &\leq \alpha_i, i = 0, 1, \dots, m, \\ \sum_{\beta=0}^{\alpha} &:= \sum_{\beta_0=0}^{\alpha_0} \sum_{\beta_1=0}^{\alpha_1} \dots \sum_{\beta_m=0}^{\alpha_m}, \\ \binom{\alpha}{\beta} &= \begin{cases} \binom{\alpha_0}{\beta_0} \binom{\alpha_1}{\beta_1} \dots \binom{\alpha_m}{\beta_m} = \frac{\alpha!}{(\alpha-\beta)!} & \text{if } \beta \leq \alpha \\ 0 & \text{otherwise} \end{cases} \\ \bar{\alpha} &:= (\alpha_0, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_m), \end{aligned} \quad (7)$$

for a fixed $i (0 \leq i \leq m)$

$$\begin{aligned} e^k &:= \left(\underbrace{0, \dots, 0}_k, 1, 0, \dots, 0 \right) \in \mathbf{R}^{m+1}, k = 0, 1, \dots, m, \\ r^k &:= e^k - e^i, \text{ for } k \neq i \\ D_{\bar{r}}^{\bar{\beta}} &:= \prod_{k=0, k \neq i} D_{r^k}^{\beta_k}, \end{aligned}$$

we have the following result:

Theorem 2.1

For any $\alpha \in \mathbf{Z}_+^{m+1}$, $|\alpha| = n$, and any fixed $i, 0 \leq i \leq m$, the Bézier control points b_α of the n -th degree B-B surface on the simplex can be expressed by

$$b_\alpha = \frac{1}{n!} \sum_{\bar{\beta}=0}^{\bar{\alpha}} (n - |\bar{\beta}|)! \binom{\bar{\alpha}}{\bar{\beta}} D_{\bar{r}}^{\bar{\beta}} B^n(e^i) \quad (8)$$

Proof: In fact, for any $i, 0 \leq i \leq m$, and all $\alpha \in \mathbf{Z}_+^{m+1}$, $|\alpha| = n$,

$$\begin{aligned} b_\alpha &:= E^\alpha b_0 = \prod_{k=0}^m E_k^{\alpha_k} b_0 = \prod_{k=0}^m (E_k - E_i + E_i)^{\alpha_k} b_0 \\ &= \prod_{k=0, k \neq i}^m \sum_{\beta_k=0}^{\alpha_k} \binom{\alpha_k}{\beta_k} (E_k - E_i)^{\beta_k} E_i^{n-|\bar{\beta}|} b_0 \\ &= \sum_{\bar{\beta}=0}^{\bar{\alpha}} \binom{\bar{\alpha}}{\bar{\beta}} \prod_{k=0, k \neq i}^m (E_k - E_i)^{\beta_k} E_i^{n-|\bar{\beta}|} b_0 \end{aligned} \quad (9)$$

On the other hand, from (3) we have

$$\begin{aligned}
D_{\vec{r}}^{\vec{\beta}} B^n(e^i) &:= \prod_{k=0, k \neq i}^m D_{\vec{r}^k}^{\beta^k} B^n(t)|_{t=e^i} \\
&= \frac{n!}{(n-|\vec{\beta}|)!} \prod_{k=0, k \neq i}^m (E_k - E_i)^{\beta_k} (e^i E)^{n-|\vec{\beta}|} \mathbf{b}_0 \\
&= \frac{n!}{(n-|\vec{\beta}|)!} \prod_{k=0, k \neq i}^m (E_k - E_i)^{\beta_k} (E_i)^{n-|\vec{\beta}|} \mathbf{b}_0
\end{aligned} \tag{10}$$

From (8) and (9) we obtain (7), this completes the proof of the theorem.

3. THE INTERPOLATION POLYNOMIAL ON A SIMPLEX

Let $\mathbf{f}(t)$ be a function defined on the simplex Ω_m , and $P_\alpha := \mathbf{f}(\frac{\alpha}{n}), |\alpha| = n$, the data points of $\mathbf{f}(t)$. We introduce the following notation:

$$(nt)^{(\alpha)} := (nt_0)^{(\alpha_0)} (nt_1)^{(\alpha_1)} \dots (nt_m)^{(\alpha_m)}$$

where $(nt_k)^{(\alpha_k)} = (nt_k)(nt_k - 1) \dots (nt_k - \alpha_k + 1)$ is the factorial polynomial, then the n -th degree interpolation polynomial of $\mathbf{f}(t)$ on the simplex can be written as follows:

$$P_n(t) = \sum_{|\alpha|=n} \frac{(nt)^{(\alpha)}}{\alpha!} E^\alpha P_0 \tag{11}$$

where P_0 is a nomial symbol. In fact, for any $|\beta| = n$, we have

$$P_n(\frac{\beta}{n}) = \sum_{|\alpha|=n} \frac{\beta^{(\alpha)}}{\alpha!} E^\alpha P_0 \tag{12}$$

This is the Langrage type interpolation polynomial.

Let $\bar{t} := (t_0, \dots, t_{i-1}, t_i, \dots, t_m)$ consist of the m independent components of t except t_i , for any fixed $i, 0 \leq i \leq m$. Correspondingly set $\bar{\alpha} := (\alpha_0, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_m) \in \mathbb{Z}_+^m$. Define the following difference operator

$$\bar{\Delta}^\alpha := \prod_{k=0, k \neq i}^m \Delta_k^{\alpha_k}$$

where $\Delta_k := E_k - I$, and I is the identical operator. Then the n -th degree interpolation polynomial of $\mathbf{f}(t)$ on the simplex Ω_m can be written as

follows:

$$\bar{P}_n(\bar{t}) = \sum_{|\bar{\alpha}| \leq n} \frac{(n\bar{t})^{\bar{\alpha}}}{\bar{\alpha}!} \bar{\Delta}^{\bar{\alpha}} P_{\bar{0}} \quad (13)$$

In fact, for any $\bar{\beta} \in \mathbf{Z}_+^m$, $|\bar{\beta}| \leq n$, we have

$$\begin{aligned} \bar{P}_n\left(\frac{\bar{\beta}}{n}\right) &= \sum_{|\bar{\alpha}| \leq n} \frac{(\bar{\beta})^{(\bar{\alpha})}}{\bar{\alpha}!} \bar{\Delta}^{\bar{\alpha}} P_{\bar{0}} \\ &= \prod_{k=0, k \neq i}^m \sum_{\alpha_k=0}^{\beta_k} \frac{\beta_k^{(\alpha_k)}}{\alpha_k!} \Delta_k^{\alpha_k} P_{\bar{0}} \\ &= \prod_{k=0, k \neq i}^m E_k^{\beta_k} P_{\bar{0}} = P_{\bar{\beta}} \end{aligned} \quad (14)$$

This is the Newton type interpolation polynomial.

Now we suppose $\bar{f}(\bar{t}) \equiv f(t)$, where $t_i = 1 - t_0 - \dots - t_{i-1} - t_{i+1} - \dots - t_m$. Recall $r^k := e^k - e^i$, we have:

$$\begin{aligned} D_{r^k} f(t) &:= D_{\tau} f(t + \tau r^k)|_{\tau=0} \\ &= (D_k - D_i) f(t) = D_k \bar{f}(\bar{t}) \end{aligned} \quad (15)$$

where $D_k := \frac{\partial}{\partial t_k}$, Introduce the notation

$$S_{\bar{\beta}}^{\bar{\alpha}} := \prod_{k=0, k \neq i} S_{\beta_k}^{\alpha_k} \quad (16)$$

where

$$S_{\beta_k}^{\alpha_k} := \frac{D^{\beta_k} \tau^{(\alpha_k)}}{\beta_k!} \Big|_{\tau=0} \quad (17)$$

is the Stirling numbers of the first kind [Richard,54]. They satisfy the following recurrence relation

$$S_i^{k+1} = S_{i-1}^k - k S_i^k \quad (18)$$

Here we assume that $S_0^0 = 1$, $S_0^k = 0$ if $k > 0$, and $S_i^k = 0$ if $i > k$. A table of S_i^k can be found in [Richardson, 54]. Thus we have the following theorem:

Theorem 3.1

For any fixed i , $0 \leq i \leq m$, and any $\bar{\beta} \in \mathbf{Z}_+^m$, $|\bar{\beta}| \leq n$, we have

$$D_{\bar{r}}^{\bar{\beta}} P_n(e^i) = \sum_{\bar{\alpha} \geq \bar{\beta}, |\bar{\alpha}| \leq n} \frac{n^{|\bar{\beta}|} \bar{\beta}!}{\bar{\alpha}!} S_{\bar{\beta}}^{\bar{\alpha}} \bar{\Delta}^{\bar{\alpha}} P_{\bar{0}} \quad (19)$$

where $D_{\bar{r}}^{\bar{\beta}} := \prod_{k=0, k \neq i}^m D_{r^k}^{\beta_k}$.

Proof: From (14) and (12), we have

$$\begin{aligned} D_{\bar{r}}^{\bar{\beta}} P_n(e^i) &= \prod_{k=0, k \neq i}^m D_{r^k}^{\beta_k} P_n(t)|_{t=e^i} = \prod_{k=0, k \neq i}^m D_{r^k}^{\beta_k} \bar{P}_n(\bar{t})|_{\bar{t}=\bar{0}} \\ &= \sum_{|\bar{\alpha}| \leq n} \frac{(\prod_{k=0, k \neq i}^m D_k^{\beta_k})(n\bar{t})^{(\bar{\alpha})}}{\bar{\alpha}!} \bar{\Delta}^{\bar{\alpha}} P_{\bar{0}}|_{\bar{t}=\bar{0}} \end{aligned} \quad (20)$$

According to (16),

$$\begin{aligned} (\prod_{k \neq i} D_k^{\beta_k})(n\bar{t})^{(\bar{\alpha})}|_{\bar{t}=\bar{0}} &= \prod_{k \neq i} D_k^{\beta_k} (nt_k)^{(\alpha_k)}|_{t_k=0} \\ &= \prod_{k \neq i} S_{\beta_k}^{\alpha_k} (\beta_k)! n^{\beta_k} \\ &= n^{|\beta|} \bar{\beta}! S_{\bar{\beta}}^{\bar{\alpha}} \end{aligned} \quad (21)$$

and substituting (20) into (19), we obtain (18). This completes the proof of the theorem.

4. THE RELATION BETWEEN THE BÉZIER CONTROL POINTS AND THE DATA POINTS OF THE B-B SURFACE ON THE SIMPLEX Ω_M

Let $B^n(t)$ be the n -th degree B-B surface on the simplex Ω_m, b_α , $|\alpha| = n$, corresponding to the Bézier control points, and $P_\alpha := B^n(\frac{\alpha}{n})$ the data points. Now we may represent the Bézier control points b_α by using the data points P_α . For any fixed i , $0 \leq i \leq m$, and arbitrary $\alpha \in Z_+^{m+1}$, $|\alpha| = n$, according to Theorem 2.1 and Theorem 3.1, we obtain immediately

$$\begin{aligned} b_\alpha &= \frac{1}{n!} \sum_{\bar{\beta}=\bar{0}}^{\bar{\alpha}} (n - |\bar{\beta}|)! \binom{\bar{\alpha}}{\bar{\beta}} D_{\bar{r}}^{\bar{\beta}} B^n(e^i) \\ &= \frac{1}{n!} \sum_{\bar{\beta}=\bar{0}}^{\bar{\alpha}} (n - |\bar{\beta}|)! \binom{\bar{\alpha}}{\bar{\beta}} \sum_{\bar{\lambda} \geq \bar{\beta}, |\bar{\lambda}| \leq n} \frac{n^{|\bar{\beta}|} \bar{\beta}!}{\bar{\lambda}!} S_{\bar{\beta}}^{\bar{\lambda}} \bar{\Delta}^{\bar{\lambda}} P_{\bar{0}} \\ &= \frac{1}{n!} \sum_{|\bar{\lambda}| \leq n} \sum_{\bar{\beta}=\bar{0}}^{\bar{\alpha}} \frac{\bar{\alpha}! (n - |\bar{\beta}|)! n^{|\bar{\beta}|} S_{\bar{\beta}}^{\bar{\lambda}}}{(\bar{\alpha} - \bar{\beta})! \bar{\lambda}!} \bar{\Delta}^{\bar{\lambda}} P_{\bar{0}} \end{aligned} \quad (22)$$

where

$$\begin{aligned}
\bar{\Delta}^{\bar{\lambda}} P_{\bar{0}} &= \prod_{k=0, k \neq i}^m \Delta_k^{\lambda_k} P_{\bar{0}} = \prod_{k=0, k \neq i}^m (E_k - I)^{\lambda_k} P_{\bar{0}} \\
&= \prod_{k=0, k \neq i}^m \sum_{\mu_k=0}^{\lambda_k} \binom{\lambda_k}{\mu_k} (-1)^{\lambda_k - \mu_k} E_k^{\mu_k} P_{\bar{0}} \\
&= \sum_{\bar{\mu}=\bar{0}}^{\bar{\lambda}} \binom{\bar{\lambda}}{\bar{\mu}} (-1)^{|\bar{\lambda}| - |\bar{\mu}|} \bar{E}^{\bar{\mu}} P_{\bar{0}}
\end{aligned} \tag{23}$$

Substituting (22) into (21), we have

$$\begin{aligned}
b_{\alpha} &= \frac{1}{n!} \sum_{|\bar{\lambda}| \leq n} \sum_{\bar{\beta}=\bar{0}}^{\bar{\alpha}} \frac{\bar{\alpha}! (n - |\bar{\beta}|)! n^{|\bar{\beta}|} S_{\bar{\beta}}^{\bar{\lambda}}}{(\bar{\alpha} - \bar{\beta})! \bar{\lambda}!} \sum_{\bar{\mu}=\bar{0}}^{\bar{\lambda}} \\
&\quad \binom{\bar{\lambda}}{\bar{\mu}} (-1)^{|\bar{\lambda}| - |\bar{\mu}|} \bar{E}^{\bar{\mu}} P_{\bar{0}} \\
&= \frac{1}{n!} \sum_{|\bar{\mu}| \leq n} \sum_{\bar{\lambda} \geq \bar{\mu}, |\bar{\lambda}| \leq n} \sum_{\bar{\beta}=\bar{0}}^{\bar{\alpha}} \\
&\quad \frac{(-1)^{|\bar{\lambda}| - |\bar{\mu}|} (n - |\bar{\beta}|)! n^{|\bar{\beta}|} \bar{\alpha}! S_{\bar{\beta}}^{\bar{\lambda}}}{(\bar{\alpha} - \bar{\beta})! (\bar{\lambda}! - \bar{\mu}!) \bar{\mu}!} P_{\bar{m}\bar{u}}
\end{aligned} \tag{24}$$

Thus, we obtain the following result.

Theorem 4.1

For any fixed $i, 0 \leq i \leq m$, and any $\alpha \in Z_+^{m+1}$, $|\alpha| = n$, we have

$$b_{\alpha} = \sum_{|\bar{\mu}| \leq n} C_{\bar{\mu}}(\bar{\alpha}) P_{\bar{\mu}} \tag{25}$$

where

$$\begin{aligned}
C_{\bar{\mu}} &= \frac{1}{n!} \sum_{|\bar{\mu}| \leq n} \sum_{\bar{\lambda} \geq \bar{\mu}, |\bar{\lambda}| \leq n} \sum_{\bar{\beta}=\bar{0}}^{\bar{\alpha}} \\
&\quad \frac{(-1)^{|\bar{\lambda}| - |\bar{\mu}|} (n - |\bar{\beta}|)! n^{|\bar{\beta}|} \bar{\alpha}! S_{\bar{\beta}}^{\bar{\lambda}}}{(\bar{\alpha} - \bar{\beta})! (\bar{\lambda}! - \bar{\mu}!) \bar{\mu}!}
\end{aligned} \tag{26}$$

By means of Theorem 4.1, when given data points $P_{\bar{\alpha}}$, we can construct immediately the interpolation surface on the simplex Ω_m in terms of the B-B surface. This is useful in CAGD and scientific visualization[L. Wang, 1997, Z. Wang, 2000].

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